Chapter 2 Logic

We study the mathematical language to read and write correct (logical) mathematics.

2.1 Statements

- A **statement** is a sentence/assertion which we can decide that it is true (T) or false (F).
- Examples. The integer 57 is a prime number. It is raining now. $2 + 4 = 6$.
- An **open sentence** is an assertion with one or more variables chosen from a domain $S$.
- Example. $P(x): x > 3$. Here the domain can be $\mathbb{N}$, $\mathbb{Q}$, or $\mathbb{R}$.
- A statement may be true (T) or false (F); two statements have 4 possible combination;
- 3 statements have 8 possible combination; ...; $n$ statements have $2^n$ combination.
- We may draw the truth table for that.
2.2/2.3 Negation, disjunction, and conjunction of statements

- Negation of $P$, denoted by $\sim P$;
- Disjunction: $P$ or $Q$, denoted by $P \lor Q$;
- Conjunction: $P$ and $Q$, denoted by $P \land Q$.
- Examples and truth tables.
2.4/2.5/2.6 Implication and biconditional

- Implication: If $P$ then $Q$ (also, $P$ implies $Q$), denoted by $P \Rightarrow Q$.
- The statement $P$ is the hypothesis/premise, and the statement $Q$ is the conclusion.
- Biconditional: $P$ is equivalent to $Q$ (also, $P$ if and only if $Q$), denoted by $P \iff Q$.
- Examples and truth tables.
2.7 Tautologies and contradiction

**Tautology** In a compound statement, all possible combination of the components yield T.

**Contradiction** In a compound statement, all possible combination of the components yield F.
2.8/2.9 Logical equivalence and properties

Two compound statements $R$ and $S$ are logically equivalent, denoted by $R \equiv S$ when they have the same truth values for different combination of the component statements.

Examples.

$$(P \Rightarrow Q) \equiv (\sim P \lor Q);$$

Commutative, associative, and distributive laws for $\lor$ and $\land$;

De Morgan’s Law.
2.10 Quantifiers

There exists: \( \exists \); for all: \( \forall \).

These usually go with open statements with variables from a domain.

**Examples** For every real number \( r \), \((r + 1)^2 > 0\). \( \forall r \in \mathbb{R}, (r + 1)^2 > 0 \).

There is a real number \( r \) such that \( r^2 = -1 \). \( \exists r \in \mathbb{R}, r^2 = -1 \).
2.11 Characterizations

In mathematics, we often want to give equivalent conditions for a certain concept. That is, give characterizations for a certain property or structure.