Chapter 19 Vector Spaces
A **vector space** $V$ over a field $\mathbb{F}$ is an Abelian group $(V,+)$ with a scalar multiplication $\mu v$ for any $\mu \in \mathbb{F}$ and $v \in V$ such that $1v = v$, $(ab)v = a(bv)$, $a(u + v) = au + av$ $(a + b)v = av + bv$ for any $a, b \in \mathbb{F}$ and $u, v \in V$.

**Examples** $\mathbb{F}^n$, $M_n(\mathbb{F})$, $\mathbb{F}[x]$.

**Examples** An extension field $\mathbb{E}$ over the ground field.

(a) $\mathbb{C}$ over $\mathbb{R}$. (b) $\mathbb{R}$ over $\mathbb{Q}$. (c) $\mathbb{Z}_p[x]/\langle f(x) \rangle$ over $\mathbb{Z}_p$.

- A set $S \subseteq V$ is linearly dependent if there is a nontrivial combination of a finite collection of vectors in $S$ equal to 0.
- It is a basis if $V$ if it is a spanning set of $V$. 
Theorem Every vector space has a basis. If $V$ has a basis with $n$ elements, then every basis has $n$ elements. In such a case, we say that $V$ has dimension $n$. We use the convention that dimension $V$ is 0 if $V = \{0\}$.

Question Can we say that two bases of a vector space must have the same cardinality? [A writing project?]

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