Examples (c) $P(n) : \frac{1}{3^3} + \frac{1}{3^4} + \cdots + \frac{1}{(n+1)(n+2)} = \frac{n}{2n+4}$, $n \in \mathbb{N}$.

(i) $P(n) : 4((5^n) - 1)$.

(e) $P(n) : 6(n^3 - n)$.

[We can prove (e) by the method of minimum counter example. See Section 6.3.]

(c)

1. \( P(1) : \frac{1}{2 \cdot 3} = \frac{1}{2 \cdot 1 + 4} \) is true.

2. Assume \( P(k) \) is true, i.e., \( \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k}{2k+4} \).

Consider \( P(k+1) \):

\[
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k}{2k+4} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{2(k+1)+4}
\]

i.e., \( P(k+1) \) is true.

By the principle of mathematical induction, \( P(n) \) is true for all \( n \in \mathbb{N} \).

(d) \( P(n) : 4 \mid (5^n - 1), \quad n \in \mathbb{N} \)

Prove \( P(n) \) holds for all \( n \in \mathbb{N} \) by induction.

Step 1. \( P(1) : 5^1 - 1 = 4 \) is divisible by 4, i.e., \( P(1) \) holds.

Step 2. Assume \( P(k) \) is true, i.e., \( 5^k - 1 = 4q \) for some \( q \in \mathbb{Z} \).

Consider \( P(k+1) \):

\[
5^{k+1} - 1 = 5^k \cdot 5^1 - 1 = 5^k \cdot 5 - 1 = 5^k \cdot 5^1 - 1 \cdot 5^1 + 1 \cdot 5^1 - 1
\]

\[
= (5^k - 1) \cdot 5^1 + 4 \cdot 5^1 + 4 = 4(5^k + 1)
\]

i.e., \( P(k+1) \) is true.

By PMI, \( P(n) \) is true for all \( n \in \mathbb{N} \).
(e) Consider \( P(n) : b \mid (n^3 - n) \).

Prove by induction that \( P(n) \) holds for all \( n \in \mathbb{N} \).

**Step 1.** \( P(1) : 1^3 - 1 = 0 \) is divisible by \( 6 \).
\[ \therefore P(1) \text{ holds} \]

**Step 2.** Assume \( P(k) \) holds, \( k \geq 1 \)
\[ \therefore k^3 - k = 6b \quad \text{for some } b \in \mathbb{Z} \.

Consider \( P(k+1) \)
\[
(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1)
\]
\[
= k^3 + 3k^2 + 3k - k
\]
\[
= (k^3 - k) + 3k(k+1)
\]
\[
= 6b + 3(k)(k+1)
\]
\[
= 6b + 3 \cdot 2 \cdot r \quad \text{because } \frac{k(k+1)}{2} = r \in \mathbb{Z}
\]
\[
= 6(3+r)
\]
\[ \text{is divisible by } 6. \]
\[ \therefore P(k+1) \text{ holds} \]

By P M I, \( P(n) \) holds, the result follows.
Principle of MI - 3 Suppose we can establish the statements.

(a) \( P(m) \) is true for a certain \( m \in \mathbb{Z} \).

(b) For \( k \geq m \) if \( P(j) \) is true for all \( j = m, \ldots, k \), then \( P(k+1) \) is true.

Then \( P(n) \) is true for all \( n \in \mathbb{Z} \) with \( n \geq m \).

Remark We use all the previous fallen dominoes to ensure the next domino would fall also.

Example Every integer \( n \geq 2 \) is a prime or a product of primes.
6.2/6.4 More general principles

**Principle of MI - 2** Suppose we can establish the statements.

(a) $P(m)$ is true for a certain $m \in \mathbb{Z}$.

(b) For $k \geq m$, if $P(k)$ is true then $P(k + 1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{Z}$ with $n \geq m$.

**Remark** This follows from the fact that $S = \{ n \in \mathbb{Z} : n \geq m \}$ is well ordered.

**Examples**

(a) For every $n \geq 5$, $2^n > n^2$.

(b) Let $A_1, \ldots, A_n$ be sets with $n \geq 2$. Then $\bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \overline{A_i}$.

(c) Suppose $n \geq 0$. If a set with $n$ elements then its power set has $2^n$ elements.
(c) Prove by induction for $\forall n \in \mathbb{N}$:

$$2^n > n^2$$ for $n \geq 5$.

**Proof**

Step 1. $P(5)$:

$$2^5 = 32 > 25 = 5^2.$$ 

$\therefore P(5)$ holds.

Step 2. Assume $P(k)$ holds, $k \geq 5$.

$2^k > k^2$, i.e., $2^k - k^2 > 0$.

Consider $P(k+1)$

$$2^{k+1} = 2^k \cdot 2 > (k^2 + 2k + 1)$$

$$= (2^k - k^2) + (2k - 2k - 1)$$

$$> 2^k - 2k - 1 > k^2 - 2k - 1 \geq 5k - 2k - 1 = 3k - 1 > 0 \implies 15 - 1 = 14 > 0.$$ 

$$\therefore P(k+1)$$
**Recursive sequences** Induction is useful in proving formulas and properties of recursive sequences, i.e., sequences \( \{a_1, a_2, \ldots\} \) defined by specification of \( a_1, \ldots, a_k \) and a relation/formula expressing \( a_m \) in terms of the previous \( a_1, \ldots, a_k \), for \( m > k \).

**Examples** (a) Prove that \( a_n = n^2 \) for all \( n \in \mathbb{N} \) if the sequence \( \{a_1, a_2, \ldots\} \) is defined recursively by

\[
a_1 = 1, \quad a_2 = 4, \quad \text{and} \quad a_n = 2a_{n-1} - a_{n-2} + 2 \quad \text{for} \quad n \geq 3.
\]

(b) Find a formula for \( a_n \) with a proof if the sequence \( \{a_1, a_2, \ldots\} \) is defined recursively by

\[
a_1 = 1, \quad a_2 = 2, \quad \text{and} \quad a_n = a_{n-1} + 2a_{n-2} \quad \text{for} \quad n \geq 3.
\]

---

\( (a) \) **Conjecture:** \( P(n) : a_n = n^2 \), \( n \in \mathbb{N} \)

\[
\begin{align*}
\text{Step 1.} & \quad P(1) ; P(2) \\
& \quad a_1 = 1^2, \quad a_2 = 2^2 \quad \text{are true.}
\end{align*}
\]

\[
\begin{align*}
\text{Step 2.} & \quad \text{Assume} \ P(1), \ldots, P(k) \\
& \quad \text{are true,} \quad k \geq 2
\end{align*}
\]

\[
\begin{align*}
& \quad a_1 = 1^2, \quad a_2 = 2^2, \ldots, \quad a_k = k^2
\end{align*}
\]

**Consider** \( P(k+1) \).

\[
\begin{align*}
a_{k+1} & = 2 \cdot a_k - a_{k-1} + 2 \\
& = 2 \cdot k^2 - (k-1)^2 + 2 \\
& = 2k^2 - (k^2 - 2k + 1) + 2 = k^2 + 2k + 1 \\
& = (k+1)^2
\end{align*}
\]

\[
\begin{align*}
& \quad P(k+1) \text{ is true.}
\end{align*}
\]

**By P.M.I., the result follows.**