Constructing a straight line intersecting four lines

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Abstract

In this paper, we determine the set $\mathcal{S}$ of straight lines L_0 that have intersections with four given distinct lines $L_1, \ldots, L_4$ in $\mathbb{R}^3$. If any two of the four given lines are skew, i.e., not co-planar, Bielinski and Lapinska used techniques in projective geometry to show that there are either zero, one, or two elements in the set $\mathcal{S}$. Using linear algebra techniques, we determine $\mathcal{S}$ and show that there are no, one, two or infinitely many elements L_0 in $\mathcal{S}$, where the last case was overlooked in the earlier paper. For the sake of completeness, we provide a comprehensive determination of all the elements L_0 in $\mathcal{S}$ if at least two of the four given lines are co-planar. In this scenario, there may also be zero, one, two, or infinitely many solutions.

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1 Introduction

Affine maps are a fundamental concept in mathematics that find applications in various fields such as computer graphics, computer vision, physics, and engineering; see [3, 4, 5, 7]. They are especially critical in deep learning and neural networks, with an affine transformation being the most commonly type of linear transformation in neural networks (see [7]). In addition to their use in geometric contexts, affine maps are also important in linear algebra and functional analysis. In linear algebra, affine maps are used to study the geometry of vector spaces and the properties of linear transformations. In this paper, we use affine transformations to determine straight lines that intersect four different lines in $\mathbb{R}^3$. In the formal mathematical setting, we solve the following problem.

Problem 1.1. *For $j = 1, 2, 3, 4$, let $L_j = \{\mathbf{u}_j + t\mathbf{v}_j : t \in \mathbb{R}\}$ be straight lines in $\mathbb{R}^3$ such that $\mathbf{v}_j$ is nonzero. Determine the set $\mathcal{S}$ of all straight lines $L_0 = \{\mathbf{u}_0 + t\mathbf{v}_0 : t \in \mathbb{R}\}$, which will intersect L_j for $j = 1, 2, 3, 4$. Here L_0 may not exist.*

Previous work by Bielinski and Lapinska [1] provided a solution to this problem using Monge's projections. Their approach involves selecting an appropriate system of projection planes and applying Steiner's construction of common lines of two projective pencils of lines. The authors showed that there can be either no, one, or two lines in the solution set $\mathcal{S}$. Furthermore, the authors used these results in [2] to study the problem of finding a circle that orthogonally intersects two non-coplanar circles. In [6], the authors studied the general problem of "finding best lines passing through a set of straight lines", and discussed the applications of such results in archaeological pottery analysis, precision manufacturing, and 3D modelling. In particular, they referred to the work in [1] showing that it is possible that there is no straight line intersecting 4 given straight lines.

In Section 2, we use elementary linear algebra techniques to give a descriptions of all the elements L_0 in $\mathcal{S}$. In particular, we show that there are none, one, two or infinitely many elements L_0 in $\mathcal{S}$, where the last case was missed in the paper [1] by Bielinski and Lapinska. Furthermore, concrete examples are given for the four cases when $\mathcal{S}$ has none, one, two or infinitely many solutions.

In Section 3, we consider the case where two of the four given lines lie in the same plane and provide a description of the solutions $L_0 \in \mathcal{S}$ for this scenario. Similar to the case where the given lines are in general positions, there can be no, one, two, or infinitely many solutions.

Given the four lines $L_1, \dots, L_4$, and let $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\ \mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3\ \mathbf{u}_4] \in M_{3,8}$ and $V = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4] \in M_{3,4}$, where $\mathbf{v}_j = (v_{j1}, v_{j2}, v_{j3})^t$ and $\mathbf{u}_j = (u_{j1}, u_{j2}, u_{j3})^t$ for $j = 1, 2, 3, 4$. We can simplify the problem by performing the following operations on the given lines $L_1, \dots, L_4$.

(1) We may relabel the indices of $L_1, \dots, L_4$, i.e., permute the first four columns of A and the same permutation to the last four columns of A.

(2) For $j = 1, 2, 3, 4$, we may modify the $\mathbf{v}_j$ and $\mathbf{u}_j$ in the definition of L_j as follows:

(2.a) Replace $\mathbf{v}_j$ by $\pm\mathbf{v}_j/\|\mathbf{v}_j\|$ and assume that $\mathbf{v}_j$ has unit length and the first nonzero entry is positive.

(2.b) We may assume $\mathbf{u}_j^t\mathbf{v}_j = 0$ by replacing $\mathbf{u}_j$ with $\mathbf{u}_j - \mathbf{u}_j^t\mathbf{v}_j\mathbf{v}_j$, the projection of u_j to the orthogonal complement of v_j. In particular, if $\mathbf{v}_j = \mathbf{e}_j$ is one of the basic vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3$, we may assume that $\mathbf{u}_j$ has a zero entry at the jth position.

These operations will simplify the problem by reducing the number of degrees of freedom in the vectors $\mathbf{v}_j$ and $\mathbf{u}_j$, while preserving the essential geometric properties of the lines

$L_1, \ldots, L_4$. In particular, the operation (2.b) ensures that the vectors $\mathbf{u}_j$ are orthogonal to the vectors $\mathbf{v}_j$ and have a zero component in the direction of $\mathbf{v}_j$.

In our proofs, we apply invertible affine transformations $T(\mathbf{x}) = R\mathbf{x} - \mathbf{x}_0$ to transform the four given lines to a set of lines $T(L_j) = \tilde{L}_j$ with simple forms, $j = 1, 2, 3, 4$. Here, $R \in M_3$ is an invertible matrix, and $\mathbf{x}_0 \in \mathbb{R}^3$ is a vector. Then we identify the lines $\tilde{L}_0$ (if they exist) that intersect with $\tilde{L}_1, \ldots, \tilde{L}_4$ and obtain the solutions $L_0 = T^{-1}(\tilde{L}_0)$ under the inverse map $T^{-1}(\mathbf{x}) = R^{-1}\mathbf{x} + R^{-1}\mathbf{x}_0$. Notice that

$$T(\mathbf{u}_j + t\mathbf{v}_j) = R(\mathbf{u}_j + t\mathbf{v}_j) - \mathbf{x}_0 = R\mathbf{u}_j - \mathbf{x}_0 + tR\mathbf{v}_j \quad \text{for} \quad j = 1, 2, 3, 4.$$

The affine transformation T will transform A to $RA - [\mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{x}_0\ \mathbf{x}_0\ \mathbf{x}_0\ \mathbf{x}_0]$. In many cases, we can set $\mathbf{x}_0 = R\mathbf{u}_j$ and assume that $\mathbf{u}_j$ is the zero vector after the affine transformation.

The following simple result determines whether two given lines are co-planar, and it is useful in our discussion.

Lemma 1.2. *Let $L = \{\mathbf{u} + t\mathbf{v} : t \in \mathbb{R}\}$ and $\hat{L} = \{\hat{\mathbf{u}} + t\hat{\mathbf{v}} : t \in \mathbb{R}\}$. The two lines L and $\hat{L}$ lie in the same plane if and only if*

$$\det\left(\begin{bmatrix}\mathbf{v} & \hat{\mathbf{v}} & \hat{\mathbf{u}}\end{bmatrix}\right) = \det\left(\begin{bmatrix}\mathbf{v} & \hat{\mathbf{v}} & \mathbf{u}\end{bmatrix}\right). \tag{1}$$

Furthermore, L intersects $\hat{L}$ at one point only if and only if Eq(1) holds and $\mathbf{v}$ and $\hat{\mathbf{v}}$ are not multiple of each other.

Proof. Note that L and $\hat{L}$ lie in the same plane if and only if the two lines $L' = \{t\mathbf{v} : t \in \mathbb{R}\}$ and $\hat{L}' = \{(\hat{\mathbf{u}} - \mathbf{u}) + t\hat{\mathbf{v}} : t \in \mathbb{R}\}$ lie in a 2-dimensional plane containing the origin. Thus the 3×3 matrix $T = \begin{bmatrix}\mathbf{v} & \hat{\mathbf{v}} & \hat{\mathbf{u}} - \mathbf{u}\end{bmatrix}$ formed by the three vectors $\mathbf{v}$, $\hat{\mathbf{v}}$ and $\hat{\mathbf{u}} - \mathbf{u}$ is has rank at most two. Therefore, the matrix T has zero determinant, i.e., $\det T = 0$; equivalently,

$$0 = \det\left(\begin{bmatrix}\mathbf{v} & \hat{\mathbf{v}} & \hat{\mathbf{u}} - \mathbf{u}\end{bmatrix}\right) = \det\left(\begin{bmatrix}\mathbf{v} & \hat{\mathbf{v}} & \hat{\mathbf{u}}\end{bmatrix}\right) - \det\left(\begin{bmatrix}\mathbf{v} & \hat{\mathbf{v}} & \mathbf{u}\end{bmatrix}\right).$$

The last assertion is clear. □

2 Solution when no two lines are co-planner

In this section, we present the solution of Problem 1.1 when no two of the four given lines $L_1, \ldots, L_4$ are co-planar. Let $\mathcal{S}$ be the set of lines L_0 having nonempty intersection with $L_1, \ldots, L_4$ in $\mathbb{R}^3$.

Theorem 2.1. *Suppose $L_1, \ldots, L_4$ are four lines in $\mathbb{R}^3$ such that no two of them lie in the same plane. Apply a suitable affine transformation and assume that*

$$L_1 = \{t\mathbf{e}_1 : t \in \mathbb{R}\},\ L_2 = \{t\mathbf{e}_2 + \mathbf{e}_3 : t \in \mathbb{R}\},\ \textit{and}\ L_j = \{t\mathbf{v}_j + \mathbf{u}_j : t \in \mathbb{R}\}\ \textit{for}\ j = 3, 4,$$

where $\mathbf{v}_j = (v_{j1}, v_{j2}, v_{j3})^t$ *and* $\mathbf{u}_j = (u_{j1}, u_{j2}, u_{j3})^t$*. For* $j = 3, 4$*, let*

$$(a_j, b_j, c_j) = \mathbf{v}_j \times \mathbf{u}_j = \left(\begin{vmatrix} v_{j2} & u_{j2} \\ v_{j3} & u_{j3} \end{vmatrix}, - \begin{vmatrix} v_{j1} & u_{j1} \\ v_{j3} & u_{j3} \end{vmatrix}, \begin{vmatrix} v_{j1} & u_{j1} \\ v_{j2} & u_{j2} \end{vmatrix} \right).$$

Then $\mathcal{S}$ *consists of elements of the form*

$$L_0 = \{t(t_1, t_2, 1) + (-t_1, 0, 0) : t \in \mathbb{R}\},$$

where (t_1, t_2) *satisfies* $v_{j3}t_1 + b_j \neq 0$ *for* $j = 3, 4$*, and*

$$t_2 = -\frac{(a_3 - v_{32})t_1 + c_3}{v_{33}t_1 + b_3} = -\frac{(a_4 - v_{42})t_1 + c_4}{v_{43}t_1 + b_4}. \tag{2}$$

Consequently, $|\mathcal{S}|$ *can be zero, one, two, or infinity, which equals the number of roots of the polynomial*

$$p(z) = (v_{33}z + b_3)((a_4 - v_{42})z + c_4) - (v_{43}z + b_4)((a_3 - v_{32})z + c_3)$$

in $\mathbb{R} \setminus \cup_{j=3}^4 \{z : v_{j3}z + b_j = 0\}$*.*

Proof. Suppose $L_j = \{\mathbf{u}_j + t\mathbf{v}_j : t \in \mathbb{R}\}$, $j = 1, 2, 3, 4$, are given such that L_1 and L_2 do not lie in the same plane. Then by Lemma 1.2, $R = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{u}_2 - \mathbf{u}_1 \end{bmatrix}$ has rank three. By the affine transform $\mathbf{x} \mapsto R^{-1}(\mathbf{x} - \mathbf{u}_1)$ we may assume that

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{0} & \mathbf{e}_3 \end{bmatrix}.$$

Thus, the lines take the form $L_1 = \{t\mathbf{e}_1 : t \in \mathbb{R}\}$, $L_2 = \{t\mathbf{e}_2 + \mathbf{e}_3 : t \in \mathbb{R}\}$ and $L_j = \{t\mathbf{v}_j + \mathbf{u}_j : t \in \mathbb{R}\}$ for $j = 3, 4$ with $\mathbf{v}_j = (v_{j1}, v_{j2}, v_{j3})^t$ and $\mathbf{u}_j = (u_{j1}, u_{j2}, u_{j3})^t$.

If $L_0 = \{t\mathbf{v}_0 + \mathbf{u}_0 : t \in \mathbb{R}\}$ is a solution in $\mathcal{S}$, then $L_0 \cap L_1$ is nonempty. We may assume that $\mathbf{u}_0 = -t_1\mathbf{e}_1 \in L_0 \cap L_1$ for some $t_1 \in \mathbb{R}$. On the other hand, $L_0 \cap L_2$ is nonempty implies $s_2\mathbf{v}_0 - t_1\mathbf{e}_1 = t_2\mathbf{e}_2 + \mathbf{e}_3$ for some $s_2, t_2 \in \mathbb{R}$. Replacing $\mathbf{v}_0$ by $s_2\mathbf{v}_0$, if necessary, one can assume that $s_2 = 1$ and $\mathbf{v}_0 = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + \mathbf{e}_3 = (t_1, t_2, 1)^t$.

Let $j \in \{3, 4\}$. The line L_j intersects neither L_1 nor L_2. Thus, L_0 cannot be L_j. So L_0 is a solution if and only if L_0 intersects L_j at one point only. By Lemma 1.2 $\mathbf{v}_0$ is not parallel to $\mathbf{v}_j$ and the cofactor expansion along the first row

$$a_jt_1 + b_jt_2 + c_j = \begin{vmatrix} t_1 & v_{j1} & u_{j1} \\ t_2 & v_{j2} & u_{j2} \\ 1 & v_{j3} & u_{j3} \end{vmatrix} = \begin{vmatrix} t_1 & v_{j1} & -t_1 \\ t_2 & v_{j2} & 0 \\ 1 & v_{j3} & 0 \end{vmatrix} = -v_{j3}t_1t_2 + v_{j2}t_1,$$

where $a_j \equiv \begin{vmatrix} v_{j2} & u_{j2} \\ v_{j3} & u_{j3} \end{vmatrix}$, $b_j \equiv - \begin{vmatrix} v_{j1} & u_{j1} \\ v_{j3} & u_{j3} \end{vmatrix}$, and $c_j \equiv \begin{vmatrix} v_{j1} & u_{j1} \\ v_{j2} & u_{j2} \end{vmatrix}$. Or equivalently,

$$0 = v_{j3}t_1t_2 + (a_j - v_{j2})t_1 + b_jt_2 + c_j = (v_{j3}t_1 + b_j)t_2 + (a_j - v_{j2})t_1 + c_j. \tag{3}$$

We **claim** that *for a fixed* $t_1 \in \mathbb{R}$*, there is at most one* t_2 *satisfying* (3).

To prove our claim, suppose there exist two distinct $\hat{t}_2, \tilde{t}_2 \in \mathbb{R}$ such that both the two lines $\hat{L}_0 = \{t\hat{\mathbf{v}}_0 + \mathbf{u}_0 : t \in \mathbb{R}\}$ and $\tilde{L}_0 = \{t\tilde{\mathbf{v}}_0 + \mathbf{u}_0 : t \in \mathbb{R}\}$, with $\mathbf{u}_0 = -t_1\mathbf{e}_1$, $\hat{\mathbf{v}}_0 = (t_1, \hat{t}_2, 1)^t$ and $\tilde{\mathbf{v}}_0 = (t_1, \tilde{t}_2, 1)^t$, are in the solution set $\mathcal{S}$. Since both $\hat{L}_0$ and $\tilde{L}_0$ intersect L_1, L_2, and L_j, they intersect L_1 at a common point $\mathbf{u}_0$. Then $\hat{L}_0$ and $\tilde{L}_0$ intersect L_2 and L_j at distinct points, and thus, the four lines $\hat{L}_0$, $\tilde{L}_0$, L_2, and L_j must lie in the same plane. But this contradicts the assumption that L_2 and L_j are not co-planer. Therefore, for each $t_1 \in \mathbb{R}$, there exists at most one $t_2 \in \mathbb{R}$ such that $L_0 = \{t\mathbf{v}_0 + \mathbf{u}_0 : t \in \mathbb{R}\}$ with $\mathbf{v}_0 = (t_1, t_2, 1)^t$ is in the solution set $\mathcal{S}$.

By the above **claim** and (3), L_0 intersecting L_j leads to $v_{j3}t_1 + b_j \neq 0$, since $v_{j3}t_1 + b_j = 0$ generates either none or infinite solutions t_2 for (3). It follows that t_2 is uniquely determined from t_1 such that $t_2 = -\frac{(a_j - v_{j2})t_1 + c_j}{v_{j3}t_1 + b_j}$.

Since the above arguments hold for $j = 3, 4$, one can conclude that L_0 intersects both L_3 and L_4 if and only if there exists $t_1 \in \mathbb{R}$ such that $v_{j3}t + b_j \neq 0$ for $j = 3, 4$, and

$$-\frac{(a_3 - v_{32})t_1 + c_3}{v_{33}t_1 + b_3} = t_2 = -\frac{(a_4 - v_{42})t_1 + c_4}{v_{43}t_1 + b_4}.$$

Equivalently, t_1 is a root of the polynomial

$$p(z) = (v_{33}z + b_3)((a_4 - v_{42})z + c_4) - (v_{43}z + b_4)((a_3 - v_{32})z + c_3)$$

in $\mathbb{R} \setminus \cup_{j=3}^{4}\{z : v_{j3}z + b_j = 0\}$.

On the other hand, for each root $z = t_1$ of the polynomial $p(z)$ in $\mathbb{R} \setminus \cup_{j=3}^{4}\{z : v_{j3}z + b_j = 0\}$, the line $L_0 = \{t\mathbf{v}_0 + \mathbf{u}_0 : t \in \mathbb{R}\}$ with $\mathbf{v}_0 = \left(t_1, -\frac{(a_3 - v_{32})t_1 + c_3}{v_{33}t_1 + b_3}, 1\right)^t$ and $\mathbf{u}_0 = (-t_1, 0, 0)^t$, intersects L_1, L_2, L_3, L_4. So the solution set $\mathcal{S}$ can have no, one, two, or infinitely many solutions, depending on the number of roots of the polynomial $p(z)$ in $\mathbb{R} \setminus \cup_{j=3}^{4}\{z : v_{j3}z + b_j = 0\}$. $\square$

Remark 2.2. Using the notation in the Theorem 2.1, under the assumption that $v_{j3}t + b_j \neq 0$ for $j = 3, 4$, $\mathcal{S}$ has infinitely many solutions if $p(z)$ is the zero polynomial; $\mathcal{S}$ has one solution if $p(z)$ reduces to a linear equation, or the discriminant of $p(z)$ is zero. So, in the generic case, there should be zero or two solutions.

The following examples show that the solution set $\mathcal{S}$ of L_0 can indeed have none, one, two or infinitely many elements.

Example 2.3. Let

$$L_1 = \{t\mathbf{e}_1 : t \in \mathbb{R}\}, \quad L_2 = \{t\mathbf{e}_2 + \mathbf{e}_3 : t \in \mathbb{R}\}, \quad \text{and} \quad L_3 = \{t\mathbf{e}_3 + (\mathbf{e}_1 + \mathbf{e}_2) : t \in \mathbb{R}\}.$$

Notice that $(a_3, b_3, c_3) = (-1, 1, 0)$.

1. Suppose $L_4 = \{\mathbf{u}_4 + t\mathbf{v}_4 : t \in \mathbb{R}\}$ with $\mathbf{v}_4 = (1,2,-2)^t$ and $\mathbf{u}_4 = (0,0,2)^t$. Then $(a_4, b_4, c_4) = (4,-2,0)$ and $p(z) \equiv 0$. In this case, for any $t_1 \neq -1, -\frac{1}{2}$, the line $L_0 = \{\mathbf{u}_0 + t\mathbf{v}_0 : t \in \mathbb{R}\}$ with $\mathbf{u}_0 = -t_1\mathbf{e}_1$ and $\mathbf{v}_0 = \left(t_1, \frac{t_1}{t_1+1}, 1\right)^t$, intersects L_1, L_2, L_3, L_4 at the points

   $$(-t_1, 0, 0)^t, \quad \left(0, \frac{t_1}{t_1+1}, 1\right)^t, \quad \left(1, 1, \frac{t_1+1}{t_1}\right)^t, \quad \text{and} \quad \left(\frac{t_1}{2t_1+1}, \frac{2t_1}{2t_1+1}, \frac{2(t_1+1)}{2t_1+1}\right)^t,$$

   respectively. Therefore, $\mathcal{S}$ has infinitely many elements.

2. Suppose $L_4 = \{\mathbf{u}_4 + t\mathbf{v}_4 : t \in \mathbb{R}\}$ with $\mathbf{v}_4 = (1,2,-2)^t$ and $\mathbf{u}_4 = (0,0,3)^t$. Then $(a_4, b_4, c_4) = (6,-3,0)$ and $p(z) = z(2z+1)$, which has no solution in $\mathbb{R}\setminus\left\{-\frac{3}{2}, -1, -\frac{1}{2}, 0\right\}$. Therefore, $\mathcal{S}$ is an empty set.

3. Suppose $L_4 = \{\mathbf{u}_4 + t\mathbf{v}_4 : t \in \mathbb{R}\}$ with $\mathbf{v}_4 = (1,2,-2)^t$ and $\mathbf{u}_4 = (-1,1,-2)^t$. Then $(a_4, b_4, c_4) = (-2,4,3)$ and $p(z) = -3(2z+1)(z-1)$. Now $t_1 = 1$ is the only root of $p(z)$ in $\mathbb{R} \setminus \left\{-1, -\frac{1}{2}, 0, 2\right\}$. Then the line

   $$L_0 = \{-\mathbf{e}_1 + t(2\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_3) : t \in \mathbb{R}\}$$

   intersects L_1, L_2, L_3, L_4. Therefore, $\mathcal{S}$ has one element.

4. Suppose $L_4 = \{\mathbf{u}_4 + t\mathbf{v}_4 : t \in \mathbb{R}\}$ with $\mathbf{v}_4 = (2,1,-2)^t$ and $\mathbf{u}_4 = (1,-2,1)^t$. Then $(a_4, b_4, c_4) = (-3,-4,-5)$, $p(z) = -(3z+5)(2z+1)$ and $t_1 = -\frac{1}{2}, -\frac{5}{3}$ are two roots of $p(z)$ in $\mathbb{R} \setminus \{-2,-1,0\}$. Therefore, the two lines

   $$L_0' = \left\{\frac{1}{2}\mathbf{e}_1 + t(\mathbf{e}_1 + 2\mathbf{e}_2 - 2\mathbf{e}_3) : t \in \mathbb{R}\right\} \text{ and } L_0'' = \left\{\frac{5}{3}\mathbf{e}_1 + t(6\mathbf{e}_1 + 15\mathbf{e}_2 - 10\mathbf{e}_3) : t \in \mathbb{R}\right\}$$

   intersect L_1, L_2, L_3, L_4. Therefore, $\mathcal{S}$ has two elements.

3 At least two lines are coplanar

To provide a comprehensive solution to the problem, we also describe the solution set $\mathcal{S}$ of lines L_0 having intersection with four given lines $L_1, \dots, L_4$ that are not in generic positions, i.e., at least two of them lie in the same plane in this section.

Suppose two lines belong to the same plane, we may relabel the lines and assume that L_1, L_2 are the two lines. Then we may apply an affine transform to $\mathbb{R}^3$, and assume that the plane containing the two lines are the xy-plane, i.e., the plane containing points of the form $(a,b,0)^t$ with $a, b, \in \mathbb{R}$. In particular, we may assume that $L_j = \{\mathbf{u}_j + t\mathbf{e}_j : t \in \mathbb{R}\}$ for

$j = 1, 2$. Moreover, if L_1 and L_2 intersect, we may assume that they intersect at the origin so that $L_j = \{t\mathbf{e}_j : t \in \mathbb{R}\}$ for $j = 1, 2$.

In the following, we will construct the solutions L_0 that lie in the xy-plane, and solutions L_0 that do not lie in the xy-plane. Clearly, a solution $L_0 = \{\mathbf{u}_0 + t\mathbf{v}_0 : t \in \mathbb{R}\}$, with $\mathbf{u}_0 = (u_{01}, u_{02}, u_{03})^t$ and $\mathbf{v}_0 = (v_{01}, v_{02}, v_{03})^t$, belongs to the xy-plane if and only if $\mathbf{v}_0, \mathbf{u}_0$ belong to the xy-plane, i.e., $u_{03} = v_{03} = 0$. On the other hand, a solution L_0 that does not lie in the xy-plane must have intersection with the xy-plane at a specific point $\mathbf{w}_0 = (w_1, w_2, 0)^t = \mathbf{u}_0 + t_0\mathbf{v}_0$ with $t_0 = u_{03}/v_{03}$. In particular, $v_{03} \neq 0$. By these observations, we can describe the solution set of lines L_0 intersecting L_j for $j = 1, \ldots, 4$. The statements and proof are lengthy due to the many cases need to be considered. We will continue to use $\mathcal{S}$ to denote the set of lines L_0 intersecting the four given lines.

Theorem 3.1. *Suppose $L_1, \ldots, L_4$ are four lines in $\mathbb{R}^3$, and assume that L_1 and L_2 lie in the xy-plane. Furthermore, if L_1 and L_2 intersect, then we assume that they intersect at the origin. Let $\mathcal{P}$ be the intersection of $\mathcal{S}$ and the xy-plane and $\mathcal{N} = \mathcal{S} \setminus \mathcal{P}$. We consider the following cases:*

(1) *L_3 and L_4 both lie in the xy-plane.*

(1.a) *The set $\mathcal{P}$ is infinite consisting of elements of the form $L_0 = \{\mathbf{u}_0 + t\mathbf{v}_0 : t \in \mathbb{R}\}$, where $\mathbf{u}_0$ and $\mathbf{v}_0$ lie in the xy-plane, and $\mathbf{v}_0 \neq \mathbf{v}_j$ for $j = 1, \ldots, 4$. Furthermore, $\mathcal{P}$ also contains L_ℓ if and only if $\mathbf{v}_\ell \neq \mathbf{v}_j$ for $j \in \{1, \ldots, 4\} \setminus \{\ell\}$.*

(1.b) *The set $\mathcal{N}$ is nonempty if and only if all the lines $L_1, \ldots, L_4$ intersect at the origin. In such a case, $\mathcal{N}$ is infinite consisting of lines of the form $\{t\mathbf{v}_0 : t \in \mathbb{R}\}$ with $\mathbf{v}_0$ not lying in the xy-plane.*

(2) *L_3 lies in the xy-plane, but L_4 does not. (We may relabel the lines in case L_4 lies in the xy-plane, but L_3 does not.)*

(2.a) *The set $\mathcal{P}$ is nonempty if and only if L_4 intersect the xy-plane at a point $\mathbf{u}_0$. If $\mathbf{u}_0$ exists, then $\mathcal{P}$ is infinite consisting of elements of the form $L_0 = \{\mathbf{u}_0 + t\mathbf{v}_0 : t \in \mathbb{R}\}$, where $\mathbf{v}_0$ lies in the xy-plane, and $\mathbf{v}_0 \neq \mathbf{v}_j$ for $j = 1, 2, 3$. Furthermore, $\mathcal{P}$ also contains L_ℓ with $\ell \in \{1, 2, 3\}$ if and only if $\mathbf{u}_0 \in L_\ell$ and $\mathbf{v}_\ell \neq \mathbf{v}_j$ for $j \in \{1, 2, 3\} \setminus \{\ell\}$.*

(2.b) *The set $\mathcal{N}$ is nonempty if and only if L_1, L_2, L_3 intersect at the origin. In such a case, $\mathcal{N}$ is infinite consisting of elements of the form $L_0 = \{t\mathbf{v}_0 : t \in \mathbb{R}\}$ with $\mathbf{v}_0 \in L_4$ and not lying in the xy-plane. In addition, if L_4 also passes through the origin, then $\mathcal{N}$ is infinite consisting of elements of the form $L_0 = \{t\mathbf{v}_0 : t \in \mathbb{R}\}$ with any vector $\mathbf{v}_0$ not lying in the xy-plane.*

(3) *L_3 and L_4 do not lie in the xy-plane.*

(3.a) *The set $\mathcal{P}$ is nonempty only if each L_j intersects the xy-plane at a point $\tilde{\mathbf{u}}_j$ for $j = 3, 4$. Suppose $\tilde{\mathbf{u}}_3$ and $\tilde{\mathbf{u}}_4$ exist. If $\tilde{\mathbf{u}}_3 = \tilde{\mathbf{u}}_4 = \mathbf{u}_0$, then $\mathcal{P}$ is infinite and consists of elements of the form $L_0 = \{\mathbf{u}_0 + t\mathbf{v}_0 : t \in \mathbb{R}\}$, where $\mathbf{v}_0$ lies in the xy-plane, and $\mathbf{v}_0 \neq \mathbf{v}_j$ for $j = 1, 2$. Furthermore, $\mathcal{P}$ also contains L_ℓ, $\ell = 1, 2$, if and only if $\mathbf{u}_0 \in L_\ell$ and $\mathbf{v}_1 \neq \mathbf{v}_2$. If $\tilde{\mathbf{u}}_3 \neq \tilde{\mathbf{u}}_4$ and $\tilde{L}_0 = \{\tilde{\mathbf{u}}_3 + t(\tilde{\mathbf{u}}_4 - \tilde{\mathbf{u}}_3) : t \in \mathbb{R}\}$ is the line passing through $\tilde{\mathbf{u}}_3$ and $\tilde{\mathbf{u}}_4$, then $\mathcal{P}$ is empty if $\tilde{L}_0 \cap L_1 = \emptyset$ or $\tilde{L}_0 \cap L_2 = \emptyset$; otherwise, $\mathcal{P} = \{\tilde{L}_0\}$.*

(3.b) *If L_1 and L_2 have no intersection, then $\mathcal{N}$ is empty. Assume L_1 and L_2 intersect at the origin.*

(i) *Suppose both L_3 and L_4 pass through the origin. Then $\mathcal{N}$ is infinite consisting of lines of the form $L_0 = \{t\mathbf{v}_0 : t \in \mathbb{R}\}$ with $\mathbf{v}_0$ not lying in the xy-plane.*

(ii) *Suppose L_3 passes through origin but L_4 does not. (We may relabel the lines if L_4 lies in the xy-plane, while L_3 does not). Then $\mathcal{N}$ is infinite consisting of elements of the form $L_0 = \{t\mathbf{v}_0 : t \in \mathbb{R}\}$ with $\mathbf{v}_0 \in L_4$ and not lying in the xy-plane.*

(iii) *Suppose both L_3 and L_4 do not pass through the origin. Then $L_0 \in \mathcal{N}$ if and only if $L_0 = \{t\mathbf{v}_0 : t \in \mathbb{R}\}$, where $\mathbf{v}_0 = s_3\mathbf{u}_3 + t_3\mathbf{v}_3 = s_4\mathbf{u}_4 + t_4\mathbf{v}_4$ for some $s_3, s_4, t_3, t_4 \in \mathbb{R}$ and $s_3, s_4 \neq 0$. That is, $\mathbf{v}_0$ is a vector in the intersection of the plane containing $\{\mathbf{u}_3, \mathbf{v}_3\}$ and the plane containing $\{\mathbf{u}_4, \mathbf{v}_4\}$, and $\mathbf{v}_0$ is not a multiple of $\mathbf{v}_3$ and $\mathbf{v}_4$.*

Proof. We consider the above three cases according to the theorem.

(1) Suppose L_1, L_2, L_3, and L_4 lie in the xy-plane. Then any line lying in the xy-plane and not parallel to L_1, L_2, L_3, L_4 will intersect all of them. That is, any line of the form $L_0 = \{\mathbf{u}_0 + t\mathbf{v}_0 : t \in \mathbb{R}\}$ with $\mathbf{u}_0$ and $\mathbf{v}_0$ lie in the xy-plane and $\mathbf{v}_0 \neq \mathbf{v}_j$, will intersect L_1, L_2, L_3, L_4. Furthermore, the line L_ℓ will intersect the other 3 lines if and only if $\mathbf{v}_\ell \neq \mathbf{v}_j$ for all $j \in \{1, 2, 3, 4\} \setminus \{\ell\}$. Thus, the (1.a) case holds.

Suppose L_0 exists and does not lie in the xy-plane, i.e., $L_0 \in \mathcal{N}$. Then L_0 must intersect the xy-plane at one point only. In this case, L_0 must intersect all the four given lines at the same point in the xy-plane. Thus, L_1, L_2, L_3, L_4 have a common intersection point, at the origin. This case happens if and only if $\mathbf{u}_3$ is a multiple of $\mathbf{v}_3$ and $\mathbf{u}_4$ is a multiple of $\mathbf{v}_4$. Then any line passing through the origin will intersect L_1, L_2, L_3, L_4, i.e., L_0 can be chosen to be any line of the form $L_0 = \{t\mathbf{v}_0 : t \in \mathbb{R}\}$ for any $\mathbf{v}_0$ that is not in the xy-plane. Thus, the (1.b) case holds.

(2) Suppose L_1, L_2, L_3 lie in the xy-plane and L_4 does not. If L_4 does not intersect the xy-plane. Then it is clear that $\mathcal{P}$ is empty. Suppose L_4 intersect the xy-plane. Then the intersection point is $\mathbf{u}_0 = \mathbf{u}_4 - (u_{43}/v_{43})\mathbf{v}_4$. Then all the lines of the form $L_0 = \{\mathbf{u}_0 + t\mathbf{v}_0 : t \in \mathbb{R}\}$ with $\mathbf{v}_0 \neq \mathbf{v}_j$ for $j = 1, 2, 3$ will intersect all four given lines. Furthermore, if $\mathbf{u}_0$ lies in L_ℓ for some $\ell \in \{1, 2, 3\}$ and $\mathbf{v}_\ell \neq \mathbf{v}_j$ for $j \in \{1, 2, 3\} \setminus \{\ell\}$, then L_ℓ will intersect all the four lines including itself. Thus, the (2.a) case holds.

Suppose L_0 exists and does not lie in the xy-plane, i.e., $L_0 \in \mathcal{N}$. Then L_0 must intersect the xy-plane at one point only. In this case, L_0 must intersect L_1, L_2, L_3 at the same point in xy-plane, which is the origin. This case happens if and only if $\mathbf{u}_3$ is a multiple of $\mathbf{v}_3$. In this case, L_0 can be chosen to be any line passing through the origin and intersecting L_4. That is, any line of the form $L_0 = \{t\mathbf{v}_0 : t \in \mathbb{R}\}$ with $\mathbf{v}_0 \in L_4$ will intersect L_1, L_2, L_3, L_4. In addition, if L_4 also passes through the origin, i.e., $\mathbf{u}_4$ is not a multiple of $\mathbf{v}_4$, then all L_1, L_2, L_3, L_4 have a common intersection point, which is the origin. In this case, any line passing through the origin will intersect all four given lines. That is, $\mathcal{N}$ contains all the lines of the form $L_0 = \{t\mathbf{v}_0 : t \in \mathbb{R}\}$, where $\mathbf{v}_0$ is not in the xy-plane. Thus, the (2.b) case holds.

(3) Suppose L_1 and L_2 lie in the xy-plane but L_3 and L_4 do not. If either L_3 or L_4 does not intersect the xy-plane. Then it is clear that $\mathcal{P}$ is empty. Suppose both L_3 and L_4 intersect the xy-plane. Then the intersection points are $\tilde{\mathbf{u}}_3 = \mathbf{u}_3 - (u_{33}/v_{33})\mathbf{v}_3$ and $\tilde{\mathbf{u}}_4 = \mathbf{u}_4 - (u_{43}/v_{43})\mathbf{v}_4$ respectively. If $\tilde{\mathbf{u}}_3 = \tilde{\mathbf{u}}_4 = \mathbf{u}_0$, then any line $L_0 = \{\mathbf{u}_0 + t\mathbf{v}_0 : t \in \mathbb{R}\}$ with $\mathbf{v}_0 \neq \mathbf{v}_j$ for $j \in \{1, 2\}$ will intersect L_1, L_2, L_3, L_4. Furthermore, if $\mathbf{u}_0 \in L_\ell$ with $\ell \in \{1, 2\}$ and $\mathbf{v}_1 \neq \mathbf{v}_2$, then L_ℓ will intersect all the four lines including itself too. Now suppose $\tilde{\mathbf{u}}_3 \neq \tilde{\mathbf{u}}_4$. Then the line $\tilde{L}_0 = \{\tilde{\mathbf{u}}_3 + t(\tilde{\mathbf{u}}_4 - \tilde{\mathbf{u}}_3) : t \in \mathbb{R}\}$ will intersect L_1, L_2, L_3, L_4 if and only if $\tilde{L}_0$ intersects both L_1 and L_2, i.e., $\tilde{L}_0 \cap L_1$ and $\tilde{L}_0 \cap L_2$ are both empty. In this case $\mathcal{P} = \{\tilde{L}_0\}$. Thus, the (3.a) case holds.

If L_1 and L_2 has no intersection, then L_0, if exists, must lie in the xy-plane. Then $\mathcal{N}$ must be empty. Assume L_1 and L_2 intersect at the origin. Suppose both L_3 and L_4 pass through origin. Then any line passing through the origin will intersect all four given lines. Then $\mathcal{N}$ is infinite consisting of lines of the form $L_0 = \{t\mathbf{v}_0 : t \in \mathbb{R}\}$ with $\mathbf{v}_0$ not lying in the xy-plane. Thus, the (3.b.i) case holds.

Suppose L_3 passes through the origin but L_4 does not. Then any line passing through the origin and intersecting L_4 will intersect all four given lines. So $\mathcal{N}$ is infinite consisting of elements of the form $L_0 = \{t\mathbf{v}_0 : t \in \mathbb{R}\}$ with $\mathbf{v}_0 \in L_4$ and not lying in the xy-plane. Thus, the (3.b.ii) case holds.

Finally, suppose both L_3 and L_4 do not pass through the origin. Then $L_0 \in \mathcal{N}$, if exists, must pass through the origin, i.e., L_0 has the form $L_0 = \{t\mathbf{v}_0 : t \in \mathbb{R}\}$. Furthermore,

L_0 intersects L_3 and L_4, so $t_1\mathbf{v}_0 = \mathbf{u}_3 + t_3\mathbf{v}_3$ and $t_2\mathbf{v}_0 = \mathbf{u}_4 + t_4\mathbf{v}_4$ for some t_1, t_2, t_3, t_4 with $t_1, t_2 \neq 0$. Then $\mathbf{v}_0 = (1/t_1)\mathbf{u}_3 + (t_3/t_1)\mathbf{v}_3 = (1/t_2)\mathbf{u}_4 + (t_4/t_2)\mathbf{v}_4$. Thus, the (3.b.iii) case holds. □

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