COMPLEXIFICATIONS OF REAL BANACH SPACES AND THEIR ISOMETRIES

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Abstract. Every norm $\|\cdot\|$ on a real Banach space $\mathcal{X}$ induces a minimal norm on the complex linear space $\mathbb{C}\mathcal{X} = \mathcal{X} + i\mathcal{X} = \{x + iy : x, y \in \mathcal{X}\}$ by

$$\|x + iy\|_{\mathbb{C}} = \sup\{\|x\cos\theta + y\sin\theta\| : \theta \in [0, 2\pi]\}.$$

In this note we show that if $\mathcal{X}$ is finite-dimensional there is a decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_k$ into subspaces such that the isometry group of $\|\cdot\|_{\mathbb{C}}$ is generated by that of $\|\cdot\|$ and operators of the form $e^{i\theta_1} I_{n_1} \oplus \cdots \oplus e^{i\theta_k} I_{n_k}$ acting on $\mathbb{C}\mathcal{X} = \mathbb{C}\mathcal{X}_1 \oplus \cdots \oplus \mathbb{C}\mathcal{X}_k$. Various applications are given, in particular to isometries of numerical radius.

Keywords: Isometry, complexification, dual norm, extreme point.

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1. Introduction

If A is an isometry on a finite-dimensional real Banach space $(\mathbb{R}^n, \|\cdot\|)$, then $e^{i\theta}A$, $\theta \in [0, 2\pi]$ is not always an isometry on its complexified space $(\mathbb{C}^n, \|\cdot\|)$. However, with the Taylor minimal complexified norm defined for $x + iy \in \mathbb{R}^n + i\mathbb{R}^n = \mathbb{C}^n$ by

$$\|x + iy\|_{\mathfrak{T}} = \sup_{\theta\in\mathbb{R}} \|x\cos\theta + y\sin\theta\|$$

one readily verifies that if A is a real isometry, then $e^{i\theta}A$ is a complex isometry for $\|\cdot\|_{\mathfrak{T}}$. In some cases these are all the possible isometries for $\|\cdot\|_{\mathfrak{T}}$. In other cases, the isometries for $\|\cdot\|_{\mathfrak{T}}$ could be more complicated. For example (as we show below) for the ℓ_p-norm with $p \geq 1$ all the isometries for its Taylor complexification have the form $e^{i\theta}A$, where A is a real isometry for ℓ_p. On the other hand if $p = \infty$, then the isometries for the Taylor complexification of ℓ_∞ have the form PA, where P acts as a scalar operator on each co-ordinate space. So, how can we decide whether the Taylor complexified norm always has the form $e^{i\theta}A$, and how much more complicated could the isometries of the complexified norm be? In this paper, we give a complete answer for this question for the finite-dimensional case, and a partial answer for the infinite dimensional case. In particular, in the finite-dimensional case, we show that the isometries for the Taylor complexified norm always have the form PA. We also show that our results apply to the Bochnak maximal complexified norm and discuss the limitation of the techniques to other complexified norms (see Example 6.2 which also demonstrates that for general complexified norms it is impossible to characterize its isometries with the help of isometries on the real space).

To facilitate our discussion, we give the formal definition and background of our problem in the following. Suppose $(\mathcal{X}, \|\cdot\|)$ is a real normed space. A complexification of $\mathcal{X}$ is a complex vector space $\mathcal{X}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{X}$ equipped with a complex norm $\|\cdot\|_{\mathbb{C}}$ that agrees with $\|\cdot\|$ on the real subspace $1 \otimes_{\mathbb{R}} \mathcal{X}$. We will freely identify $1 \otimes x$ with x and $i \otimes x$ with ix (throughout, $i = \sqrt{-1}$ except when appearing as index) and we will identify $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{X}$ with

$\mathbb{C}\mathcal{X}$. Thus, when considering $\mathbb{C}\mathcal{X}$ as a real vector space it decomposes into $\mathbb{C}\mathcal{X} = \mathcal{X} \oplus i\mathcal{X}$. With this in mind, the complexified norm must satisfy $\|x\|_{\mathbb{C}} = \|x\|$ for $x \in \mathcal{X}$. Observe that with $\lambda \in \mathbb{C}$ and $x \in \mathcal{X}$ one has $|\lambda| \cdot \|x\| = \|\lambda x\|_{\mathbb{C}} = \|\lambda \cdot (1 \otimes x)\|_{\mathbb{C}} = \|\lambda \otimes x\|_{\mathbb{C}}$. Thus, $\|\cdot\|_{\mathbb{C}}$ is a cross norm (we assume without further notice that $\mathbb{C}$ comes equipped with the usual normalized complex norm). It is well-known (see [13]) that the largest possible cross norm on a tensor product is the projective cross-norm and hence

$$\|x + iy\|_{\mathbb{C},\pi} = \inf\left\{\sum\nolimits_k |\lambda_k| \cdot \|x_k\| : \ x + iy = \sum\nolimits_k \lambda_k x_k\right\}$$

is the largest possible complexification norm. This norm was termed Bochnak's norm by Muñoz, Sarantopoulos, and Tonge [9] in honor of Bochnak's introductory paper [1]. We remark in passing that a complexification of a real normed algebra $\mathcal{A}$ (that is, a submultiplicative complex norm on $\mathbb{C}\mathcal{A}$ which extends a given submultiplicative norm on $\mathcal{A}$) coincides with the Bochnak complexification (see Remark 6.3 for more details).

Observe that the Bochnak norm satisfies

$$\|x + iy\|_{\mathbb{C},\pi} = \|x - iy\|_{\mathbb{C},\pi}.$$

A complexification norm with this additional property is called a reasonable complexification [9]. For any reasonable complexification one has

$$2\|x\|_{\mathbb{C}} = \|(x + iy) + (x - iy)\|_{\mathbb{C}} \le \|x + iy\|_{\mathbb{C}} + \|x - iy\|_{\mathbb{C}} = 2\|x + iy\|_{\mathbb{C}}$$

and consequently,

$$\|x+iy\|_{\mathbb{C}} = \|e^{-i\theta}(x+iy)\|_{\mathbb{C}} = \|(x\cos\theta + y\sin\theta) + i(y\cos\theta - x\sin\theta)\|_{\mathbb{C}} \ge \|x\cos\theta + y\sin\theta\|.$$

Thus, each reasonable complexification norm satisfies

$$\sup_{\theta\in[0,2\pi]} \|x\cos\theta + y\sin\theta\| \le \|x + iy\|_{\mathbb{C}} \le \|x + iy\|_{\mathbb{C},\pi}$$

The expression on the left is easily seen to be a reasonable complexification norm and by the above inequality it is the minimal possible reasonable complexification. Following [9] we call it the Taylor complexification norm and denote it by

$$\|x + iy\|_{\mathfrak{T}} := \sup_{\theta\in[0,2\pi]} \|x\cos\theta + y\sin\theta\|. \tag{1}$$

Again we remark in passing that the numerical radius is the Taylor complexification of the spectral norm restricted to Hermitian matrices (see Example 3.15 for more details).

The Taylor norm coincides with the injective norm on the tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{X}$ and, unlike the Bochnak norm, behaves well on subspaces (but unlike the Bochnak norm, does not behave well under quotient spaces). We refer to [9] for much more information along these lines.

Below (see Theorem 3.2) we give another description of the Taylor complexification norm in terms of extreme points of the unit ball for the dual of the original norm. This result is then used in our main result (see Theorem 3.10) where we determine the group of isometries of the Taylor complexification norm with the help of the group of isometries of the original norm on a real space. As an application we compute the group of isometries for the numerical radius and related norms (see Example 3.15 and remarks below it). Previously, more elaborate techniques were required to find these groups. The proofs of main results are given in section 4 and an application to Hermitian operators in Taylor's complexification in section 5. The last section gives some concluding remarks, including a partial extension to infinite-dimensional Banach spaces.

2. Preliminaries

Given a real/complex Banach space $\mathcal{X}$ we let $\mathcal{X}^*$ be its dual, that is, the space of all bounded $\mathbb{R}$-linear/$\mathbb{C}$-linear functionals on $\mathcal{X}$. The dual of a linear operator $A\colon \mathcal{X} \to \mathcal{X}$ is denoted by A^* and by definition maps a functional $f \in \mathcal{X}^*$ into the functional given by $(A^*f)\colon x \mapsto f(Ax)$. For a subset $\Omega \subseteq \mathcal{X}$ we let $\operatorname{conv}\Omega \subseteq \mathcal{X}$ be its convex hull (the intersection of all convex sets which contain Ω) and we let $\operatorname{span}\Omega$ be its linear span (the intersection of all linear spaces that contain Ω).

Recall that in finite-dimensional Banach spaces $(\mathbb{F}^n, \|\cdot\|)$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, the unit ball $\mathcal{B} = \{x \in \mathbb{F}^n : \|x\| \leq 1\}$ is a compact convex subset and hence, by Steinitz's theorem [14, p. 16] (see also [10, Corollary 18.5.1] for a modern treatment), $\mathcal{B}$ equals the convex hull of its extreme points, i.e., $\mathcal{B} = \operatorname{conv}\mathcal{E}$ where $\mathcal{E} = \operatorname{Ext}\mathcal{B}$ is the set of vectors u in $\mathcal{B}$ such that $u \neq (u_1 + u_2)/2$ for two different $u_1, u_2 \in \mathcal{B}$. For brevity we shall abuse notation and call $\operatorname{Ext}\mathcal{B}$ the set of the norm's extreme points.

We will frequently rely on the following well-known and easy to prove fact: Let

$$\|x\|^* = \max\{|\langle x, y\rangle| : y \in \mathbb{F}^n,\ \|y\| \leq 1\}$$

be the dual norm $\|\cdot\|$ on $\mathbb{F}^n$; here $\langle x, y\rangle = y^*x$ is the standard inner product on $\mathbb{F}^n$ with the elements of $\mathbb{F}^n$ being column vectors (n-by-1 matrices). If $\mathcal{E}^*$ is the set of extreme points of the dual norm unit ball $\mathcal{B}^*$, then (c.f. also Duality Theorem [3, 5.5.14]),

$$\begin{aligned} \|x\|^* &= \sup\{|\langle x, y\rangle| : y \in \mathcal{E}\} = \sup\{\operatorname{Re}\langle x, y\rangle : y \in \mathcal{E}\} \\ \|y\| &= \sup\{|\langle y, x\rangle| : x \in \mathcal{E}^*\} = \sup\{\operatorname{Re}\langle y, x\rangle : x \in \mathcal{E}^*\}. \end{aligned} \tag{2}$$

The Taylor norm is not easy to compute from the definition. However, when $\mathcal{X}$ is identified with $\mathbb{R}^n$ and its complexification, $\mathcal{X}_{\mathbb{C}}$, with $\mathbb{C}^n$ ($n = \dim\mathcal{X}$) there is a shortcut by using a dual norm and (2):

$$\begin{aligned} \|x + iy\|_{\mathfrak{T}} &= \max\{\langle x\cos\theta + y\sin\theta, v\rangle : \theta \in [0, 2\pi], v \in \mathbb{R}^n, \|v\|^* \leq 1\} \\ &= \max\{|\langle x + iy, e^{i\theta}v\rangle| : \theta \in [0, 2\pi], v \in \mathbb{R}^n, \|v\|^* \leq 1\} \\ &= \max\{|\langle x + iy, v\rangle| : v \in \mathbb{R}^n, \|v\|^* \leq 1\}. \end{aligned} \tag{3}$$

It follows that $|\langle z, v\rangle| \leq \|z\|_{\mathfrak{T}} \cdot \|v\|^*$ for all $z = x + iy \in \mathbb{C}^n$ and $v \in \mathbb{R}^n$ so, by (2),

$$\|v\|_{\mathfrak{T}}^* \leq \|v\|^*; \qquad v \in \mathbb{R}^n. \tag{4}$$

Example 2.1. *Consider the ℓ_p-norm of $x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n$ defined by*

$$\ell_\infty(x) = \max\{|x_j| : 1 \leq j \leq n\} \quad \text{and} \quad \ell_p(x) = (\sum_{j=1}^n |x_j|^p)^{1/p} \ \text{ for } p \in [1, \infty).$$

The Taylor complexification of the ℓ_p-norm on $\mathbb{R}^n$ is

$$\ell_{p,\mathfrak{T}}(z) = \max\{|\langle z, v\rangle| : v \in \mathbb{R}^n, \ell_q(v) \leq 1\}; \qquad z = x + iy \in \mathbb{C}^n,$$

where $1/p + 1/q = 1$ with the convention that $(p, q) = (1, \infty)$ or $(p, q) = (\infty, 1)$ in the limiting cases. It is interesting to note that $\ell_{p,\mathfrak{T}}$ is different from the standard ℓ_p-norm on $\mathbb{C}^n$ for all

$p \in [1,\infty)$, whereas $\ell_{\infty,\mathfrak{T}}$ coincides with the ℓ_∞-norm on $\mathbb{C}^n$. Namely, with the standard basis $e_1,\dots,e_n \in \mathbb{R}^n$ one has

$$\begin{aligned}\ell_{1,\mathfrak{T}}(e_1+ie_2) &= \max\{|v_1+iv_2| :\ |v_1|,|v_2|\le 1,\ v_1,v_2\in\mathbb{R}\}\\ &= \max\{\sqrt{v_1^2+v_2^2} :\ |v_1|,|v_2|\le 1,\ v_1,v_2\in\mathbb{R}\} = \sqrt{2}\neq |1|+|i| = \ell_1(e_1+ie_2)\end{aligned}$$

and similarly, for $1<p<\infty$ one calculates $\ell_{p,\mathfrak{T}}(e_1+ie_2) = \max\{\sqrt{v_1^2+v_2^2} :\ |v_1|^q+|v_2|^q\le 1,\ v_1,v_2\in\mathbb{R}\} = \max\{1, 2^{\frac{q-2}{2q}}\} \neq \sqrt[p]{|1|^p+|i|^p} = \ell_p(e_1+ie_2)$. However, with ℓ_∞ one sees that

$$\ell_{\infty,\mathfrak{T}}(\sum z_ie_i) = \max\{|z_1v_1+\dots+z_nv_n| :\ v_i\in\mathbb{R}, |v_1|+\dots+|v_n|\le 1\} = \ell_\infty(\sum z_ie_i). \tag{5}$$

3. The Main Results and Consequences

Let us start with a general result about extreme points of tensor products.

Lemma 3.1. *Let $(\mathbb{R}^n, \|\cdot\|_1)$ and $(\mathbb{R}^m, \|\cdot\|_2)$ be two normed spaces with closed unit balls $\mathcal{B}_1$ and $\mathcal{B}_2$, and let $\mathcal{E}_i = \operatorname{Ext}\mathcal{B}_i$ denote the sets of their extreme points. Then $\mathcal{E}_1\otimes\mathcal{E}_2 := \{s_1\otimes s_2 :\ s_i\in\mathcal{E}_i\} = \operatorname{Ext}(\operatorname{conv}\mathcal{E}_1\otimes\mathcal{E}_2)$.*

Proof. Clearly, $\operatorname{Ext}(\operatorname{conv}\mathcal{E}_1\otimes\mathcal{E}_2)\subseteq\mathcal{E}_1\otimes\mathcal{E}_2$. Conversely, let $a\otimes b\in\mathcal{E}_1\otimes\mathcal{E}_2$. Suppose there exist $a_1\otimes b_1,\dots,a_n\otimes b_n\in\mathcal{E}_1\otimes\mathcal{E}_2$ and positive scalars $\lambda_1,\dots,\lambda_n$ with $\sum_k\lambda_k = 1$ such that

$$a\otimes b = \sum_k \lambda_k a_k\otimes b_k. \tag{6}$$

Since $b\in\operatorname{Ext}\mathcal{B}_2$ and $\mathcal{B}_2$ is balanced there exists a supporting functional $f\colon\mathbb{R}^m\to\mathbb{R}$ such that $|f(b_k)|\le f(b) = 1$ for all k. Evaluate $I\otimes f\colon \mathbb{R}^n\otimes\mathbb{R}^m\to\mathbb{R}^n\otimes\mathbb{R} = \mathbb{R}^n$ at $a\otimes b$ to get

$$a = (I\otimes f)(a\otimes b) = \sum_k \lambda_k f(b_k)a_k.$$

Since $|f(b_k)|\le 1$ and $\mathcal{B}_1$ is balanced we have that $f(b_k)a_k\in\mathcal{B}_1$, and hence the extreme point $a\in\operatorname{Ext}\mathcal{B}_1$ is a convex linear combination of $f(b_k)a_k\in\mathcal{B}_1$. Since $\lambda_k>0$ we deduce that $f(b_k)a_k = a$ for every k. Now $a_k\in\operatorname{Ext}\mathcal{B}_1$ and so $\|a_k\|_1 = \|a\|_1 = 1$ and so $f(b_k) = \pm 1$ giving $a_k = \pm a$. Likewise we show that $b_k = \pm b$. Thus, $a_k\otimes b_k = \pm a\otimes b$ for each index k and so $a\otimes b$ is an extreme point of $\operatorname{conv}(\mathcal{E}_1\otimes\mathcal{E}_2)$. □

The result below provides an alternative description for the Taylor complexification of a norm $\|\cdot\|$ in terms of extreme points. Loosely speaking, it asserts that the unit ball of *the dual of* the Taylor complexified norm $\|\cdot\|_{\mathfrak{T}}^*$ is the convex hull of $\mathbb{T}\mathcal{E}^*$, where $\mathbb{T} = \{e^{i\theta} :\ \theta\in[0,2\pi]\}\subseteq\mathbb{C}$ is the unit circle, $\mathcal{E}^*$ is the set of extreme points for the dual norm $\|\cdot\|^*$, and $\mathbb{T}\mathcal{E}^* = \{e^{i\theta}y :\ \theta\in[0,2\pi],\ y\in\mathcal{E}^*\}$. Let us emphasize that this result characterizes the Taylor complexification since two norms whose dual norms have the same extreme points must coincide. We denote the Taylor complexification of a normed space $(\mathcal{X},\|\cdot\|)$ by $\mathcal{X}_{\mathfrak{T}}$ thus $\mathcal{X}_{\mathfrak{T}} = \mathbb{C}\otimes_{\mathbb{R}}\mathcal{X}$ equipped with Taylor norm $\|\cdot\|_{\mathfrak{T}}^*$.

Theorem 3.2. *Let $(\mathcal{X},\|\cdot\|)$ be a finite dimensional Banach space. Let $\mathcal{B}^*\subseteq\mathcal{X}^*$ and $\mathcal{B}_{\mathfrak{T}}^*\subseteq\mathcal{X}_{\mathfrak{T}}^*$ be the closed unit balls of its dual norm $\|\cdot\|^*$ and the dual of its Taylor complexification $\|\cdot\|_{\mathfrak{T}}^*$. Then*

$$\operatorname{Ext}(\mathcal{B}_{\mathfrak{T}}^*) = \mathbb{T}\operatorname{Ext}(\mathcal{B}^*).$$

Remark 3.3. *The reference to the dual norms is rather critical here. For example, in the case of Euclidean real space* $(\mathbb{R}^2, \ell_2)$, *with* $\ell_2(x) := \sqrt{x^*x}$, *the Taylor complexification norm of* $x + iy \in \mathbb{C}^2$, *where* $x = (x_1, x_2)^t$ *and* $y = (y_1, y_2)^t$, *coincides with the operator norm of a matrix* $\left(\begin{smallmatrix} x_1 & y_1 \\ x_2 & y_2 \end{smallmatrix}\right)$. *The extreme points of the complexified unit ball are thus orthogonal* 2*-by-*2 *matrices; however* $\mathbb{T}\,\mathrm{Ext}(\mathcal{B}) = \{(\cos\alpha, \sin\alpha) \otimes (\cos\theta, \sin\theta) : \ \alpha, \theta \in \mathbb{R}\}$ *coincides with partial isometries of rank-one. Hence* $\mathrm{Ext}(\mathcal{B}_{\mathfrak{T}}) \neq \mathbb{T}\,\mathrm{Ext}(\mathcal{B})$.

Proof. Identify $\mathcal{X}$ with $\mathbb{R}^n$ and $\mathcal{X}_{\mathfrak{T}}$ with $\mathbb{C}^n$. Let $\mathcal{E}^* := \mathrm{Ext}(\mathcal{B}^*)$. The Kronecker product establishes that, with each nonempty $S \subseteq \mathbb{R}^n$, the set $\mathbb{T}S \subseteq \mathbb{C}^n = \mathbb{R}^{2n}$ coincides with $\mathbb{T}_{\mathbb{R}} \otimes S \subseteq \mathbb{R}^{2n}$ where $\mathbb{T}_{\mathbb{R}} = \{(\cos\theta, \sin\theta) : \ \theta \in [0, 2\pi]\} \subseteq \mathbb{R}^2$ is the unit circle. Hence, it suffices to prove

$$\mathcal{B}^*_{\mathfrak{T}} = \mathrm{conv}(\mathbb{T}\mathcal{E}^*) \tag{7}$$

because once this is established, Lemma 3.1 finishes the proof.

By (4) we have $\mathbb{T}\mathcal{B}^* \subseteq \mathcal{B}^*_{\mathfrak{T}}$, thus $\mathrm{conv}(\mathbb{T}\mathcal{E}^*) \subseteq \mathcal{B}^*_{\mathfrak{T}}$. Note that $\mathbb{T}\mathcal{B}^*$ is compact, hence $\mathrm{conv}(\mathbb{T}\mathcal{E}^*) = \mathrm{conv}(\mathbb{T}\mathcal{B}^*)$ is compact (see also [12, Theorem 3.25]). By way of contradiction, suppose there exists $z \in \mathcal{B}^*_{\mathfrak{T}} \setminus \mathrm{conv}(\mathbb{T}\mathcal{E}^*)$. Choose a separating hyperplane, i.e., choose $t \in \mathbb{R}^{2n} = \mathbb{C}^n$ such that

$$\mathrm{Re}\,\langle t, z\rangle > 1, \quad \text{and } \mathrm{Re}\,\langle t, y\rangle < 1 \text{ for every } y \in \mathrm{conv}(\mathbb{T}\mathcal{E}^*).$$

Then we have

$$\begin{aligned} 1 < \mathrm{Re}\,\langle t, z\rangle &\leq \|z\|^*_{\mathfrak{T}}\|t\|_{\mathfrak{T}} \leq \|t\|_{\mathfrak{T}} = \sup_{\substack{\theta\in[0,2\pi] \\ v\in\mathcal{B}^*}} \left|\langle t, e^{i\theta}v\rangle\right| \qquad \text{by (3)} \\ &= \sup_{\substack{\theta\in[0,2\pi] \\ v\in\mathcal{B}^*}} \mathrm{Re}\,\langle t, e^{i\theta}v\rangle = \sup_{\substack{\theta\in[0,2\pi] \\ v\in\mathcal{E}^*}} \mathrm{Re}\,\langle t, e^{i\theta}v\rangle = \sup_{y\in\mathbb{T}\mathcal{E}^*} \mathrm{Re}\,\langle t, y\rangle \leq \max_{y\in\mathrm{conv}(\mathbb{T}\mathcal{E}^*)} \mathrm{Re}\,\langle t, y\rangle < 1, \end{aligned}$$

a contradiction. □

We continue by defining an equivalence relation which will partition $\mathcal{E}^*$ into disjoint subsets $\mathcal{E}^* = \bigcup_{j\in\mathcal{J}} \mathcal{E}^*_j$ such that their linear span will form a direct sum space decomposition

$$\mathrm{span}\,\mathcal{E}^* = \bigoplus_{j\in\mathcal{J}} \mathrm{span}\,\mathcal{E}^*_j.$$

Definition 3.4. *Let* $\mathcal{S}$ *be a set of nonzero vectors in a real vector space* $\mathcal{X}$. *Define a pre-equivalence relation* $\sim_p$ *on* $\mathcal{S}$ *by* $x_1 \sim_p x_2$ *if*

(1) $\{x_1, x_2\}$ *is linearly dependent, or*
(2) $\{x_1, x_2\}$ *can be enlarged to a linearly independent subset* $\{x_1, x_2, \ldots, x_\ell\} \subseteq \mathcal{S}$ *such that for some* $\alpha_1 \ldots, \alpha_\ell \in \mathbb{R} \setminus \{0\}$ *we have*

$$x_0 = \alpha_1 x_1 + \cdots + \alpha_\ell x_\ell \in \mathcal{S}.$$

Define an equivalence relation $\sim$ *on* $\mathcal{S}$ *as the transitive closure of* $\sim_p$*; in other words,* $x \sim y$ *in* $\mathcal{S}$ *if there is a sequence* $u_1, \ldots, u_r \in \mathcal{S}$ *such that* $x = u_1 \sim_p u_2 \sim_p \cdots \sim_p u_r = y$.
We say that $\sim$ *is* a relation of spatial partition on $\mathcal{S}$ *and we call the partition of* $\mathcal{S}$ *into equivalence sets* a spatial partition of $\mathcal{S}$.

Let us illustrate these notions with two extreme examples.

Example 3.5. *Let* $\mathcal{S} = \{e_1, \ldots, e_n\}$ *be the standard basis for* $\mathbb{R}^n$. *Then the spatial partition of* $\mathcal{S}$ *consists of singletons, that is,* $\mathcal{S} = \{e_1\} \cup \cdots \cup \{e_n\}$.

Example 3.6. *Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{R}^n$, and let*

$$\mathcal{S} = \{e_j : 1 \le j \le n\} \cup \{e_j + e_{j+1} : 1 \le j < n\}.$$

Then $e_j \sim e_{j+1}$, $j = 1 \dots, n-1$, so the spatial partition of $\mathcal{S}$ consists of a single equivalence class.

We next show that Definition 3.4 achieves what was intended.

Proposition 3.7. *Let $\mathcal{S}$ be a set of nonzero vectors in a vector space $\mathcal{X}$. Partition $\mathcal{S}$ into a collection of equivalence classes $\{\mathcal{S}_j : j \in \mathcal{J}\}$ for the relation $\sim$. Then*

$$\operatorname{span} \mathcal{S} = \oplus_{j \in \mathcal{J}} \operatorname{span} \mathcal{S}_j.$$

Proof. If $|\mathcal{J}| = 1$ there is nothing to prove, so suppose $|\mathcal{J}| \ge 2$. Let $a \in \mathcal{J}$. Suppose, by way of contradiction, that there exists a nonzero vector $z \in \operatorname{span} \mathcal{S}_a \cap \operatorname{span}(\mathcal{S} \setminus \mathcal{S}_a)$. Then we may write

$$\sum_{i=1}^{m} \alpha_i x_i = z = \sum_{j=1}^{n} \beta_j y_j$$

where $\{x_1, \dots, x_m\} \subseteq \mathcal{S}_a$ is linearly independent, $\{y_1, \dots, y_n\} \subseteq \cup_{j \ne a} \mathcal{S}_j$ is linearly independent, and $\alpha_i, \beta_j \ne 0$. Thus $\{x_1, \dots, x_m, y_1, \dots, y_n\}$ is linearly dependent; let $\mathcal{T}$ be a subset of these $m+n$ vectors that is minimal with respect to the property:

(*) A linear combination (with nonzero coefficients) of the vectors in $\mathcal{T}$ is zero.

Without loss of generality we may suppose $x_1, y_1 \in \mathcal{T}$. Since y_j and x_i lie in different equivalence classes, the set $\{x_i, y_j\}$ is linearly independent for all i, j; in particular $|\mathcal{T}| \ge 3$. By the minimality of $\mathcal{T}$ it follows that $x_1 \sim_p y_1$. But this is a contradiction since x_1, y_1 lie in different equivalence classes.

Thus we have shown that, for all $a \in \mathcal{J}$,

$$\operatorname{span} \mathcal{S}_a \cap \operatorname{span}(\cup_{j \ne a} \mathcal{S}_j) = \{0\}$$

and the result follows. □

Note that if $\dim \mathcal{X} < \infty$ then $\mathcal{S}$ must have a finite number of equivalence classes.

In the Lemma below we recall the shortcut $\mathbb{C}\mathcal{X} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{X}$ for the complexification of a real space $\mathcal{X}$. Hence, if $M^* \subseteq \mathcal{X}^*$ is a subspace of the dual of $\mathcal{X}$, i.e., a subspace of $\mathbb{R}$-linear real-valued functionals on $\mathcal{X}$, then $\mathbb{C}M^* = \mathbb{C} \otimes_{\mathbb{R}} M^*$ is the subspace of $\mathbb{C}$-linear functionals on $\mathbb{C}\mathcal{X}$ with the standard identifications. That is, $(\lambda \otimes_{\mathbb{R}} f) \colon x = 1 \otimes_{\mathbb{R}} x \mapsto \lambda f(x) \in \mathbb{C}$ ($f \in M^*$, $x \in \mathcal{X}$, and $\lambda \in \mathbb{C}$). Also, a decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \dots \oplus \mathcal{X}_k$ of real vector spaces induces the decomposition $\mathbb{C}\mathcal{X} = \mathbb{C}\mathcal{X}_1 \oplus \dots \oplus \mathbb{C}\mathcal{X}_k$ of complex vector spaces.

Lemma 3.8. *Let $\mathcal{X}$ be a finite-dimensional real Banach space and let $\mathcal{E}^* \subseteq \mathcal{X}^*$ be the set of extreme points of its dual norm. Partition $\mathcal{E}^*$ into equivalence classes $\mathcal{E}_1^*, \dots, \mathcal{E}_k^*$ as in Proposition 3.7, and let $M_j^* = \operatorname{span}_{\mathbb{R}} \mathcal{E}_j^*$. Define*

$$\mathcal{X}_j = \ker(\oplus_{i \ne j} M_i^*) = \{x \in \mathcal{X} : f(x) = 0 \textit{ for all } f \in \oplus_{i \ne j} M_i^*\}.$$

Then $\mathcal{X} = \oplus_{j=1}^k \mathcal{X}_j$. Moreover, if a $\mathbb{C}$-linear $D \colon \mathbb{C}\mathcal{X} \to \mathbb{C}\mathcal{X}$ is defined by

$$D(x_1 + \dots + x_k) = \omega_1 x_1 + \dots + \omega_k x_k; \qquad (\omega_i \in \mathbb{T},\ x_i \in \mathcal{X}_i)$$

with respect to the decomposition $\mathbb{C}\mathcal{X} = \mathbb{C}\mathcal{X}_1 \oplus \dots \oplus \mathbb{C}\mathcal{X}_k$, then the dual map satisfies

$$D^*(f_1 + \dots + f_k) = \omega_1 f_1 + \dots + \omega_k f_k$$

with respect to the decomposition $(\mathbb{C}\mathcal{X})^* = (\mathbb{C}M_1^*) \oplus \cdots \oplus (\mathbb{C}M_k^*)$.

Proof. Suppose $x \in \mathcal{X}_\alpha \cap \sum_{\beta\neq\alpha} \mathcal{X}_\beta$. Since $\mathcal{X}_\beta \subseteq \ker M_\alpha^*$ for all $\beta \neq \alpha$ we have

$$x \in \mathcal{X}_\alpha \cap \ker M_\alpha^* = \Big(\bigcap_{\beta\neq\alpha} \ker M_\beta^*\Big) \cap \ker M_\alpha^* = \bigcap_{\beta} \ker M_\beta^* = \ker \mathcal{X}^*,$$

so $x = 0$. Thus $\sum_{i=1}^k \mathcal{X}_i = \oplus_{i=1}^k \mathcal{X}_i$.

Let $n = \dim \mathcal{X} = \dim \mathcal{X}^*$ and let $n_j = \dim M_j^*$. Clearly $\dim \mathcal{X}_j = n_j$, so $\dim \oplus_{i=1}^k \mathcal{X}_i = \sum_{i=1}^k n_i = n$. Thus $\mathcal{X} = \oplus_{i=1}^k \mathcal{X}_i$.

Given $D\colon \mathbb{C}\mathcal{X} \to \mathbb{C}\mathcal{X}$ as in the statement, we have, for any $f = \sum f_j$, $f_j \in \mathbb{C}M_j^*$, and any $x = \sum x_j$, $x_j \in \mathcal{X}_j$,

$$(D^* f)(x) = f(Dx) = f\Big(\sum_j \omega_j x_j\Big) = \sum_i \sum_j \omega_j f_i(x_j) = \sum_j \omega_j f_j(x)$$

since $f_i(x_j) = 0$ if $i \neq j$. □

Lemma 3.9. *Let* $(\mathcal{X}, \mathcal{E}^*)$ *be as in Lemma 3.8. Partition* $\mathcal{E}^*$ *into equivalence classes* $\mathcal{E}_1^*, \dots, \mathcal{E}_k^*$ *as in Proposition 3.7, and let* $M_j^* = \operatorname{span} \mathcal{E}_j^*$. *Then for any* $f = \sum f_j$ *with* $f_j \in M_j^*$ *we have that* $\|f\|^* = \sum \|f_j\|^*$. *That is, the norm on* $\mathcal{X}^*$ *is the* ℓ_1*-norm of the norms on the subspaces* M_j^*.

Proof. Let $f \in \mathcal{X}^*$. Let $\widehat{f} = f/\|f\|^*$, so $\widehat{f} \in \mathcal{B}^*$, the closed unit ball of the dual norm. By Minkowski's theorem $\widehat{f}$ is a convex combination of extreme points of $\mathcal{B}^*$, so we may write $f = \sum_{i=1}^n \|f\|^* \lambda_i g_i$, where $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$, and $g_i \in \mathcal{E}^*$. By gathering those extreme points which lie in the same equivalence class together, we may write $f = \sum f_j$, where $f_j = \sum \|f\|^* \lambda_i g_i \in M_j^*$ and the sum is over those indices i for which $g_i \in \mathcal{E}_j^*$.

Then

$$\|f\|^* = \Big\|\sum_{j=1}^k f_j\Big\|^* \leq \sum_{j=1}^k \|f_j\|^* \leq \sum_{i=1}^n \lambda_i \|f\|^* \|g_i\|^* = \|f\|^*$$

since $\|g_i\|^* = 1$ for each extreme point. Since $\mathcal{X}^* = \oplus_{j=1}^k M_j^*$, the decomposition $f = \sum_{j=1}^k f_j$ is unique and $\|f\|^* = \sum_{j=1}^k \|f_j\|^*$ as desired. □

Using the above definitions and results, we can describe the relation between the isometries for $\|\cdot\|$ and the isometries for the Taylor complexified norm $\|\cdot\|_{\mathfrak{T}}$ if $\dim \mathcal{X}$ is finite. Clearly, if A is an isometry for the real Banach space $(\mathcal{X}, \|\cdot\|)$, then it is straightforward (see Eq. (1) or see [11]) that its complexification, $1 \otimes_{\mathbb{R}} A\colon \mathbb{C} \otimes_{\mathbb{R}} \mathcal{X} \to \mathbb{C} \otimes_{\mathbb{R}} \mathcal{X}$, which for brevity we again denote by A, satisfies

$$\|A(x+iy)\|_{\mathfrak{T}} = \|x+iy\|_{\mathfrak{T}}.$$

In the next theorem we completely determine the relation between the isometries for a given norm $\|\cdot\|$ on $\mathcal{X}$ and the isometries for the Taylor complexified norm $\|\cdot\|_{\mathfrak{T}}$ on $\mathbb{C}\mathcal{X}$.

Theorem 3.10. *Let* $(\mathcal{X}, \|\cdot\|)$ *be a finite-dimensional real normed space with decomposition* $\mathcal{X} = \oplus_{j=1}^k \mathcal{X}_j$ *as defined in Lemma 3.8 and let* $(\mathcal{X}_{\mathfrak{T}}, \|\cdot\|_{\mathfrak{T}})$ *be its minimal complexification. Then* T *is a complex isometry for* $\|\cdot\|_{\mathfrak{T}}$ *if and only if* $T = DA = A\widehat{D}$, *where* A *is (the complexification of) a real isometry for* $\|\cdot\|$ *while* $D = \gamma_1 I_{n_1} \oplus \cdots \oplus \gamma_k I_{n_k}$ *and* $\widehat{D} = \widehat{\gamma}_1 I_{n_1} \oplus \cdots \oplus \widehat{\gamma}_k I_{n_k}$ *act on* $\mathcal{X}_{\mathfrak{T}} = (\mathbb{C}\mathcal{X}_1) \oplus \cdots \oplus (\mathbb{C}\mathcal{X}_k)$, *with* $\gamma_j, \widehat{\gamma}_j \in \mathbb{T}$ *and* $n_j = \dim \mathcal{X}_j$, $j = 1, \dots, k$.

We have a similar result for another extremal complexification, that is, for the Bochnak or maximal complexification. The only difference is that now we are partitioning the extreme points of the original norm rather than its dual. Recall from the Introduction that the maximal (or Bochnak) complexification of $\mathcal{X}$ equals $\mathcal{X}_{\mathfrak{B}} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{X}$ equipped with the projective tensor norm $\|x+iy\|_{\mathfrak{B}} = \inf\{\sum_k |\lambda_k| \cdot \|x_k\| : \ x+iy = \sum_k \lambda_k x_k\}$ on $\mathcal{X}_{\mathfrak{B}}$.

Theorem 3.11. *Let $(\mathcal{X}, \|\cdot\|)$ be a finite-dimensional real Banach space and let $(\mathcal{X}_{\mathfrak{B}}, \|\cdot\|_{\mathfrak{B}})$ be its Bochnak complexification with $\mathcal{B}$ and $\mathcal{B}_{\mathfrak{B}}$ the corresponding unit balls. Partition $\mathcal{E} = \operatorname{Ext}(\mathcal{B})$ into equivalence classes $\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_k$ as in Definition 3.4 and let $\mathcal{X}_i := \operatorname{span} \mathcal{E}_i$. Then T is a complex isometry for $\|\cdot\|_{\mathfrak{B}}$ if and only if $T = DA = A\widehat{D}$, where A is (the complexification of) a real isometry for $\|\cdot\|$ while $D = \gamma_1 I_{n_1} \oplus \cdots \oplus \gamma_k I_{n_k}$ and $\widehat{D} = \widehat{\gamma}_1 I_{n_1} \oplus \cdots \oplus \widehat{\gamma}_k I_{n_k}$ act on $\mathcal{X}_{\mathfrak{B}} = (\mathbb{C}\mathcal{X}_1) \oplus \cdots \oplus (\mathbb{C}\mathcal{X}_k)$, with $\gamma_j, \widehat{\gamma}_j \in \mathbb{T}$ and $n_j = \dim \mathcal{X}_j$, $j = 1, \ldots, k$. Moreover,*

$$\operatorname{Ext}(\mathcal{B}_{\mathfrak{B}}) = \mathbb{T} \operatorname{Ext}(\mathcal{B}). \tag{8}$$

The proofs of both theorems will be given in the next section. Below, we mention some consequences and examples connected to the first theorem, and also make some remarks to put the results in perspective.

Corollary 3.12. *Suppose $\|\cdot\|$ is a norm on $\mathbb{R}^n$ such that the set of extreme points $\mathcal{E}^*$ of the norm ball of $\|\cdot\|^*$ cannot be partitioned into $\mathcal{E}_1^* \cup \mathcal{E}_2^*$ such that $\operatorname{span} \mathcal{E}_1^* \cap \operatorname{span} \mathcal{E}_2^* = \{0\}$. Then T is an isometry for $\|\cdot\|_{\mathfrak{T}}$ if and only if $T = \gamma A$ for a complex unit γ and a real isometry A for $\|\cdot\|$.*

Example 3.13. Consider the ℓ_∞-norm on $\mathbb{R}^n$. Its dual norm is $\ell_\infty^* = \ell_1$ with extreme points equal to $\mathcal{E}^* = \{\pm e_j : 1 \le j \le n\}$, where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$. It partitions into $\mathcal{E}^* = \bigcup_j \{\pm e_j\}$. The complex isometry group $G_{\mathfrak{T}}$ is the set of generalized permutations in $M_n(\mathbb{C})$; the real isometry group $G_{\mathbb{R}}$ is the set of generalized permutations in $M_n(\mathbb{R})$ (see, e.g. [7]). Clearly, $G_{\mathfrak{T}} \neq \mathbb{T} G_{\mathbb{R}}$.

Example 3.14. *In contrast, for $p \in [1, \infty)$ the dual of the ℓ_p-norm on $\mathbb{R}^n$ is $\ell_p^* = \ell_q$ ($\frac{1}{p} + \frac{1}{q} = 1$) and, except for $(q, n) = (\infty, 2)$ the spatial partition of its extreme points consists of a single class. This is easy to see except, perhaps, for $q = \infty$ and $n \ge 3$ when $\mathcal{E}^* = \{(\pm 1, \ldots, \pm 1)^t\} \subseteq \mathbb{R}^n$. To see that vectors in $\mathcal{E}^*$ are equivalent, pick any $x, y \in \mathcal{E}^*$. By applying a suitable diagonal matrix $\operatorname{diag}(\pm 1, \ldots, \pm 1)$ we may assume $x = (1, \ldots, 1)^t$. By replacing, if needed, y with $-y$ we may also assume that at most half of the entries in y equal (-1). By permuting entries, if needed, we may further suppose $y = (-1_k, 1_{n-k})^t$ where the index indicates the number of occurrences. Then, $x - y - (1_k, -1, 1_{n-k-1})$ is a sum of linearly independent vectors from $\mathcal{E}^*$ and equals $x_0 = (1_k, 1, -1_{n-k-1}) \in \mathcal{E}^*$, so $x \sim_p y$ (see Definition 3.4).*

Hence, Corollary 3.12 applies to $\ell_{p,\mathfrak{T}}$, the complexified ℓ_p-norm, for every $p < \infty$ and $n \ge 2$ except for $(p, n) = (1, 2)$.

Example 3.15. *The numerical radius*

$$w(A) = \max\{|x^* A x| : x \in \mathbb{C}^n, x^* x = 1\}; \qquad A \in M_n(\mathbb{C})$$

is the Taylor complexification of the restriction of the spectral norm $\|A\|_s = \sqrt{\lambda_{\max}(A^*A)}$ to the real subspace of Hermitian matrices $H_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$. To see this recall that $M_n(\mathbb{C}) =$

$H_n(\mathbb{C}) + iH_n(\mathbb{C})$ and that the restriction of the spectral norm to $H_n(\mathbb{C})$ coincides with the numerical radius. Then, let $x_A \in \mathbb{C}^n$ be a unit vector which maximizes the modulus of x^*Ax subject to $x^*x = 1$, and let $\phi_A \in [0, 2\pi]$ be the argument of $x_A^*Ax_A$. It follows that the numerical radius of $A = S_1 + iS_2 \in H_n(\mathbb{C}) + iH_n(\mathbb{C})$ equals

$$w(A) = |x_A^*Ax_A| = e^{-i\phi_A}x_A^*Ax_A = x_A^*(S_1 \cos\phi_A + S_2 \sin\phi_A)x_A$$

while, with fixed $\phi \in [0, 2\pi]$,

$$w(S_1 \cos\phi + S_2 \sin\phi) = \max_{\|x\|=1} x^*(S_1 \cos\phi + S_2 \sin\phi)x = \max_{\|x\|=1} \mathrm{Re}(x^*(e^{-i\phi}A)x) \le \max_{\|x\|=1} |x^*Ax|.$$

The dual norm of w restricted to $H_n(\mathbb{C})$ is the restriction of the trace norm $\|A\|_t = \mathrm{tr}\sqrt{A^*A}$ to $H_n(\mathbb{C})$ and so its set of extreme points equals

$$\mathcal{E}^* = \{\pm xx^* : x \in \mathbb{C}^n, \|x\| = 1\}.$$

Clearly, one cannot partition $\mathcal{E}^*$ into two subsets whose real linear spans have trivial intersection. By Corollary 3.12, T is an isometry for w on $M_n(\mathbb{C})$ if and only if $T = \xi A$ for some $\xi \in \mathbb{T}$ and some isometry A for w on $H_n(\mathbb{C})$. We remark that the isometries of the restriction of the spectral norm to $H_n(\mathbb{C})$ are known; see [4] where also isometries for the numerical radius on $M_n(\mathbb{C})$ were determined by an induction argument.

Remarks.

(1) The same result holds for the c-numerical radius $w_c(A)$ for $c \in \mathbb{R}^n$ not equal to a multiple of $\mathbf{1} = (1, \ldots, 1)^t$ nor satisfying $c^t\mathbf{1} = 0$, where the dual norm ball of $w_c(A)$ on $H_n(\mathbb{C})$ has the set of extreme points

$$\mathcal{E}^* = \{\pm U^* \,\mathrm{diag}(c_1, \ldots, c_n)U : U \text{ unitary}\}$$

satisfying the condition of Corollary 3.12. The proof of this result in [5] uses the convexity of the c-numerical range.

(2) More generally, it is known [8, 6] that for every unitary similarity invariant norm on $M_n(\mathbb{C})$, there is a bounded subset $\mathcal{S} \subseteq M_n(\mathbb{C})$ such that

$$\|A\| = \sup\{w_C(A) : C \in \mathcal{S}\},$$

where

$$w_C(A) = \sup\{|\mathrm{tr}\,(CU^*AU)| : U \text{ unitary}\}.$$

We can assume [6, Remark II, p. 186] that $\mathcal{U}_{\mathcal{S}} = \{e^{i\theta}UCU^* : C \in \mathcal{S}, \theta \in \mathbb{R}, U \text{ unitary}\}$ is the set of extreme points of the dual norm. Suppose that the set $\mathcal{S}$ can be chosen to be a subset of $H_n(\mathbb{C})$, and that $\mathcal{S}$ contains a nonscalar matrix C with nonzero trace. Since $\mathcal{U}_C = \{UCU^* : U \text{ unitary}\}$ is path connected, $\mathcal{U}_C$ lies in a single equivalence class of $\mathcal{E}^*$; since $\mathcal{U}_C$ spans $H_n(\mathbb{C})$ (see [6, p. 184]), $\mathcal{E}^*$ has a single equivalence class and Corollary 3.12 applies. Then T is an isometry for $\|\cdot\|$ on $M_n(\mathbb{C})$ if and only if $T = e^{is}A$ for some isometry A for $\|\cdot\|$ on $H_n(\mathbb{C})$.

(3) The description of the isometries relies on the structure of the set $\mathcal{E}^*$. It is not hard to prove that the decomposition $\mathbb{R}^n = \mathrm{span}\,\mathcal{E}_1^* \oplus \cdots \oplus \mathrm{span}\,\mathcal{E}_k^*$ into maximal possible summands, where $\mathcal{E}_1^*, \ldots, \mathcal{E}_k^*$ partition $\mathcal{E}^*$, is unique. To see this, suppose $\{\mathcal{E}_1^*, \ldots, \mathcal{E}_k^*\}$ and

$\{\hat{\mathcal{E}}_1^*, \ldots, \hat{\mathcal{E}}_\ell^*\}$ are two different partitions of $\mathcal{E}^*$ such that $\mathbb{R}^n = \operatorname{span}\mathcal{E}_1^* \oplus \cdots \oplus \operatorname{span}\mathcal{E}_k^* = \operatorname{span}\hat{\mathcal{E}}_1^* \oplus \cdots \oplus \operatorname{span}\hat{\mathcal{E}}_\ell^*$. Then we have

$$\mathbb{R}^n = \bigoplus_{\substack{1\le r\le k\\ 1\le s\le \ell}} \operatorname{span}(\mathcal{E}_r^* \cap \widehat{\mathcal{E}}_s^*).$$

Since the number of summands k (for which the decomposition of $\mathbb{R}^n$ into spans of sets that partition $\mathcal{E}^*$ is possible) was maximal and since the same holds for ℓ one sees that

$$\{\mathcal{E}_1^*, \ldots, \mathcal{E}_k^*\} = \{\mathcal{E}_r^* \cap \hat{\mathcal{E}}_s^* : 1 \le r \le k,\ 1 \le s \le \ell\} = \{\hat{\mathcal{E}}_1^*, \ldots, \hat{\mathcal{E}}_\ell^*\}.$$

(4) By the above remark, the spatial partition of $\mathcal{E}^*$ can be done by induction as follows. If $\mathcal{E}^*$ cannot be partitioned into two subsets $\mathcal{E}_1^*$ and $\mathcal{E}_2^*$ such that $(\operatorname{span}\mathcal{E}_1^*)\cap(\operatorname{span}\mathcal{E}_2^*) = \{0\}$, then the maximum number k is 1. Otherwise, we can further partition $\mathcal{E}_1^*$ and/or $\mathcal{E}_2^*$ until we cannot further partition the subsets. Then we obtain a partition into a maximal number of subsets, say $\mathcal{E}^* = \mathcal{E}_1^* \cup \cdots \cup \mathcal{E}_k^*$ and $\mathbb{R}^n = \operatorname{span}\mathcal{E}_1^* \oplus \cdots \oplus \operatorname{span}\mathcal{E}_k^*$. It follows easily that for $x, y \in \mathcal{E}^*$, $x \sim_p y$ (see Definition 3.4) implies both x, y belong to the same subset, say $x, y \in \mathcal{E}_i^*$. Proposition 3.7 then implies that we have a spatial partition of $\mathcal{E}^*$.

(5) There is no easy way (and there should not be one) to translate the condition in Theorem 3.10 from $\mathcal{E}^*$ to $\mathcal{E}$. In fact, applying Corollary 3.12 to the ℓ_p-norm on $\mathbb{R}^n$, we see that, except for $(p, n) = (1, 2)$, the isometries of the $\ell_{p,\mathfrak{T}}$-norm on $\mathbb{C}^n$ have the form γA for some complex unit γ and real isometry A if $p \in [1, \infty)$ (see Example 3.14), whereas, by (5), the complexified norm of the ℓ_∞-norm on $\mathbb{R}^n$ is the ℓ_∞-norm on $\mathbb{C}^n$ with isometries of the form PD for a real permutation matrix P and a diagonal unitary matrix D. These differences, especially between ℓ_1- and ℓ_∞-norm, are hard to detect directly by comparing the norm's extreme points but are easily seen when considering the dual norms.

(6) Let $\mathcal{G}$ be the isometry group (of real matrices) for the norm $\|\cdot\|$ and let $\mathcal{G}_\mathfrak{T}$ be the isometry group (of complex matrices) for the complexified norm $\|\cdot\|_\mathfrak{T}$. Then Theorem 3.10 implies that $\mathcal{G}_\mathfrak{T}$ is a nonsplit extension of a normal subgroup $\mathcal{G}$ with the torus group of operators of the form $D = \gamma_1 I_{n_1} \oplus \cdots \oplus \gamma_k I_{n_k}$ acting on $\mathbb{C}^n = \operatorname{span}\mathcal{X}_1 \oplus \cdots \oplus \operatorname{span}\mathcal{X}_k$.

4. Proofs of Theorems 3.10 and 3.11

Proof of Theorem 3.10. In general, suppose $\|\cdot\|$ is a norm on $\mathbb{C}^n$. Then T is an isometry for $\|\cdot\|$ if and only if the dual operator T^* is an isometry for the dual norm $\|\cdot\|^*$ (equivalently, T^* preserves $\mathcal{E}^*$, the set of extreme points of $\|\cdot\|^*$). Translating this to the complexified norm $\|\cdot\|_\mathfrak{T}$ for a given norm $\|\cdot\|$ on a real space $\mathcal{X} = \mathbb{R}^n$, we see that T preserves $\|\cdot\|_\mathfrak{T}$ if and only if T^* preserves the set of extreme points of $\|\cdot\|_\mathfrak{T}^*$, i.e., $\mathcal{E}_\mathfrak{T}^* = \mathbb{T}\mathcal{E}^*$ (the equality follows by Theorem 3.2).

Let $\mathcal{E}^* = \mathcal{E}_1^* \cup \cdots \cup \mathcal{E}_k^*$ be the spatial partition and let $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_k$ be the associated decomposition defined in Lemma 3.8. Assume $T^* = A^*D^*$ (or $\widehat{T}^* = D^*A^*$), where A is a complexification of a real isometry for $\|\cdot\|$ and $D = \gamma_1 I_{n_1} \oplus \cdots \oplus \gamma_k I_{n_k}$ ($\gamma_j \in \mathbb{T}$) acts on

$$\mathcal{X}_\mathfrak{T} = \mathbb{C}\mathcal{X} = (\mathbb{C}\mathcal{X}_1) \oplus \cdots \oplus (\mathbb{C}\mathcal{X}_k).$$

Then by Lemma 3.8 and using $\mathbb{C}\operatorname{span}_\mathbb{R}(\mathcal{E}_i^*) = \operatorname{span}_\mathbb{C}(\mathbb{T}\mathcal{E}_i^*)$ we have that $D^* = \gamma_1 I_{m_1} \oplus \cdots \oplus \gamma_k I_{m_k}$ with respect to the decomposition

$$\mathcal{X}_\mathfrak{T}^* = \operatorname{span}_\mathbb{C}(\mathbb{T}\mathcal{E}_1^*) \oplus \cdots \oplus \operatorname{span}_\mathbb{C}(\mathbb{T}\mathcal{E}_k^*); \qquad (m_j = \dim\operatorname{span}\mathcal{E}_j^*).$$

Clearly, $A^*D^*(\mathbb{T}\mathcal{E}^*) = D^*A^*(\mathbb{T}\mathcal{E}^*) = \mathbb{T}\mathcal{E}^*$. Thus $T = DA$ and $\widehat{T} = AD$ are both isometries for $\|\cdot\|_{\mathfrak{T}}$.

Conversely, suppose T is a (complex) isometry for $\|\cdot\|_{\mathfrak{T}}$. Then $T^*(\mathbb{T}\mathcal{E}^*) = \mathbb{T}\mathcal{E}^*$. We will show that $T^* = A^*D^* = \widehat{D}^*A^*$ such that A is a complexified real isometry for $\|\cdot\|$ and $D^*, \widehat{D}^*$ have the asserted form. To this end, we first establish the following.

Claim: For each j, $1 \le j \le k$, there exists $\mu_j \in \mathbb{T}$ such that $T^*(\mathcal{E}_j^*) \in \mu_j\mathcal{E}^*$.

We shall take $j = 1$ to simplify notation. Fix $x_1 \in \mathcal{E}_1^*$. Then $T^*x_1 = \mu y_1$ for some $\mu \in \mathbb{T}$ and $y_1 \in \mathcal{E}^*$. Let x_2 be some other vector in $\mathcal{E}_1^*$. Consider the following cases.

Case 1. Suppose $\{x_1, x_2\}$ is linearly dependent, so $x_2 = -x_1$. Since $\mathcal{E}^* = -\mathcal{E}^*$, $T^*x_2 = \mu(-y_1) \in \mu\mathcal{E}^*$.

Case 2. Suppose $\{x_1, x_2\}$ is linearly independent and $x_1 \sim_p x_2$ (see Definition 3.4). Then there is a linearly independent set $\{x_1, \dots, x_\ell\}$ and x_0 in $\mathcal{E}_1^*$ such that

$$x_0 = a_1x_1 + a_2x_2 + \cdots + a_\ell x_\ell \tag{9}$$

with $a_1a_2\cdots a_\ell \neq 0$. We may write $T^*x_j = e^{i\theta_j}y_j \in \mathbb{T}\mathcal{E}^*$ with $\theta_j \in [0, 2\pi]$ and $y_j \in \mathcal{E}^*$ for $j = 0, 1, \dots, \ell$. Applying T^* to (9) and taking real and imaginary parts, we have

$$(\cos\theta_0)\, y_0 = \sum_{j=1}^{\ell} a_j(\cos\theta_j)\, y_j \quad \text{and} \quad (\sin\theta_0)\, y_0 = \sum_{j=1}^{\ell} a_j(\sin\theta_j)\, y_j. \tag{10}$$

Since T^* is an invertible $\mathbb{C}$-linear map, $\{e^{i\theta_1}y_1, \dots, e^{i\theta_\ell}y_\ell\}$ is linearly independent and spans an ℓ-dimensional subspace in $\mathcal{X}_{\mathfrak{T}} = \mathbb{C}^n$. As a result, the set of real vectors $\{y_1, \dots, y_\ell\}$ will span the same subspace in $\mathbb{C}^n$. By (10), the nonzero real vector y_0 can be written as a real linear combination of the linearly independent vectors $y_1, \dots, y_\ell$ in two ways (or else $\cos\theta_j = 0$ for all j, or $\sin\theta_j = 0$ for all j). In any case, we see that $(a_1\cos\theta_1, \dots, a_\ell\cos\theta_\ell)$ and $(a_1\sin\theta_1, \dots, a_\ell\sin\theta a_\ell)$ are parallel, so $\tan\theta_1 = \cdots = \tan\theta_\ell$. Thus $e^{i\theta_2} = e^{i\theta_1}$ or $-e^{i\theta_1}$, so $T^*x_2 = \mu y_2$ or $\mu(-y_2)$. In either case the claim holds.

Case 3. Finally, suppose $\{x_1, x_2\}$ is linearly independent and there is a set $\{u_1, \dots, u_\ell\} \subseteq \mathcal{E}_1^*$ such that $x_1 = u_1 \sim_p u_2 \sim_p \cdots \sim_p u_\ell = x_2$. Cases 1 and 2 have actually shown that whenever $z_1, z_2 \in \mathcal{E}_1^*$, $z_1 \sim_p z_2$, and $T^*z_1 = \mu y_1$ for some $y_1 \in \mathcal{E}^*$, then $T^*z_2 = \mu y_2$ for some $y_2 \in \mathcal{E}^*$. By iterating this argument, we have $T^*u_2 = \mu y_2, \dots, T^*u_\ell = \mu y_\ell \in \mu\mathcal{E}^*$ for some $y_j \in \mathcal{E}^*$. The proof of the **Claim** is complete.

By the **Claim**, there exist $\mu_1, \dots, \mu_k \in \mathbb{T}$ so that for $x_j \in \mathcal{E}_j^*$, $\overline{\mu}_jT^*x_j \in \mathcal{E}^*$. Define a linear map D^* acting on $\mathcal{X}_{\mathfrak{T}}^* = \operatorname{span}_{\mathbb{C}}\mathcal{E}_1^* \oplus \cdots \oplus \operatorname{span}_{\mathbb{C}}\mathcal{E}_k^*$ by

$$D^* = \mu_1 I_{n_1} \oplus \cdots \oplus \mu_k I_{n_k}.$$

Writing D^{-*} for $(D^*)^{-1}$ we have $T^*D^{-*}(\mathcal{E}^*) = \mathcal{E}^*$, and since $\mathcal{E}^*$ spans $\mathbb{R}^n$, $T^*D^{-*}(\mathbb{R}^n) \subseteq \mathbb{R}^n$. Thus T^*D^{-*} is a real isometry for $\|\cdot\|^*$, so it equals A^* for some real isometry A for $\|\cdot\|$. Thus $T^* = A^*D^*$ and so $T = DA$.

For the reversed product, note that the restriction $A^*|_{\mathbb{R}^n} = T^*D^{-*}|_{\mathbb{R}^n} \colon \mathbb{R}^n \to \mathbb{R}^n$ preserves the equivalence relation from Definition 3.4. Thus, it permutes the equivalence classes $\mathcal{E}_1^*, \dots, \mathcal{E}_k^*$ that form the spatial partition of $\mathcal{E}^*$. Being $\mathbb{R}$-linear it must also permute their real linear spans, that is, it permutes the subspaces $\operatorname{span}_{\mathbb{R}}\mathcal{E}_1^*, \dots, \operatorname{span}_{\mathbb{R}}\mathcal{E}_k^* \subseteq \mathcal{X}^*$. It follows that relative to the decomposition $\mathcal{X}^* = \operatorname{span}_{\mathbb{R}}\mathcal{E}_1^* \oplus \cdots \oplus \operatorname{span}_{\mathbb{R}}\mathcal{E}_k^*$ the operator T^*D^{-*} is represented by a block-permutational matrix $\big(B_{i\sigma(i)}^*\big)_i$ for some permutation

σ on $\{1,\dots,k\}$, where $B^*_{i\sigma(i)} \in M_{n_i}(\mathbb{R})$, $i=1,\dots,k$ are the corresponding blocks. Hence, $T^* = \big(B^*_{i\sigma(i)}\big)_i D^* = \big(\mu_{\sigma(i)}\, B^*_{i\sigma(i)}\big)_i = \widehat{D}^*\big(B^*_{i\sigma(i)}\big)_i$ where

$$\widehat{D}^* = \mu_{\sigma(1)} I_{n_1} \oplus \cdots \oplus \mu_{\sigma(k)} I_{n_k}.$$

So $T^* = \widehat{D}^* A^*$ and the result follows. □

Remark 4.1. *The proof of Theorem 3.10 establishes even more: If A is an isometry for $(\mathcal{X}, \|\cdot\|)$, then its dual A^* is represented by a block-permutational matrix with respect to decomposition*

$$\mathcal{X}^* = \mathrm{span}_{\mathbb{R}}\, \mathcal{E}_1^* \oplus \cdots \oplus \mathrm{span}_{\mathbb{R}}\, \mathcal{E}_k^*$$

where $\mathcal{E}^* = \mathcal{E}_1^* \cup \cdots \cup \mathcal{E}_k^*$ is the spatial decomposition of the extreme points for the dual norm $\|\cdot\|^*$.

Proof of Theorem 3.11. It is known (see [13, Proposition 2.2, p. 17]) that the closed unit ball of the projective tensor product norm $\|\cdot\|_{\mathbb{C},\pi}$ is the closed convex hull of the set $\Delta \otimes_{\mathbb{R}} \mathcal{B}_{\mathcal{X}} = \{\omega \otimes_{\mathbb{R}} x : |\omega|, \|x\| \le 1\}$. Using polar coordinates for complex number $\omega = |\omega| e^{it}$ we see that $\Delta \otimes_{\mathbb{R}} \mathcal{B}_{\mathcal{X}} = \mathbb{T} \otimes_{\mathbb{R}} \mathcal{B}_{\mathcal{X}}$ so that

$$\mathcal{B}_{\mathbb{C}} = \overline{\mathrm{conv}}(\mathbb{T} \otimes \mathcal{B}_{\mathcal{X}}). \tag{11}$$

Identify $\mathcal{X}$ with $\mathbb{R}^n$, $n = \dim \mathcal{X}$, and $\mathbb{T} \otimes \mathcal{B}_{\mathcal{X}}$ with a closed and bounded, hence compact, subset $\mathbb{T}\mathcal{B}_{\mathcal{X}} \subseteq \mathbb{C}^n = \mathbb{R}^{2n}$. It follows that $\mathrm{conv}(\mathbb{T}\mathcal{B}_{\mathcal{X}})$ is also compact [12, Theorem 3.25] so from (11) we get $\mathcal{B}_{\mathbb{C}} = \mathrm{conv}(\mathbb{T} \otimes \mathcal{B}_{\mathcal{X}})$. Lemma 3.1 then implies that in Bochnak's maximal complexification norm, (8) holds.

The rest of the proof is the same as for Theorem 3.10, except that we now are partitioning $\mathcal{E} = \mathrm{Ext}(\mathcal{X})$ into equivalence classes $\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_k$ given from Definition 3.4 instead of the extreme points $\mathcal{E}^* = \mathrm{Ext}(\mathcal{X}^*)$ of the dual norm and we decompose $\mathcal{X}$ into $\mathcal{X} = \mathrm{span}\, \mathcal{E}_1 \oplus \cdots \oplus \mathrm{span}\, \mathcal{E}_k$. □

5. Hermitian operators on the Taylor complexification

Recall that an operator T on a complex Banach space $\mathcal{X}_{\mathbb{C}}$ is Hermitian if its numerical range $V(T) := \{f(Tx) :\ x \in \mathcal{X}_{\mathbb{C}},\ f \in \mathcal{X}_{\mathbb{C}}^*,\ \|x\| = f(x) = \|f\|^* = 1\} \subseteq \mathbb{R}$. By the famous Vidav-Palmer theorem, this is equivalent to the fact that $\|e^{i\theta T}\| = 1$ for every real θ, which, due to $1 = \|I\| \le \|e^{i\theta T}\| \cdot \|e^{-i\theta T}\|$ is further equivalent to the fact that $e^{i\theta T}$ is an isometry for every real θ.

Proposition 5.1. *Let $(\mathcal{X}, \|\cdot\|)$ be a finite-dimensional real Banach space and let $(\mathcal{X}_{\mathbb{C}}, \|\cdot\|_{\mathfrak{T}})$ be the Taylor minimal complexification. Assume that $A\colon \mathcal{X} \to \mathcal{X}$ becomes a Hermitian operator when extended to $\mathcal{X}_{\mathbb{C}}$. Then relative to the decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_k$ from Theorem 3.10, $A = \lambda_1 I_{m_1} \oplus \cdots \oplus \lambda_k I_{m_k}$ for some $\lambda_i \in \mathbb{R}$.*

Proof. Denote the complexified operator $1 \otimes A$ on $\mathcal{X}_{\mathbb{C}}$ again by A. By definition, $e^{i\theta A}$ is an isometry of $\|\cdot\|_{\mathfrak{T}}$ for every real θ. Clearly, its dual, $A^*\colon \mathcal{X}_{\mathbb{C}}^* \to \mathcal{X}_{\mathbb{C}}^*$ is also Hermitian and leaves the set of functionals which are real-valued on $\mathcal{X}$ invariant. Hence, A^* maps the real vector space $\mathcal{X}^* \subseteq \mathcal{X}_{\mathbb{C}}^*$ back to itself. Let $\mathcal{E}^* = \mathcal{E}_1^* \cup \cdots \cup \mathcal{E}_k^* \subseteq \mathcal{X}^*$ be the spatial decomposition of extreme points of the dual norm $\|\cdot\|^*$ and let

$$\mathcal{X}^* = \mathrm{span}_{\mathbb{R}}\, \mathcal{E}_1^* \oplus \cdots \oplus \mathrm{span}_{\mathbb{R}}\, \mathcal{E}_k^*, \tag{12}$$

$$\mathcal{X}_{\mathbb{C}}^* = \mathrm{span}_{\mathbb{C}}\, \mathcal{E}_1^* \oplus \cdots \oplus \mathrm{span}_{\mathbb{C}}\, \mathcal{E}_k^*. \tag{13}$$

It follows from the proof of Theorem 3.10 (see also Remark 4.1) that, relative to the decomposition (13), the isometry $e^{i\theta A^*}$ can be written as $e^{i\theta A^*} = D_\theta^* B_\theta^*$ where $D_\theta^* = \mu_1(\theta) I_{n_1} \oplus \cdots \oplus \mu_k(\theta) I_{n_k}$ acts on $\mathcal{X}_{\mathbb{C}}^* = \operatorname{span}_{\mathbb{C}} \mathcal{E}_1^* \oplus \cdots \oplus \operatorname{span}_{\mathbb{C}} \mathcal{E}_k^*$ and the real isometry B_θ^* acts as a block-permutational operator on (12), that is, there exists a permutation $\sigma\colon \{1,\dots,k\} \to \{1,\dots,k\}$ (which depends on θ) such that $B_\theta^*(\mathcal{X}_i^*) = \mathcal{X}_{\sigma(i)}^*$. As such, relative to decomposition (12), the operator B_θ^* is represented with a block-permutational real matrix (having exactly k blocks), while $D_\theta^* B_\theta^*$ is represented by the same matrix, except that the block in the i-th row is multiplied with a unimodular number μ_i. It follows that, if $s \geq 1$ is the exponent of the symmetric group Sym_k, then $\sigma^s = \mathrm{id}$ and so $e^{is\theta A^*} = (D_\theta^* B_\theta^*)^s$ is a block-diagonal isometry of the form

$$e^{is\theta A^*} = \widehat{D}_\theta^* \widehat{B}_\theta^*, \tag{14}$$

for a suitable $\widehat{D}_\theta^* = \widehat{\mu}_1(\theta) I_{n_1} \oplus \cdots \oplus \widehat{\mu}_k(\theta) I_{n_k}$ acting on (13) and a suitable real isometry $\widehat{B}_\theta^*$ leaving the real spaces $\mathcal{X}_i^*$ invariant.

We claim that A^* leaves each $\mathcal{X}_j^*$ invariant. To see this, consider the restriction of both sides in (14) to an invariant subspace $\mathbb{C}\mathcal{X}_j^* \subseteq \mathbb{C}\mathcal{X}^* = \mathcal{X}_{\mathbb{C}}^*$. If $\theta \in \mathbb{R}$ is sufficiently close to 0, then the spectrum $\mathrm{Sp}(is\theta A^*)$ lies in the rectangle $(-1,1) \times (-\pi/2, \pi/2) \subseteq \mathbb{C}$ which the exponential function, $z \mapsto e^z$, maps bijectively onto a simply connected region in $\mathbb{C}$. Hence, one may find a holomorphic logarithm ln such that $\ln(\exp(z)) = z$ for $z \in (-1,1) \times (-\pi/2, \pi/2) \subseteq \mathbb{C}$. Then holomorphic calculus [12, Theorem 10.29] implies $\ln e^{is\theta A^*} = is\theta A^*$ for $\theta \in \mathbb{R}$ sufficiently close to 0. Since $\mathbb{C}\mathcal{X}_j^*$ is invariant for $e^{is\theta A^*}$, it must be also invariant for every holomorphic function in $e^{is\theta A^*}$ and in particular it must be invariant for $\ln e^{is\theta A^*} = is\theta A^*$ and hence for A^* as well. Finally, A^* leaves invariant $\mathcal{X}^*$, the real part of $\mathcal{X}_{\mathbb{C}}^*$, and hence also $\mathcal{X}^* \cap \mathbb{C}\mathcal{X}_j^* = \mathcal{X}_j^*$ as claimed.

In the sequel we concentrate solely on the first summand and denote the corresponding restrictions by $A_1^* = A^*|_{\mathcal{X}_1^*}\colon \mathcal{X}_1^* \to \mathcal{X}_1^*$ and by $\widehat{B}_{1,\theta}^* = \widehat{B_\theta}^*|_{\mathcal{X}_1^*}$, and write $\widehat{\mu}_1(\theta) = e^{i\xi_\theta}$ for suitable $\xi_\theta \in [0, 2\pi]$. In this way, (14) becomes

$$e^{is\theta A_1^*} = e^{is\theta A^*}|_{\mathbb{C}\mathcal{X}_1^*} = \widehat{\mu}_1(\theta) \widehat{B}_{1,\theta}^* = e^{i\xi_\theta} \widehat{B}_{1,\theta}^*. \tag{15}$$

Using functional calculus we may decompose

$$\cos(s\theta A_1^*) + i \sin(s\theta A_1^*) = e^{is\theta A_1^*} = e^{i\xi_\theta} \widehat{B}_{1,\theta}^* = (\cos \xi_\theta) \widehat{B}_{1,\theta}^* + i (\sin \xi_\theta) \widehat{B}_{1,\theta}^*.$$

Relative to (13), A_1^* and $\widehat{B}_{1,\theta}^*$ are represented by real matrices. Thus, we see that

$$\cos(s\theta A_1^*) = (\cos \xi_\theta) \widehat{B}_{1,\theta}^*,$$
$$\sin(s\theta A_1^*) = (\sin \xi_\theta) \widehat{B}_{1,\theta}^*.$$

Summing up the squares of both sides gives

$$I_{n_1} = (\widehat{B}_{1,\theta}^*)^2 \tag{16}$$

wherefrom, by squaring (15), one gets

$$e^{2is\theta A_1^*} = e^{2i\xi_\theta} (\widehat{B}_{1,\theta}^*)^2 = e^{2i\xi_\theta I_{n_1}}. \tag{17}$$

It is a well-known fact that each isometry $e^{i\theta A^*}$ is diagonalizable (with spectrum lying on the unit disk). Hence, there exist $n = \dim \mathcal{X}_{\mathbb{C}}$ complex one-dimensional subspaces invariant for $e^{i\theta A}$ whose sum equals $\mathcal{X}_{\mathbb{C}}$. Using holomorphic calculus, as above, we see that A^*, and hence also A_1^*, is itself a diagonalizable operator.

Consequently, we may choose a suitable basis in $\mathbb{C}\mathcal{X}_1^*$ such that A_1^* is represented by a diagonal n_1-by-n_1 matrix $\operatorname{diag}(\lambda_1, \dots \lambda_{n_1})$, which for brevity we again denote by A_1^*. Rewrite (17) into

$$I_{n_1} = e^{2i(s\theta A_1^* - \xi_\theta I_{n_1})} = \operatorname{diag}(e^{2i(s\theta\lambda_1 - \xi_\theta)}, \dots, e^{2i(s\theta\lambda_{n_1} - \xi_\theta)}).$$

It follows that $\xi_\theta = s\theta\lambda_1 - K_1(\theta)\pi = \cdots = s\theta\lambda_{n_1} - K_{n_1}(\theta)\pi$ for suitable integer-valued functions $K_1(\theta), \dots, K_{n_1}(\theta)$. Subtracting the first and the j-th of these identities gives

$$\tfrac{\lambda_1 - \lambda_j}{\pi}\, s\theta = K_1(\theta) - K_j(\theta).$$

On the left-hand side there is a continuous function in θ, but on the right-hand side there is an integer-valued function. Therefore, $\lambda_1 = \lambda_j$ for every j and so $A_1^* = \lambda_1 I_{n_1}$. Clearly, $\lambda_1 \in \mathbb{R}$ because A_1^* leaves the real space $\mathcal{X}_1^*$ invariant.

Similar arguments apply for restrictions of A^* to $\mathbb{C}\mathcal{X}_j$ for every j. Therefore, relative to the decomposition (12),

$$A^* = \lambda_1 I_{n_1} \oplus \dots, \oplus \lambda_k I_{n_k}.$$

By Lemma 3.8, $A = \lambda_1 I_{m_1} \oplus \dots, \oplus \lambda_k I_{m_k}$ relative to decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_k$. □

We say an $\mathbb{R}$-linear operator A acting on a real Banach space is *real-Hermitian* if its complexification $1 \otimes A$ is Hermitian in the minimal complexification norm, i.e., if $\|e^{i\theta(1\otimes A)}\|_{\mathfrak{T}} = 1$ for every real θ.

Corollary 5.2. *Let $(\mathcal{X}, \|\cdot\|)$ be a finite-dimensional real Banach space and let $(\mathcal{X}_{\mathbb{C}}, \|\cdot\|_{\mathfrak{T}})$ be its minimal (Taylor) complexification. Then there exists a fixed decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_k$ such that every real Hermitian operator $A\colon \mathcal{X} \to \mathcal{X}$ leaves each summand $\mathcal{X}_j$ invariant and acts as a scalar operator on it. Hence, the set of real-Hermitian operators with respect to Taylor complexification is an abelian algebra. Moreover, the number of different blocks in this algebra equals the number of equivalence classes in the spatial decomposition of $\mathcal{E}^*$ (the extreme points of the dual norm $\|\cdot\|^*$).*

Example 5.3. *The minimal complexification of the Euclidean norm ℓ_2 on $\mathbb{R}^n$ differs from the complex Euclidean norm ℓ_2 on $\mathbb{C}^n$.*

Namely, in $(\mathbb{C}^n, \ell_2)$ the Hermitian operators coincide with the self-adjoint ones. Clearly, one can find two non-commuting real self-adjoint operators. Hence, there exists no fixed basis relative to which all self-adjoint operators would be simultaneously diagonal.

Example 5.4. *It is known (see [15]) that the Hermitian operators on $(\mathbb{C}^n, \ell_p)$, $1 \le p \le \infty$, are precisely the multiplication operators induced by n-tuples of real numbers when $p \ne 2$, and the self-adjoint operators when $p = 2$. However, since the spatial partition of extreme points of the norm ball of the dual of $(\mathbb{R}^n, \ell_p)$, for $p \ne \infty$ and $(n,p) \ne (2,1)$, consists of a single class (Example 3.14), the only real-Hermitian operators on $(\mathbb{C}^n, \ell_{p,\mathfrak{T}})$, are the real scalar multiples of the identity. Let us note that Hermitian operators on $(\mathbb{C}^n, \ell_p)$ restricted to $\mathbb{R}^n$ are real-valued. This is yet another argument showing that $\ell_{p,\mathfrak{T}}$ and ℓ_p are different norms on $\mathbb{C}^n$ for $p \ne \infty$. If $p = \infty$ we recall that the extreme points of the norm ball of $(\mathbb{R}^n, \ell_1)$ decompose into n classes, so the real-Hermitian operators on $(\mathbb{C}^n, \ell_{\infty,\mathfrak{T}})$ are the same as real-Hermitian operators on $(\mathbb{C}^n, \ell_\infty)$. Therefore, $(\mathbb{C}^n, \ell_\infty)$ does not admit additional Hermitian operators besides those extended from $(\mathbb{R}^n, \ell_\infty)$.*

Remark 5.5. *A finite-dimensional real Banach space $(\mathcal{X}, \|\cdot\|)$ admits exactly 2^k real-Hermitian projections (that is, idempotents that are real-Hermitian operators, including 0 and I) if and only if the spatial partition of $\mathcal{E}^*$ consists of k equivalence classes.*

6. Concluding examples and remarks

Given that the Taylor complexification is extremal in some sense, one may wonder whether the isometry groups of other complexifications will contain or be contained by that of the Taylor complexification. The following shows that neither conclusion holds.

Example 6.1. *The ℓ_p-norm on $\mathbb{C}^n$ is a complexification of the ℓ_p-norm on $\mathbb{R}^n$, but the isometry group of $(\mathbb{C}^n, \ell_p)$ is much larger than $\{e^{i\theta}A : \theta \in [0, 2\pi], A$ an isometry of $(\mathbb{R}^n, \ell_p)\}$.*

Example 6.2. *Consider the ℓ_2-norm on $\mathbb{R}^2$, and a complexification on $\mathbb{C}^2$ such that the dual norm ball of the complexified norm has the set of extreme points equal to:*

$$\left\{e^{i\theta}v : v \in \mathbb{R}^2, \ell_2(v) = 1, \theta \in [0, 2\pi]\right\} \cup \left\{\tfrac{1}{2}e^{i\theta}(\sqrt{3}, i)^t : \theta \in [0, 2\pi]\right\}.$$

Then the isometry group for the complexified norm is just $\{zI : |z| = 1\}$.

Remark 6.3. *In case $(\mathcal{A}, \|\cdot\|)$ is a (possibly infinite-dimensional) real normed* algebra *its Bochnak or maximal complexification coincides with a standard complexification procedure, described in §13 of Bonsall and Duncan's book* [2]*, which makes $\mathcal{A}$ into a normed algebra over the complex field. Namely this procedure builds a submultiplicative complexified norm $\|\cdot\|_{\mathbb{C}}$ on $\mathcal{A}_{\mathbb{C}} = \mathcal{A} + i\mathcal{A}$ with the help of Minkowski's functional p attached to a convex, balanced, absorbing, and radially bounded set*

$$V := \left\{\sum_{k=1}^{n}(\alpha_k u_k + i\beta_k u_k) : n \in \mathbb{N},\, \alpha_k, \beta_k \in \mathbb{R},\, u_k \in \mathcal{A},\, \|u_k\| < 1,\, \sum |\alpha_k + i\beta_k| \le 1\right\}.$$

As shown in [2, §13] *the Minkowski functional p is a norm and V is its open unit ball. Clearly, the closure of V coincides with the closed convex hull of $\Delta \otimes \mathcal{B}_{\mathcal{A}}$ ($\Delta \subseteq \mathbb{C}$ is the closed unit disc) which, by* [13, Proposition 2.2, p. 17] *is the closed unit ball of a projective tensor space $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{A}$, that is, of the Bochnak complexification of $\mathcal{A}$.*

Remark 6.4. *Assume $(\mathcal{X}, \|\cdot\|)$ is an infinite-dimensional real Banach space. By the classical Banach-Alaoglu and Krein-Milman Theorems, the dual norm's unit ball $\mathcal{B}^*$ is the weak-star closure of* $\mathrm{conv}(\mathrm{Ext}\,\mathcal{B}^*)$. *This allows one to derive the same result as in Theorem 3.10 in the case that the equivalence relation from Definition 3.4 partitions the set of extreme points* $\mathrm{Ext}(\mathcal{B}^*)$ *into finitely many classes.*

As for the Bochnak complexification the situation is much more involved because infinite-dimensional Banach spaces may have no extreme points (consider, e.g, $\mathbf{c_0}$*).*

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