Extremality of Bounds for Numerical Radii of Foguel Operators

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Abstract

For any operator T on ℓ^2, its associated Foguel operator F_T is $\begin{bmatrix} S^* & T \\ 0 & S \end{bmatrix}$ on $\ell^2 \oplus \ell^2$, where S is the (simple) unilateral shift. It is easily seen that the numerical radius $w(F_T)$ of F_T satisfies $1 \le w(F_T) \le 1 + (1/2)\|T\|$. In this paper, we study when such upper and lower bounds of $w(F_T)$ are attained. For the upper bound, we show that $w(F_T) = 1 + (1/2)\|T\|$ if and only if $w(S + T^*S^*T) = 1 + \|T\|^2$. When T is a diagonal operator with nonnegative diagonals, we obtain, among other results, that $w(F_T) = 1 + (1/2)\|T\|$ if and only if $w(ST) = \|T\|$. As for the lower bound, it is shown that any diagonal T with $w(F_T) = 1$ is compact. Examples of various T's are given to illustrate such attainments of $w(F_T)$.

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1. Introduction

A *Foguel operator* F_T is one of the form $\begin{bmatrix} S^* & T \\ 0 & S \end{bmatrix}$, where T is some operator on ℓ^2 and S is the unilateral shift $S(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots)$ on ℓ^2. Such operators were first considered by Foguel [2] as an example of a power-bounded operator not similar to a contraction (cf. also [5]). The *numerical range* $W(A)$ of a (bounded linear) operator A on a complex Hilbert space H is the subset $\{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$ of the complex plane, where $\langle \cdot, \cdot \rangle$ is the inner product in H and $\| \cdot \|$ its associated norm, and the *numerical radius* $w(A)$ is $\sup\{|z| : z \in W(A)\}$. After some preliminary results below, including the inequalities $1 \le w(F_T) \le 1 + (1/2)\|T\|$ for $w(F_T)$, we consider in subsequent sections when such upper and lower bounds are attained. We start with the upper bound in Section 2. It is shown that $w(F_T) = 1 + (1/2)\|T\|$ if and only if $w(S + T^*S^*T) = 1 + \|T\|^2$ (Theorem 2.1). For a diagonal operator $T = \operatorname{diag}(a_1, a_2, \ldots)$, the attainment for the upper

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bound can be expressed in terms of the diagonal entries of T. One such condition is the existence, for each $n \geq 1$, of positive integers n_j, $j \geq 1$, with $1 \leq n_1 < n_2 < \cdots$ such that $\lim_j T_{n,n_j} = \lambda_1\|T\|\mathrm{diag}\,(1, \lambda_2, \ldots, \lambda_2^n)$ for some λ_1 and λ_2 satisfying $|\lambda_1| = |\lambda_2| = 1$, where $T_{n,n_j} = \mathrm{diag}\,(a_{n_j}, a_{n_j+1}, \ldots, a_{n_j+n})$ for each j (Theorem 2.6). A necessary condition for $w(F_T) = 1 + (1/2)\|T\|$ is the normaloidity of the unilateral weighted shift ST, that is, ST satisfies $w(ST) = \|ST\|$ (Theorem 2.4 (b)). If $a_n \geq 0$ for all n, then, the condition $w(ST) = \|ST\|$ is also sufficient, and is equivalent to several other numerical radius and norm equality conditions (Theorem 2.12). In Section 3, we move to consider the attainment of the lower bound for $w(F_T)$. For a diagonal T, we show that the condition $w(F_T) = 1$ implies that T is compact (Theorem 3.4). In particular, if $T = \mathrm{diag}\,(1, a, a^2, \ldots)$ with $|a| < 1$, then $w(F_T) = 1$ is equivalent to $a = 0$ (Proposition 3.6).

For an operator A, $\sigma(A)$ and $\rho(A)$ denote its spectrum and spectral radius, and $\mathrm{Re}\,A$ and $\mathrm{Im}\,A$ its *real part* $(A + A^*)/2$ and *imaginary part* $(A - A^*)/(2i)$, respectively. The identity operator (resp., zero operator) on a space is denoted by I (resp., 0). If the space is identified as $\mathbb{C}^n$, then they are denoted by I_n and 0_n, respectively. An operator A is *positive semidefinite*, denoted by $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for all vectors x. A real matrix $A = [a_{ij}]_{i,j=1}^n$, $1 \leq n \leq \infty$, is *nonnegative*, denoted by $A \succcurlyeq 0$, if $a_{ij} \geq 0$ for all i and j. For two real matrices A and B of the same size, $A \preccurlyeq B$ means that $B - A \succcurlyeq 0$. For any m-by-n (complex) matrix $A = [a_{ij}]$, $|A|$ denotes the nonnegative matrix $[|a_{ij}|]$. We use S_n to denote the n-by-n matrix

$$\begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

and $\mathbb{D}$ the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$. For any real t, $\lfloor t \rfloor$ is the largest integer smaller than or equal to t.

Properties of the numerical range and numerical radius can be found in [8]. For properties of operators and finite matrices in general, consult [6] and [7], respectively.

To conclude this section, we give some basic properties of nonnegative matrices and Foguel operators for easier later reference.

Proposition 1.1. *Let A be an n-by-n matrix* $(1 \leq n \leq \infty)$.

(a) *If B is a real matrix of the same size as A and $|A| \preccurlyeq B$, then $w(A) \leq w(B)$.*

(b) *If $A \succcurlyeq 0$, then $w(A) = w(\mathrm{Re}\,A)$ and $w(A) \in \overline{W(A)}$.*

Proof. (a) If x is any unit vector, then so is $|x|$. We infer from

$$|\langle Ax, x\rangle| \leq \langle |A||x|, |x|\rangle \leq \langle B|x|, |x|\rangle \leq w(B)$$

that $w(A) \leq w(B)$.

(b) From $A \succcurlyeq 0$, we have $|\mathrm{Re}\,(\lambda A)| \preccurlyeq \mathrm{Re}\, A$ for any λ, $|\lambda| = 1$. Hence $w\big(\mathrm{Re}\,(\lambda A)\big) \leq w(\mathrm{Re}\, A)$ by (a). It follows that $w(A) = \max\{w\big(\mathrm{Re}\,(\lambda A)\big) : |\lambda| = 1\} \leq w(\mathrm{Re}\, A)$. On the other hand, the inequality $w(\mathrm{Re}\, A) \leq \big(w(A) + w(A^*)\big)/2 = w(A)$ also holds. These together prove that $w(A) = w(\mathrm{Re}\, A)$.

To show that $w(A) \in \overline{W(A)}$, let $\{x_k\}_{k=1}^{\infty}$ be a sequence of unit vectors such that $\lim_k |\langle Ax_k, x_k\rangle| = w(A)$. Passing to a subsequence, we may assume that $\langle A|x_k|, |x_k|\rangle$ converges, say, to a. From $|\langle Ax_k, x_k\rangle| \leq \langle A|x_k|, |x_k|\rangle$ for all k, we obtain $w(A) \leq a$. Since a is in $\overline{W(A)}$, we also have $a \leq w(A)$. Hence $w(A) = a$ is in $\overline{W(A)}$. ■

Proposition 1.2. *Let T be an operator on ℓ^2. Then* (a) $1 \leq w(F_T^n) \leq 1 + (n/2)\|T\|$ *for* $n \geq 1$, *and* (b) $\sigma(F_T) = \overline{\mathbb{D}}$.

Proof. (a) As

$$F_T^n = \begin{bmatrix} S^{*n} & \sum_{j=0}^{n-1} S^{*j} T S^{n-1-j} \\ 0 & S^n \end{bmatrix},$$

we obtain $W(F_T^n) \supseteq W(S^{*n}) = W(S^*) = \mathbb{D}$ by the fact that S^{*n} is unitarily similar to S^* together with [8, Lemma 1.4.2]. Thus $w(F_T^n) \geq 1$. On the other hand, we also have

$$\begin{aligned} w(F_T^n) &\leq w\Big(\begin{bmatrix} S^{*n} & 0 \\ 0 & S^n \end{bmatrix}\Big) + w\Big(\begin{bmatrix} 0 & \sum_{j=0}^{n-1} S^{*j} T S^{n-1-j} \\ 0 & 0 \end{bmatrix}\Big) \\ &= 1 + \frac{1}{2}\|\sum_{j=0}^{n-1} S^{*j} T S^{n-1-j}\| \leq 1 + \frac{1}{2}\sum_{j=0}^{n-1} \|S^{*j} T S^{n-1-j}\| \\ &\leq 1 + \frac{1}{2}\sum_{j=0}^{n-1} \|T\| = 1 + \frac{n}{2}\|T\|, \end{aligned}$$

where we used the fact that $w\Big(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\Big) = \|A\|/2$ (cf. [8, Corollary 2.1.3 (a)]).

(b) To prove $\sigma(F_T) = \overline{\mathbb{D}}$, we deduce from above that

$$\rho(F_T) = \lim_{n\to\infty} w(F_T^n)^{1/n} \leq \lim_{n\to\infty} (1 + \frac{n}{2}\|T\|)^{1/n} = 1,$$

where the first equality is by [8, Proposition 1.5.1 (g)]. Thus $\sigma(F_T) \subseteq \overline{\mathbb{D}}$. For the converse containment, let z be a point not in $\sigma(F_T)$. Then $F_T - zI$ is invertible. If $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is its inverse, then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} S^* - zI & T \\ 0 & S - zI \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

from which follows $A(S^* - zI) = I$. This shows that $S^* - zI$ is left invertible. Thus z is not in $\overline{\mathbb{D}}$, the left spectrum of S^* (cf. [6, Solution 82]). Therefore, $\sigma(F_T) \supseteq \overline{\mathbb{D}}$. Our assertion follows. ■

The bounds for $w(F_T^n)$ in the preceding proposition are due to Kittaneh by private communication.

2. Upper bound of $w(F_T)$

As seen from Proposition 1.2 (a), we have $1 \le w(F_T) \le 1 + (1/2)\|T\|$ for any operator T on ℓ^2. The next theorem gives some general conditions for the attainment of this upper bound of $w(F_T)$.

Theorem 2.1. *The following conditions are equivalent for any operator T on ℓ^2:*

(a) $w(F_T) = 1 + (1/2)\|T\|$,

(b) *there is a sequence of unit vectors $\{y_n\}_{n=1}^\infty$ in ℓ^2 and a complex number λ, $|\lambda| = 1$, such that* $\lim_n\langle(\operatorname{Re}(\overline{\lambda}S))y_n, y_n\rangle = 1$ *and* $\lim_n\langle(\operatorname{Re}(\lambda S))Ty_n, Ty_n\rangle = \|T\|^2$,

(c) $w(S + T^*S^*T) = 1 + \|T\|^2$.

Proof. (a)$\Rightarrow$(b). Note that (a) implies that there are sequences of unit vectors $\{x_n\}$ and $\{y_n\}$ in ℓ^2, a sequence $\{t_n\}$ in $[0, 2\pi]$, and a sequence $\{\lambda_n\}$ with $|\lambda_n| = 1$ such that $z_n = \big((\cos t_n)x_n, (\sin t_n)y_n\big)$ in $\ell^2 \oplus \ell^2$ satisfies $\lim_n \overline{\lambda}_n\langle F_T z_n, z_n\rangle = 1 + (1/2)\|T\|$. Passing to subsequences, we may assume that $\{\langle S^*x_n, x_n\rangle\}$, $\{\langle Sy_n, y_n\rangle\}$, $\{\langle Ty_n, x_n\rangle\}$, $\{t_n\}$, and $\{\lambda_n\}$ all converge. Let $\lim_n t_n = t$ in $[0, 2\pi]$ and $\lim_n \lambda_n = \lambda$. We may further assume that $z_n = \big((\cos t)x_n, (\sin t)y_n\big)$ for all n and $\lim_n\langle F_T z_n, z_n\rangle = \lambda\big(1 + (1/2)\|T\|\big)$. If $\lim_n\langle S^*x_n, x_n\rangle = a$, $\lim_n\langle Sy_n, y_n\rangle = b$, and $\lim_n(\cos t \cdot \sin t)\langle Ty_n, x_n\rangle = c$, then

$$\lim_n\langle F_T z_n, z_n\rangle = (\cos^2 t)a + (\sin^2 t)b + c = \lambda(1 + \frac{1}{2}\|T\|). \tag{1}$$

It is easy to see that $|a|, |b| \le 1$ and hence $|(\cos^2 t)a + (\sin^2 t)b| \le 1$. Similarly, we have $|c| \le \|T\|/2$. We deduce from (1) that the latter two inequalities must actually be equalities:

$$|(\cos^2 t)a + (\sin^2 t)b| = 1 \quad \text{and} \quad |c| = \frac{1}{2}\|T\|. \tag{2}$$

From the first one, we obtain $|a| = |b| = 1$. As $(\cos^2 t)a + (\sin^2 t)b$ is a convex combination of a and b, the equalities $|(\cos^2 t)a + (\sin^2 t)b| = |a| = |b| = 1$ yield that $a = b = (\cos^2 t)a + (\sin^2 t)b$. From (1), we obtain

$$a + c = \lambda(1 + \frac{1}{2}\|T\|). \tag{3}$$

Hence

$$1 + \frac{1}{2}\|T\| = |a + c| \le |a| + |c| = 1 + |c| = 1 + \frac{1}{2}\|T\|$$

by (3) and (2). This gives $|a+c| = |a|+|c|$. Thus $c = sa$ for some $s \geq 0$ or $|c| = s|a| = s$. We infer from (3) and (2) that

$$\lambda(1+\frac{1}{2}\|T\|) = a+c = a+sa = a+|c|a = a(1+|c|) = a(1+\frac{1}{2}\|T\|).$$

Therefore, we obtain $a = \lambda$ and $c = |c|a = |c|\lambda = \lambda\|T\|/2$. These yield $\lim_n\langle S^*x_n, x_n\rangle = \lim_n\langle Sy_n, y_n\rangle = \lambda$ and $\lim_n(\cos t\cdot\sin t)\langle Ty_n, x_n\rangle = \lambda\|T\|/2$, from which we deduce that $\lim_n\langle\big(\mathrm{Re}\,(\lambda S)\big)x_n, x_n\rangle = \lim_n\langle\big(\mathrm{Re}\,(\overline{\lambda}S)\big)y_n, y_n\rangle = 1$ and $\cos t\sin t = 1/2$. Hence $\lim_n\langle Ty_n, x_n\rangle = \lambda\|T\|$. As

$$\begin{aligned}
&0 \leq \lim_n \big\|Ty_n - \lambda\|T\|x_n\big\|^2 \\
=&\ \lim_n \big(\|Ty_n\|^2 - 2\mathrm{Re}\,(\overline{\lambda}\|T\|\langle Ty_n, x_n\rangle) + |\lambda|^2\|T\|^2\|x_n\|^2\big) \\
=&\ \lim_n(\|Ty_n\|^2 - \|T\|^2) \leq 0,
\end{aligned}$$

we have $\lim_n \|Ty_n - \lambda\|T\|x_n\| = 0$. Finally, replacing Ty_n by $\lambda\|T\|x_n$ in $\langle\big(\mathrm{Re}\,(\lambda S)\big)Ty_n, Ty_n\rangle$, taking the limit, and using $\lim_n\langle\big(\mathrm{Re}\,(\lambda S)\big)x_n, x_n\rangle = 1$, we conclude that $\lim_n\langle\big(\mathrm{Re}\,(\lambda S)\big)Ty_n, Ty_n\rangle = \|T\|^2$, completing the proof.

(b)⇒(c). Note that

$$w(S+T^*S^*T) \leq \|S+T^*S^*T\| \leq \|S\| + \|T^*S^*T\| \leq 1+\|T\|^2.$$

On the other hand, (b) implies that $\lim_n \big\langle\big(\mathrm{Re}\,(\overline{\lambda}(S+T^*S^*T))\big)y_n, y_n\big\rangle$ equals

$$\lim_n \big(\langle(\mathrm{Re}\,(\overline{\lambda}S))y_n, y_n\rangle + \langle T^*(\mathrm{Re}\,(\overline{\lambda}S^*))Ty_n, y_n\rangle\big) = \ 1+\|T\|^2.$$

Hence

$$1+\|T\|^2 \leq w\big(\mathrm{Re}\,(\overline{\lambda}(S+T^*S^*T))\big) \leq w\big(\overline{\lambda}(S+T^*S^*T)\big) = w(S+T^*S^*T).$$

Therefore, $w(S+T^*S^*T) = 1+\|T\|^2$ holds.

(c)⇒(a). From (c), we argue as in the proof of (a)⇒(b) to obtain a sequence of unit vectors $\{y_n\}$ and a complex number λ with $|\lambda| = 1$ such that $\lim_n\langle(S+T^*S^*T)y_n, y_n\rangle = \lambda(1+\|T\|^2)$. As before, this yields $\lim_n\langle Sy_n, y_n\rangle = \lambda$ and $\lim_n\langle S^*Ty_n, Ty_n\rangle = \lambda\|T\|^2$. Since $|\langle S^*Ty_n, Ty_n\rangle| \leq \|S^*Ty_n\|\|Ty_n\| \leq \|Ty_n\|^2 \leq \|T\|^2$ for all n, we have $\lim_n \|Ty_n\| = \|T\|$. Let $x_n = Ty_n/\|Ty_n\|$ in ℓ^2 and $z_n = (1/\sqrt{2})(x_n, y_n)$ in $\ell^2\oplus\ell^2$. Then $\|z_n\| = 1$ for all n and

$$\begin{aligned}
&\lim_n\langle F_{\lambda T}z_n, z_n\rangle = \frac{1}{2}\lim_n(\langle S^*x_n, x_n\rangle + \langle Sy_n, y_n\rangle + \lambda\langle Ty_n, x_n\rangle) \\
=&\ \frac{1}{2}\lim_n\big(\frac{1}{\|Ty_n\|^2}\langle S^*Ty_n, Ty_n\rangle + \langle Sy_n, y_n\rangle + \frac{\lambda}{\|Ty_n\|}\langle Ty_n, Ty_n\rangle\big) \\
=&\ \frac{1}{2}\big(\frac{1}{\|T\|^2}\lambda\|T\|^2 + \lambda + \frac{\lambda}{\|T\|}\|T\|^2\big) = \lambda(1+\frac{1}{2}\|T\|).
\end{aligned}$$

It follows that $1+(1/2)\|T\| = \lim_n |\langle F_{\lambda T} z_n, z_n\rangle| \le w(F_{\lambda T}) = w(F_T)$, where the last equality is a consequence of the unitary similarity of $F_{\lambda T}$ and F_T:

$$\begin{bmatrix} S^* & \lambda T \\ 0 & S \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \overline{\lambda} I \end{bmatrix} \begin{bmatrix} S^* & T \\ 0 & S \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \lambda I \end{bmatrix}.$$

Since $w(F_T) \le 1 + (1/2)\|T\|$ always holds, this proves (a). ■

The following examples are easy consequences of the preceding theorem. The first one appeared before in [3, Corollary 2.9].

Example 2.2. (a) If $T = S$, then $w(F_T) = 1 + (1/2)\|T\| = 3/2$ since

$$w(S + S^*S^*S) = w(S + S^*) = 2w(\operatorname{Re} S) = 2 = 1 + \|S\|^2.$$

(b) If $T = \operatorname{diag}(1, 0, 1, 0, \dots)$, then $w(F_T) < 1 + (1/2)\|T\|$ since $T^*S^*T = 0$ and hence $w(S + T^*S^*T) = w(S) = 1 < 1 + \|T\|^2$.

Corollary 2.3. *Let $\mathcal{S}$ be the set of all T's on ℓ^2 which satisfy $w(F_T) = 1 + (1/2)\|T\|$.*

(a) *For any nonzero complex number z, T is in $\mathcal{S}$ if and only if zT is.*
(b) *Let $A = \operatorname{diag}(1, a, a^2, \dots)$, where $|a| = 1$. Then T is in $\mathcal{S}$ if and only if A^*TA^* is.*

Proof. (a) is an easy consequence of Theorem 2.1 (b).

(b) Since A is unitary, $A^*S^*A = aS^*$, and $ASA^* = aS$, we have

$$(A^* \oplus A)F_T(A \oplus A^*) = \begin{bmatrix} aS^* & A^*TA^* \\ 0 & aS \end{bmatrix} = a \begin{bmatrix} S^* & \overline{a}A^*TA^* \\ 0 & S \end{bmatrix} = aF_{\overline{a}A^*TA^*}.$$

The assertion then follows from (a). ■

We now consider $w(F_T)$ for a diagonal operator T.

Theorem 2.4. *Let $T = \operatorname{diag}(a_1, a_2, \dots)$.*

(a) *$w(F_T) = 1 + (1/2)\|T\|$ if and only if $w(S + \lambda T^*ST) = 1 + \|T\|^2$ for some λ, $|\lambda| = 1$.*
(b) *If $w(F_T) = 1 + (1/2)\|T\|$, then $w(ST) = \|T\|$.*

Note that the converse of the implication in (b) is in general false. One example is $T = \operatorname{diag}(1, 1, -1, -1, 1, 1, -1, -1, \dots)$. Since ST is unitarily similar to S, we have $w(ST) = w(S) = 1 = \|T\|$, but $w(F_T) = \sqrt{5 + 2\sqrt{2}}/2 < 3/2$ by [3, Proposition 3.4].

Proof. [Proof of Theorem 2.4] (a) By Theorem 2.1, we need only prove

$$w(S + T^*S^*T) = w(S + \lambda T^*ST) \quad \text{for some } \lambda \ hboxwith\ |\lambda| = 1.$$

Indeed, we have $w(S+T^*S^*T) = \max\{w\big(\mathrm{Re}\,(\lambda(S+T^*S^*T))\big) : |\lambda|=1\}$ and

$$w\big(\mathrm{Re}\,(\lambda(S+T^*S^*T))\big) = w\big(\mathrm{Re}\,(\lambda S) + \mathrm{Re}\,(\overline{\lambda}T^*ST)\big) = w(S+\overline{\lambda}^2 T^*ST),$$

where the last equality follows from the fact that

$$S+\overline{\lambda}^2T^*ST = \begin{bmatrix} 0 & & & & \\ 1+\overline{\lambda}^2 a_1\overline{a}_2 & 0 & & & \\ & 1+\overline{\lambda}^2 a_2\overline{a}_3 & 0 & & \\ & & \ddots & \ddots & \end{bmatrix}$$

is a unilateral weighted shift whose numerical range is an (open or closed) circular disc centered at the origin. Thus

$$\begin{aligned} w(S+T^*S^*T) &= \max\{w(S+\overline{\lambda}^2T^*ST) : |\lambda|=1\} = \max\{w(S+\lambda T^*ST) : |\lambda|=1\} \\ &= w(S+\lambda T^*ST) \end{aligned}$$

for some λ, $|\lambda|=1$.

(b) If $w(F_T) = 1+(1/2)\|T\|$, then $w(S+T^*S^*T) = 1+\|T\|^2$ by Theorem 2.1. Hence

$$\begin{aligned} 1+\|T\|^2 &\le w(S)+w(T^*S^*T) \le 1+w(|T^*|S^*)\|T\| \\ &= 1+w(T^*S^*)\|T\| \le 1+\|T^*S^*\|\|T\| \le 1+\|T\|^2, \end{aligned}$$

where the second inequality follows from $|T^*S^*T| \preccurlyeq |T^*|S^*\|T\|$ (cf. Proposition 1.1 (a)), and the equality $w(|T^*|S^*) = w(T^*S^*)$ follows from the unitary similarity of $|T^*|S^*$ and T^*S^*. This yields equalities throughout and, in particular, $w(ST) = w(T^*S^*) = \|T\|$ holds. ■

The next two examples illustrate the usefulness of the preceding theorem. The first appeared before in [3, Corollary 3.6].

Example 2.5. (a) Let $T = \mathrm{diag}\,(1, a, a^2, \dots)$, where $|a|=1$. Then $w(F_T) = 1+(1/2)\|T\| = 3/2$ since $w(S+aT^*ST) = w(S+S) = 2 = 1+\|T\|^2$.

(b) Let $T = \mathrm{diag}\,(a_1, a_2, \dots)$ with $\lim_n a_n = a$ and $|a| = \|T\|$. Then $w(F_T) = 1+(1/2)\|T\|$ since, in this case, $S+T^*ST$ is a unilateral weighted shift with weights $\{1+a_n\overline{a}_{n+1}\}_{n=1}^{\infty}$ satisfying $\lim_n |1+a_n\overline{a}_{n+1}| = 1+|a|^2$ and hence $w(S+T^*ST) = 1+|a|^2 = 1+\|T\|^2$ by [8, Proposition 2.4.2].

For a diagonal T, the condition in Theorem 2.4 (a) involves the numerical radius of the unilateral weighted shift $S+\lambda T^*ST$. In the following, we express the condition for $w(F_T) = 1+(1/2)\|T\|$ in terms of the diagonals of T more explicitly.

Theorem 2.6. *Let* $T = \operatorname{diag}(a_1, a_2, \dots)$ *on* ℓ^2 *and* $T_{n,k} = \operatorname{diag}(a_k, a_{k+1}, \dots, a_{k+n})$ *on* $\mathbb{C}^{n+1}$ *for* $n, k \geq 1$. *Then* $w(F_T) = 1 + (1/2)\|T\|$ *if and only if for any* $n \geq 1$ *there are integers* $1 \leq n_1 < n_2 < \cdots$ *such that* $\lim_{j\to\infty} T_{n,n_j} = \lambda_1\|T\|\operatorname{diag}(1, \lambda_2, \dots, \lambda_2^n)$ *for some* λ_1 *and* λ_2 *with* $|\lambda_1| = |\lambda_2| = 1$. *Moreover, in this case,* λ_2 *can be chosen to satisfy* $w(S + \lambda_2 T^* S T) = 1 + \|T\|^2$.

The proof is facilitated by the next proposition on unilateral weighted shift.

Proposition 2.7. *Let*

$$A = \begin{bmatrix} 0 & & & \\ w_1 & 0 & & \\ & w_2 & 0 & \\ & & \ddots & \ddots \end{bmatrix} \quad \text{on } \ell^2$$

and

$$A_{n,k} = \begin{bmatrix} 0 & & & & \\ w_k & 0 & & & \\ & w_{k+1} & 0 & & \\ & & \ddots & \ddots & \\ & & & w_{k+n-1} & 0 \end{bmatrix} \quad \text{on } \mathbb{C}^{n+1} \text{ for } n, k \geq 1.$$

Then $\max\{w(S + \lambda A) : |\lambda| = 1\} = 1 + \|A\|$ *if and only if for any* $n \geq 1$ *there are integers* $1 \leq n_1 < n_2 < \cdots$ *such that* $\lim_{j\to\infty} A_{n,n_j} = \overline{\lambda}_0\|A\|S_{n+1}$ *for some* λ_0, $|\lambda_0| = 1$. *Moreover,* λ_0 *may be chosen to satisfy* $w(S + \lambda_0 A) = 1 + \|A\|$.

An operator A is *normaloid* if it satisfies $w(A) = \|A\|$. For a unilateral weighted shift, normaloidity can be characterized in terms of its weights (cf. [9, Theorem 4.6] or [8, Problem 3.4]).

Lemma 2.8. *A unilateral weighted shift with weights* $\{w_n\}_{n=1}^{\infty}$ *is normaloid if and only if* $\sup_{n\geq 1}|w_n| = \lim_{j\to\infty}\sup_{k\geq 1}|w_k w_{k+1}\cdots w_{k+j-1}|^{1/j}$.

Proof. [Proof of Proposition 2.7] First assume that $\max\{w(S + \lambda A) : |\lambda| = 1\} = 1 + \|A\|$. Let λ_0, $|\lambda_0| = 1$, be such that $w(S + \lambda_0 A) = 1 + \|A\|$. Then $\|S + \lambda_0 A\| \leq 1 + \|A\| = w(S + \lambda_0 A)$, which implies that $w(S + \lambda_0 A) = \|S + \lambda_0 A\|$ or $S + \lambda_0 A$ is normaloid. Let $u_n = 1 + \lambda_0 w_n$ for $n \geq 1$. As $S + \lambda_0 A$ is a unilateral weighted shift with weights $\{u_n\}_{n=1}^{\infty}$, Lemma 2.8 yields that $\lim_{j\to\infty}\sup_{k\geq 1}|u_k u_{k+1}\cdots u_{k+j-1}|^{1/j} = \|S + \lambda_0 A\| = 1 + \|A\|$. We now show that for any $n \geq 1$ there are integers $1 \leq n_1 < n_2 < \cdots$ such that $\lim_j u_{n_j+s} = 1 + \|A\|$ for all s, $0 \leq s \leq n-1$. This is done by first checking that $\lim_j |u_{n_j+s}| = 1 + \|A\|$ for all s. Indeed, assume otherwise that, for some $n \geq 1$, we have

$\limsup_{k\to\infty} \min\{|u_k|, \ldots, |u_{k+n-1}|\} < 1 + \|A\|$. Then, under $A \neq 0$, there is an $N \geq 1$ and an ε, $0 < \varepsilon < \|A\|$, such that $\min\{|u_k|, \ldots, |u_{k+n-1}|\} \leq 1 + \|A\| - \varepsilon$ for all $k \geq N$. For any $j \geq n + N$, let $\alpha_k = \lfloor (k + j - N)/n \rfloor$ if $1 \leq k < N$, and $\lfloor (j - N)/n \rfloor$ if $k \geq N$. We have

$$
\begin{aligned}
|u_k u_{k+1} \cdots u_{k+j-1}| &= \begin{cases} \left(\prod_{l=k}^{N-1} |u_l|\right)\left(\prod_{m=0}^{\alpha_k - 1} \left(\prod_{l=N+mn}^{N+(m+1)n-1} |u_l|\right)\right)\left(\prod_{l=N+\alpha_k n}^{k+j-1} |u_l|\right) & \text{if } 1 \leq k < N, \\ \left(\prod_{m=0}^{\alpha_k - 1} \left(\prod_{l=k+mn}^{k+(m+1)n-1} |u_l|\right)\right)\left(\prod_{l=k+(m+1)n}^{k+j-1} |u_l|\right) & \text{if } k \geq N \end{cases} \\
&\leq (1 + \|A\| - \varepsilon)^{\alpha_k} (1 + \|A\|)^{j - \alpha_k} \\
&\leq (1 + \|A\| - \varepsilon)^{\lfloor (j-N)/n \rfloor} (1 + \|A\|)^{j - \lfloor (j-N)/n \rfloor},
\end{aligned}
$$

where the first inequality is because at least one of the $|u_l|$'s in each of the α_k many products $\prod_{l=N+mn}^{N+(m+1)n-1} |u_l|$ or $\prod_{l=k+mn}^{k+(m+1)n-1} |u_l|$ is at most $1 + \|A\| - \varepsilon$, and the second inequality results from $\big((1 + \|A\|)/(1 + \|A\| - \varepsilon)\big)^{\alpha_k - \lfloor (j-N)/n \rfloor} \geq 1$ since $\alpha_k \geq \lfloor (j - N)/n \rfloor$ and $(1 + \|A\|)/(1 + \|A\| - \varepsilon) > 1$. As $j - N = \lfloor (j - N)/n \rfloor n + r$ for some r, $0 \leq r < n$, we obtain $\lim_j \lfloor (j - N)/n \rfloor / j = (1/n) \lim_j \big(((j - N)/j) - (r/j)\big) = 1/n$. Thus, from the above inequalities, we further deduce that

$$
\begin{aligned}
&\lim_{j\to\infty} \sup_{k\geq 1} |u_k u_{k+1} \cdots u_{k+j-1}|^{1/j} \leq \lim_{j\to\infty} (1 + \|A\| - \varepsilon)^{\lfloor (j-N)/n \rfloor / j} (1 + \|A\|)^{(j - \lfloor (j-N)/n \rfloor)/j} \\
&\leq (1 + \|A\| - \varepsilon)^{1/n} (1 + \|A\|)^{1-(1/n)} < (1 + \|A\|)^{1/n} (1 + \|A\|)^{1-(1/n)} = 1 + \|A\|.
\end{aligned}
$$

This contradicts our previous condition for the normaloidity of $S + \lambda_0 A$. Thus we have proved $\lim_j |u_{n_j+s}| = 1 + \|A\|$ for all s, $0 \leq s \leq n - 1$.

The next step is to show that $\lim_j u_{n_j+s} = 1 + \|A\|$ for all s. If $s = 0$, then, from $\lim_j |u_{n_j}| = 1 + \|A\|$ and $|u_{n_j}| \leq 1 + |w_{n_j}| \leq 1 + \|A\|$, we also have $\lim_j |w_{n_j}| = \|A\|$. On the other hand, we deduce from

$$
(1 + \|A\|)^2 = \lim_j |u_{n_j}|^2 = \lim_j \left(1 + |w_{n_j}|^2 + 2\mathrm{Re}\,(\lambda_0 w_{n_j})\right) = 1 + \|A\|^2 + 2 \lim_j \mathrm{Re}\,(\lambda_0 w_{n_j})
$$

that $\lim_j \mathrm{Re}\,(\lambda_0 w_{n_j}) = \|A\|$. Together with $\lim_j |\lambda_0 w_{n_j}| = \|A\|$, this yields $\lim_j \mathrm{Im}\,(\lambda_0 w_{n_j}) = 0$. Hence $\lim_j \lambda_0 w_{n_j} = \|A\|$. Similarly, we can prove $\lim_j \lambda_0 w_{n_j+s} = \|A\|$ for all s, $1 \leq s \leq n - 1$. Thus $\lim_j A_{n,n_j} = \overline{\lambda}_0 \|A\| S_{n+1}$ as required.

To prove the converse, assume that, for any $n \geq 1$, there is a sequence $\{n_j\}_{j=1}^{\infty}$ such that $\lim_j A_{n,n_j} = \overline{\lambda}_0 \|A\| S_{n+1}$ for some λ_0, $|\lambda_0| = 1$. Since $w(S_{n+1} + \lambda_0 A_{n,n_j}) \leq w(S + \lambda_0 A)$ for any n, letting j approach infinity, we obtain

$$
(1 + \|A\|) w(S_{n+1}) = w(S_{n+1} + \|A\| S_{n+1}) \leq w(S + \lambda_0 A) \leq \max\{w(S + \lambda A) : |\lambda| = 1\}.
$$

It follows from $\lim_j w(S_{n+1}) = \lim_j \big(\cos(\pi/(n+2))\big) = 1$ (cf. [8, Lemma 2.4.1 (a)]) that $1+\|A\| \le \max\{w(S+\lambda A) : |\lambda| = 1\}$. On the other hand, we also have $w(S+\lambda A) \le w(S) + w(A) \le 1 + \|A\|$ for any λ, $|\lambda| = 1$. This proves $\max\{w(S+\lambda A) : |\lambda| = 1\} = 1 + \|A\|$. ■

Proof. [Proof of Theorem 2.6] We need only consider $T \neq 0$. Assume first that $w(F_T) = 1 + (1/2)\|T\|$. From Theorem 2.4 (a), we have $w(S + \lambda_2 T^*ST) = 1 + \|T\|^2$ for some λ_2, $|\lambda_2| = 1$. If $A = T^*ST$, then A is a unilateral weighted shift with weights $\{a_n\overline{a}_{n+1}\}_{n=1}^{\infty}$ and $w(S + \lambda_2 A) = 1 + \|A\|$. Thus, by Proposition 2.7, for any $n \ge 1$ there is a sequence $\{n_j\}_{j=1}^{\infty}$, $1 \le n_1 < n_2 < \cdots$, such that $\lim_j A_{n+1,n_j} = \overline{\lambda}_2\|A\|S_{n+2}$. This is the same as $\lim_j a_{n_j+m}\overline{a}_{n_j+m+1} = \overline{\lambda}_2\|T\|^2$ for all m, $0 \le m \le n$. Passing to subsequences, we may assume that both $\{a_{n_j+m}\}_{j=1}^{\infty}$ and $\{a_{n_j+m+1}\}_{j=1}^{\infty}$ converge for each fixed m. As $|a_k| \le \|T\|$ for all k, we infer from $\lim_j |a_{n_j+m}||a_{n_j+m+1}| = \|T\|^2$ that $\lim_j |a_{n_j+m}| = \|T\|$. Moreover, if $a_k = \lambda_k|a_k|$, where $|\lambda_k| = 1$, for $k \ge 1$, then $\lim_j \lambda_{n_j+m}\overline{\lambda}_{n_j+m+1} = \overline{\lambda}_2$ for each m. Thus

$$\lim_j \lambda_{n_j}\overline{\lambda}_{n_j+m} = \lim_j (\lambda_{n_j}\overline{\lambda}_{n_j+1})(\lambda_{n_j+1}\overline{\lambda}_{n_j+2})\cdots(\lambda_{n_j+m-1}\overline{\lambda}_{n_j+m}) = \overline{\lambda}_2^{\,m}$$

for $0 \le m \le n$. Again, passing to a subsequence, we may assume that $\{\lambda_{n_j}\}_{j=1}^{\infty}$ converges, say, to λ_1 with $|\lambda_1| = 1$. Then we obtain $\lim_j \overline{\lambda}_{n_j+m} = \overline{\lambda}_1\overline{\lambda}_2^{\,m}$ or $\lim_j \lambda_{n_j+m} = \lambda_1\lambda_2^m$. Together with $\lim_j |a_{n_j+m}| = \|T\|$, this yields $\lim_j a_{n_j+m} = \lambda_1\lambda_2^m\|T\|$ for all m, $0 \le m \le n$. In other words, we have $\lim_j T_{n,n_j} = \lambda_1\|T\|\mathrm{diag}\,(1, \lambda_2, \ldots, \lambda_2^n)$ as required.

To prove the converse, for any $n \ge 1$, let $\{n_j\}_{j=1}^{\infty}$ be such that

$$\lim_j T_{n,n_j} = \lambda_1\|T\|\mathrm{diag}\,(1, \lambda_2, \ldots, \lambda_2^n) \text{ for some } \lambda_1 \text{ and } \lambda_2,\ |\lambda_1| = |\lambda_2| = 1.$$

Then we have $\lim_j a_{n_j+m}\overline{a}_{n_j+m+1} = \overline{\lambda}_2\|T\|^2$ for $0 \le m \le n-1$. If

$$A = \begin{bmatrix} 0 & & & \\ a_1\overline{a}_2 & 0 & & \\ & a_2\overline{a}_3 & 0 & \\ & & \ddots & \ddots \end{bmatrix} \text{ on } \ell^2 \quad\text{and}\quad A_{n,k} = \begin{bmatrix} 0 & & & \\ a_k\overline{a}_{k+1} & 0 & & \\ & \ddots & \ddots & \\ & & a_{k+n-1}\overline{a}_{k+n} & 0 \end{bmatrix} \text{ on } \mathbb{C}^{n+1}$$

for $n, k \ge 1$, then the above limits can be expressed as $\lim_j A_{n,n_j} = \overline{\lambda}_2\|T\|^2 S_{n+1}$. It follows that $\lim_j \|A_{n,n_j}\| = \|T\|^2$. On the other hand, we also have $\|A_{n,n_j}\| \le \|A\| \le \|T\|^2$ for all j. Thus $\|A\| = \|T\|^2$ and $\lim_j A_{n,n_j} = \overline{\lambda}_2\|A\|S_{n+1}$. We obtain from Proposition 2.7 that $\max\{w(S+\lambda A) : |\lambda| = 1\} = 1 + \|A\| = 1 + \|T\|^2$. As $A = T^*ST$, the assertion $w(F_T) = 1 + (1/2)\|T\|$ then follows from Theorem 2.4 (a). ■

The next two propositions are consequences of Theorem 2.6.

Proposition 2.9. *Let* $T = \mathrm{diag}\,(a_1, a_2, \ldots)$ *and* $T(m) = \mathrm{diag}\,(a_m, a_{m+1}, \ldots)$ *for* $m \ge 1$. *Then the following conditions are equivalent:*

(a) $w(F_T) = 1 + (1/2)\|T\|$.

(b) $w(F_{T(m_0)}) = 1 + (1/2)\|T\|$ *for some* $m_0 \geq 1$.

(c) $w(F_{T(m)}) = 1 + (1/2)\|T\|$ *for all* $m \geq 1$.

Proof. We need only prove (b)⇒(c). Let $T_{n,k}(m) = \mathrm{diag}\,(a_{m+k-1}, \dots, a_{m+k+n-1})$ on $\mathbb{C}^{n+1}$ for $n, k, m \geq 1$. Assuming $w(F_{T(m_0)}) = 1 + (1/2)\|T\|$, we have

$$1 + \frac{1}{2}\|T\| = w(F_{T(m_0)}) \leq 1 + \frac{1}{2}\|T(m_0)\| \leq 1 + \frac{1}{2}\|T\|.$$

Thus $\|T(m_0)\| = \|T\|$. By Theorem 2.6, for any $n \geq 1$, there is a sequence $\{n_j\}_{j=1}^{\infty}$ such that $\lim_j T_{n,n_j}(m_0) = \lambda_1\|T\|\mathrm{diag}\,(1, \lambda_2, \dots, \lambda_2^n)$ with $|\lambda_1| = |\lambda_2| = 1$. Fixing any $m \geq 1$, let j_0 be such that $n_j \geq m$ for all $j \geq j_0$ and let $n_j' = n_j + m_0 - m$ for $j \geq j_0$. Then

$$\lim_j T_{n,n_j'}(m) = \lim_j T_{n,n_j}(m_0) = \lambda_1\|T\|\mathrm{diag}\,(1, \lambda_2, \dots, \lambda_2^n). \tag{4}$$

This yields $\lim_j \|T_{n,n_j'}(m)\| = \|T\|$. Since $\|T_{n,n_j'}(m)\| \leq \|T(m)\|$ for all j and m, we obtain $\|T\| \leq \|T(m)\|$. Hence $\|T\| = \|T(m)\|$ for all m. Therefore, (c) follows from (4) via Theorem 2.6. ■

The *period* $p\,(\geq 1)$ of a periodic sequence $\{a_n\}_{n=1}^{\infty}$ is the smallest integer for which $a_{n+p} = a_n$ for all $n \geq 1$.

Proposition 2.10. *Let* $T = \mathrm{diag}\,(a_1, a_2, \dots)$*, where* a_n*'s are periodic with period* $p\,(\geq 1)$*. Then* $w(F_T) = 1 + (1/2)\|T\|$ *if and only if* $a_n = \lambda_1\lambda_2^n\|T\|$ *for* $n \geq 1$*, where* $|\lambda_1| = 1$ *and* $\lambda_2^p = 1$*.*

Proof. For any $n, k \geq 1$, let $T_{n,k} = \mathrm{diag}\,(a_k, a_{k+1}, \dots, a_{k+n})$. If $w(F_T) = 1 + (1/2)\|T\|$, then, by Theorem 2.6, there is a sequence $\{p_j\}_{j=1}^{\infty}$ such that

$$\lim_j T_{p,p_j} = \lambda_1'\|T\|\mathrm{diag}\,(1, \lambda_2, \dots, \lambda_2^p) \text{ for some } \lambda_1' \text{ and } \lambda_2 \text{ with } |\lambda_1'| = |\lambda_2| = 1.$$

On the other hand, for the periodic a_n's, we also have

$$T_{p,\,kp+l} = T_{p,l} \quad \text{for } k \geq 1 \text{ and } 1 \leq l \leq p. \tag{5}$$

By the pigeonhole principle, there is a q, $1 \leq q \leq p$, and a subsequence of $\{p_j\}$ whose elements are all of the form $kp + q$ $(k \geq 1)$. Passing to this subsequence, we may assume that the p_j's are themselves of this form. Thus, from (5), we have $T_{p,p_j} = T_{p,q}$ for all j. This yields that

$$\mathrm{diag}\,(a_q, a_{q+1}, \dots, a_{q+p}) = T_{p,q} = T_{p,p_j} = \lambda_1'\|T\|\mathrm{diag}\,(1, \lambda_2, \dots, \lambda_2^p).$$

Hence $a_{q+m} = \lambda_1'\|T\|\lambda_2^m = a_q\lambda_2^m$ for $0 \leq m \leq p$. In particular, we have $a_q = a_{q+p} = a_q\lambda_2^p$. If $a_q = 0$, then all the a_n's are zero or $T = 0$. Otherwise, we have $\lambda_2^p = 1$ and $a_{q+m} = \lambda_1'\lambda_2^m\|T\|$ for $0 \leq m \leq p$. Let $\lambda_1 = \lambda_1'\lambda_2^{-q}$. If $1 \leq n \leq q-1$, then $0 \leq p - q + n \leq p$ and hence

$$a_n = a_{q+(p-q+n)} = \lambda_1'\lambda_2^{p-q+n}\|T\| = (\lambda_1'\lambda_2^{-q})\lambda_2^n\|T\| = \lambda_1\lambda_2^n\|T\|.$$

On the other hand, if $n \geq q$, say, $n = (k-1)p + q + m$ for some $k \geq 1$ and some m, $0 \leq m \leq p-1$, then

$$a_n = a_{q+m} = \lambda_1' \lambda_2^m \|T\| = (\lambda_1' \lambda_2^{-q}) \lambda_2^{q+m} \|T\| = \lambda_1 \lambda_2^n \|T\|.$$

These prove our assertion on the a_n's.

Conversely, if the a_n's are of the asserted form, then $a_{n+1} = \lambda_2 a_n$ for all n. Hence $T = a_1 \operatorname{diag}(1, \lambda_2, \lambda_2^2, \dots)$. Then $w(F_T) = 1 + (1/2)\|T\|$ by Example 2.5 (a) and Corollary 2.3 (a). ■

The following are examples for Proposition 2.10.

Example 2.11. (a) If $T = aI$ on ℓ^2, then $p = 1$, $\lambda_1 = a/|a|$ (for $a \neq 0$), and $\lambda_2 = 1$ yield the required expression for the diagonals of T, which implies $w(F_T) = 1 + (1/2)\|T\|$ by Proposition 2.10.

(b) If $T = \operatorname{diag}(1, -1, 1, -1, \dots)$, then $p = 2$ and $\lambda_1 = \lambda_2 = -1$ yield the required expression, which results in $w(F_T) = 1 + (1/2)\|T\|$.

(c) If $T = \operatorname{diag}(1, 0, 1, 0, \dots)$, then $w(F_T) < 1 + (1/2)\|T\|$ since no expression for the diagonals of T as in Proposition 2.10 exists.

(d) If $T = \operatorname{diag}(1, 1, -1, -1, 1, 1, -1, -1, \dots)$, then $w(F_T) < 1 + (1/2)\|T\|$ by Proposition 2.10.

We remark that the example in (a) above appeared before in [3, Theorem 3.5 (a)], (c) in Example 2.2 (b), and the exact value of $w(F_T)$ for T in (d) has been computed in [3, Proposition 3.4].

In the rest of this section, we consider $w(F_T)$ for a diagonal T with nonnegative diagonals. The next theorem gives more conditions for $w(F_T) = 1 + (1/2)\|T\|$ to hold.

Theorem 2.12. *Let $T = \operatorname{diag}(a_1, a_2, \dots)$ with $a_n \geq 0$ for all n. Then the following conditions are equivalent:*

(a) $w(F_T) = 1 + (1/2)\|T\|$,

(b) $\|S + S^* + T\| = 2 + \|T\|$,

(c) $w(ST + S^*T) = 2\|T\|$,

(d) $w(ST + TS) = 2\|T\|$,

(e) $w(S + TST) = w(S + TS^*T) = 1 + \|T\|^2$,

(f) $w(TST) = \|T\|^2$,

(g) $w(ST) = \|T\|$.

Proof. The equivalence of (a) and (b) follows from [3, Proposition 3.2 (c)]. To prove (b)⇔(c), we use the fact that, for any two operators A and B on the same space, $\|A+B\| = \|A\| + \|B\|$ if and only if $\|A\|\|B\|$ is in $\overline{W(A^*B)}$ (cf. [1, Theorem 2.1]). Indeed, the equality in (b) is the same as $\|(\mathrm{Re}\, S) + (1/2)T\| = \|\mathrm{Re}\, S\| + \|(1/2)T\|$, which is equivalent to $\|T\|$ being in $\overline{W\big((\mathrm{Re}\, S)T\big)}$ from above or to $2\|T\|$ in $\overline{W(ST + S^*T)}$. Thus $w(ST+S^*T) \geq 2\|T\|$. Together with $w(ST+S^*T) \leq \|ST+S^*T\| \leq 2\|T\|$, this yields $w(ST+S^*T) = 2\|T\|$, that is, (c) holds. Conversely, if $w(ST + S^*T) = 2\|T\|$, then, from $ST + S^*T \succcurlyeq 0$, we have $2\|T\| = w(ST + S^*T)$ belonging to $\overline{W(ST + S^*T)}$ by Proposition 1.1 (b). Thus from [1, Theorem 2.1], we obtain the equality in (b). The equivalence of (c) and (d) follows from the following equalities:

$$w(ST + S^*T) = w\big(\mathrm{Re}\,(ST + S^*T)\big) = \frac{1}{2}w\big((ST + S^*T) + (TS^* + TS)\big)$$
$$= \frac{1}{2}w\big((ST + TS) + (TS^* + S^*T)\big) = w\big(\mathrm{Re}\,(ST + TS)\big) = w(ST + TS),$$

where the first (resp., last) equality is by Proposition 1.1 (b) since $ST + S^*T \succcurlyeq 0$ (resp., $ST + TS \succcurlyeq 0$). For the equivalence of (a) and (e), note that, by Theorem 2.1, $w(F_T) = 1 + (1/2)\|T\|$ if and only if $w(S + TS^*T) = 1 + \|T\|^2$. However, we also have

$$w(S + TS^*T) = w\big(\mathrm{Re}\,(S + TS^*T)\big) = w\big(\mathrm{Re}\,(S + TST)\big) = w(S + TST)$$

via Proposition 1.1 (b). Hence (a) and (e) are equivalent.

For the proof of (e)⇒(f), since $1+\|T\|^2 = w(S+TST) \leq w(S)+w(TST) = 1+w(TST)$, we obtain $\|T\|^2 \leq w(TST)$. Together with $w(TST) \leq \|TST\| \leq \|T\|^2$, this yields (f).

For (f)⇒(g), since $0 \preccurlyeq TST \preccurlyeq \|T\|ST$, we have $\|T\|^2 = w(TST) \leq \|T\|w(ST)$ and hence $\|T\| \leq w(ST)$. Together with $w(ST) \leq \|ST\| \leq \|T\|$, this yields (g).

Finally, we prove the implication (g)⇒(d). As before, we may assume that $\|T\| = 1$. Then $0 \leq a_n \leq 1$ for all n. Let D be the unilateral weighted shift with weights $\{d_n\}_{n=1}^{\infty}$, where $d_n = \sqrt{a_n a_{n+1}}$ for $n \geq 1$. Since $ST + TS$ is also a unilateral weighted shift with weights $\{a_n + a_{n+1}\}_{n=1}^{\infty}$ and $ST + TS \succcurlyeq 2D \succcurlyeq 0$, we have $w(ST + TS) \geq 2w(D)$. We now use Lemma 2.8 to prove $w(D) = 1$. Indeed, condition (g) implies that $w(ST) = \|T\| = \|ST\| = 1$. Hence ST is normaloid. By Lemma 2.8, for any ε, $0 < \varepsilon < 1$, there is an integer N such that $\sup_{k\geq 1}(a_k a_{k+1} \cdots a_{k+j-1})^{1/j} > 1 - \varepsilon$ for all $j \geq N$. Therefore, for each $j \geq N$, there is a k_j such that $(a_{k_j} a_{k_j+1} \cdots a_{k_j+j-1})^{1/j} > 1 - \varepsilon$. As $0 \leq a_n \leq 1$ for all n, we have

$$(a_{k_j} a_{k_j+1} \cdots a_{k_j+j-2})^{1/2},\ (a_{k_j+1} a_{k_j+2} \cdots a_{k_j+j-1})^{1/2} > (1 - \varepsilon)^{j/2}.$$

It follows that

$$d_{k_j} d_{k_j+1} \cdots d_{k_j+j-2} = (a_{k_j} a_{k_j+1}^2 \cdots a_{k_j+j-2}^2 a_{k_j+j-1})^{1/2} > (1 - \varepsilon)^j.$$

Therefore, for any $j \geq N$, we have

$$\|D^{j-1}\|^{1/(j-1)} = \sup_{k\geq 1}(d_k d_{k+1}\cdots d_{k+j-2})^{1/(j-1)} \geq (1-\varepsilon)^{j/(j-1)}.$$

Hence

$$\rho(D) = \lim_{j\to\infty}\|D^{j-1}\|^{1/(j-1)} = \lim_{j\to\infty}\sup_{k\geq 1}(d_k d_{k+1}\cdots d_{k+j-2})^{1/(j-1)} \geq \lim_{j\to\infty}(1-\varepsilon)^{j/(j-1)} = 1-\varepsilon.$$

As this is true for any ε, $0<\varepsilon<1$, we obtain that

$$1 \leq \rho(D) \leq w(D) \leq \|D\| = \sup_{n\geq 1} d_n = \sup_{n\geq 1}\sqrt{a_n a_{n+1}} \leq 1.$$

This results in equalities throughout. In particular, we have $w(D)=1$ and thus

$$2 = 2w(D) \leq w(ST+TS) \leq \|ST+TS\| \leq \|ST\|+\|TS\| \leq 2.$$

Therefore, $w(ST+TS)=2$, that is, condition (d) holds as claimed. ∎

Corollary 2.13. *Let $T = \operatorname{diag}(a_1,\dots,a_n,1,1,\dots)$ with $a_k \geq 0$ for $1\leq k\leq n$. Then $w(F_T) = 1+(1/2)\|T\|$ if and only if $a_k \leq 1$ for all k.*

Proof. Assume that $w(F_T)=1+(1/2)\|T\|$ and let $a=\max_{1\leq k\leq n} a_k$. We check that $a\leq 1$. Indeed, if otherwise $a>1$, then we have $w(ST)=\|T\|=a$ from Theorem 2.12 (g) or Theorem 2.4 (b). Let $T' = \operatorname{diag}(\underbrace{a,\dots,a}_{n},1,1,\dots)$. Since $0 \preccurlyeq ST \preccurlyeq ST' \preccurlyeq aS$, we obtain $a = w(ST) \leq w(ST') \leq w(aS) = a$ by Proposition 1.1 (a). It follows that $w(ST') = a = \|T'\| = \|ST'\|$ or ST' is normaloid. Lemma 2.8 then implies that $a = \lim_{j\to\infty} a^{n/j} = 1$, which contradicts our assumption of $a>1$. Thus $a_k\leq 1$ for all k, $1\leq k\leq n$. The converse is by Example 2.5 (b). ∎

Corollary 2.14. *Let $T=\operatorname{diag}(a_1,a_2,\dots)\neq 0$ with $a_n = 0$ exactly when $n=n_j$, $1\leq n_j < n_{j+1}$, for $j\geq 1$. If $w(F_T)=1+(1/2)\|T\|$, then $\{n_{j+1}-n_j\}_{j=1}^{\infty}$ is unbounded.*

Proof. Assume the contrary that $n_{j+1}-n_j \leq M$ for all j. Then we have

$$|a_k a_{k+1}\cdots a_{k+l-1}|^{1/l} = 0 \text{ for any } k\geq 1 \text{ and } l\geq \max\{n_1, M\}.$$

Thus

$$\|ST\| = \sup_{n\geq 1}|a_n| > \lim_{l\to\infty}\sup_{k\geq 1}|a_k a_{k+1}\cdots a_{k+l-1}|^{1/l} = 0.$$

By Lemma 2.8, this says that ST is not normaloid or $w(ST) < \|ST\| = \|T\|$. Hence $w(F_T) < 1+(1/2)\|T\|$ by Theorem 2.4 (b). This proves the unboundedness of $\{n_{j+1}-n_j\}_{j=1}^{\infty}$. ∎

Example 2.15. (a) Let $T = \text{diag}\,(a_1, a_2, \dots)$, where $a_n = 0$ if $n = k(k+1)/2$ for some $k \geq 1$, and $a_n = 1$ otherwise. Then $ST = \sum_{n=1}^{\infty} \oplus S_n$. Thus

$$w(ST) = \sup_n w(S_n) = \sup_n \cos \frac{\pi}{n+1} = 1 = \|T\|$$

(cf. [8, Lemma 2.4.1 (a)]), and $w(F_T) = 1 + (1/2)\|T\| = 3/2$ by Theorem 2.12.

(b) Let $T = \text{diag}\,(a_1, a_2, \dots)$, where

$$a_n = \begin{cases} 0 & \text{if } n = k(k+1)/2 \text{ for some } k \geq 1, \\ 1 & \text{if } n = 2, \\ 1/2 & \text{otherwise.} \end{cases}$$

Then $ST = [0] \oplus S_2 \oplus (1/2) \sum_{n=3}^{\infty} \oplus S_n$. Since

$$w(ST) = \sup\{\frac{1}{2}, \frac{1}{2}\cos\frac{\pi}{n+1} : n \geq 3\} = \frac{1}{2} < 1 = \|T\|,$$

we have $w(F_T) < 1 + (1/2)\|T\| = 3/2$ by Theorem 2.12. This shows that the converse of the assertion in Corollary 2.14 is false.

Finally, for a nonnegative diagonal T, Theorem 2.6 has the following analogue.

Corollary 2.16. *Let $T = \text{diag}\,(a_1, a_2, \dots)$ with $a_n \geq 0$ for all n, and*

$$T_{n,k} = \text{diag}\,(a_k, a_{k+1}, \dots, a_{k+n}) \text{ for } n, k \geq 1.$$

Then $w(F_T) = 1 + (1/2)\|T\|$ if and only if for any $n \geq 1$ there is a sequence $\{n_j\}_{n=1}^{\infty}$ of positive integers such that $\lim_{j\to\infty} T_{n,n_j} = \|T\| I_{n+1}$.

Proof. If $w(F_T) = 1 + (1/2)\|T\|$, then Theorem 2.12 (e) yields $w(S + TST) = 1 + \|T\|^2$. Hence the asserted condition holds since it is the one in Theorem 2.6 with $\lambda_1 = \lambda_2 = 1$. The converse is also by Theorem 2.6. ∎

3. Lower bound of $w(F_T)$

In this section, we consider conditions on T for $w(F_T)$ to be equal to 1. We start with a sufficient one.

Proposition 3.1. *Let $T = T_n \oplus 0$ on ℓ^2, where T_n is an n-by-n nonnegative symmetric matrix, and let $T_{n+1} = T_n \oplus [1]$ on $\mathbb{C}^{n+1}$. If $w(S_{n+1} + S_{n+1}^* + T_{n+1}) \leq 2$, then $w(F_T) = 1$.*

For its proof, we need the following lemma.

Lemma 3.2. *Let T be an operator on ℓ^2. Then*

(a) $w(F_T) \geq (1/2)\max\{w\big(S+S^*+\mathrm{Re}\,(\lambda T)\big) : |\lambda|=1\}$, *and*

(b) *if* T *is nonnegative symmetric, then* $w(F_T) = w(S+S^*+T)/2$.

Note that part (b) here is a generalization of [3, Proposition 3.2 (c)].

Proof. [Proof of Lemma 3.2] (a) For any unit vector x in ℓ^2, let $y = (\overline{\lambda}x \oplus x)/\sqrt{2}$, where $|\lambda| = 1$. Then y is also a unit vector in $\ell^2 \oplus \ell^2$ and

$$
\begin{aligned}
w(F_T) \geq |\langle F_T y, y\rangle| \geq \mathrm{Re}\,\langle F_T y, y\rangle = \frac{1}{2}\langle (F_T + F_T^*)y, y\rangle \\
= \frac{1}{4}\Big\langle \begin{bmatrix} S^*+S & T \\ T^* & S+S^* \end{bmatrix} \begin{bmatrix} \overline{\lambda}x \\ x \end{bmatrix}, \begin{bmatrix} \overline{\lambda}x \\ x \end{bmatrix} \Big\rangle \\
= \frac{1}{4}\big(\langle \overline{\lambda}(S^*+S)x + Tx, \overline{\lambda}x\rangle + \langle \overline{\lambda}T^*x + (S+S^*)x, x\rangle\big) \\
= \frac{1}{2}\big(\langle (S+S^*)x, x\rangle + \langle (\mathrm{Re}\,(\lambda T))x, x\rangle\big) \\
= \frac{1}{2}\langle \big(S+S^*+\mathrm{Re}\,(\lambda T)\big)x, x\rangle.
\end{aligned}
$$

Since this is true for any unit vector x and any λ, $|\lambda| = 1$, the asserted inequality holds.

(b) For a nonnegative T, we have $w(F_T) = w(\mathrm{Re}\,F_T)$ by Proposition 1.1 (b). Let $U = (1/\sqrt{2})\begin{bmatrix} I & I \\ I & -I \end{bmatrix}$ on $\ell^2 \oplus \ell^2$. Then U is unitary and

$$
U^*(\mathrm{Re}\,F_T)U = \frac{1}{2}U^*\begin{bmatrix} S+S^* & T \\ T & S+S^* \end{bmatrix}U = \frac{1}{2}\begin{bmatrix} S+S^*+T & 0 \\ 0 & S+S^*-T \end{bmatrix}.
$$

For any unit vector x in ℓ^2, we have

$$
|\langle (S+S^*-T)x, x\rangle| \leq \langle (S+S^*+T)|x|, |x|\rangle \leq w(S+S^*+T).
$$

This shows that $w(S+S^*-T) \leq w(S+S^*+T)$. Thus

$$
w(F_T) = w(\mathrm{Re}\,F_T) = \frac{1}{2}\max\{w(S+S^*+T), w(S+S^*-T)\} = \frac{1}{2}w(S+S^*+T).
$$

■

Proof. [Proof of Proposition 3.1] In the following, we show that $w(S+S^*+T) \leq 2$ and then apply Lemma 3.2 (b). Let $x = (x_1, x_2, \ldots)$ be any unit vector in ℓ^2 and let

$x' = (x_1, \ldots, x_{n+1})$ in $\mathbb{C}^{n+1}$. We have

$$
\begin{aligned}
&|\langle (S+S^*+T)x, x\rangle| \le \langle (S+S^*+T)|x|, |x|\rangle \\
&= \langle (S_{n+1}+S_{n+1}^*+T_{n+1})|x'|, |x'|\rangle - |x_{n+1}|^2 + 2\sum_{j=n+1}^{\infty} |x_j x_{j+1}| \\
&\le 2\||x'|\|^2 - |x_{n+1}|^2 + \sum_{j=n+1}^{\infty} (|x_j|^2 + |x_{j+1}|^2) \quad (\text{because } w(S_{n+1}+S_{n+1}^*+T_{n+1}) \le 2) \\
&= 2\sum_{j=1}^{n+1} |x_j|^2 + 2\sum_{j=n+2}^{\infty} |x_j|^2 = 2\|x\|^2 = 2.
\end{aligned}
$$

Hence $w(S+S^*+T) \le 2$. By Lemma 3.2 (b), we obtain $w(F_T) \le 1$. As $w(F_T) \ge 1$ is always true, we conclude that $w(F_T) = 1$. ■

Example 3.3. (a) Let $T = \operatorname{diag}(0, \ldots, 0, \underset{n\text{th}}{a}, 0, 0, \ldots)$ with $|a| \le 1/n$ $(n \ge 1)$. Since $|F_T| \preccurlyeq F_{T'}$ for $T' = \operatorname{diag}(0, \ldots, 0, 1/n, 0, 0, \ldots)$, we have $w(F_T) \le w(F_{T'})$. We now check that $w(F_{T'}) = 1$. Indeed, if $T'_{n+1} = \operatorname{diag}(0, \ldots, 0, 1/n, 1)$ on $\mathbb{C}^{n+1}$, then

$$
2I_{n+1} - (S_{n+1} + S_{n+1}^* + T'_{n+1}) = \begin{bmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & 2 & -1 & \\ & & -1 & 2-(1/n) & -1 \\ & & & -1 & 1 \end{bmatrix}.
$$

It is easily shown by induction that its jth $(1 \le j \le n+1)$ leading principal submatrix,

$$
\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix} (1 \le j \le n-1), \quad \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 2-(1/n) \end{bmatrix} (j = n),
$$

or $2I_{n+1} - (S_{n+1} + S_{n+1}^* + T'_{n+1})$ $(j = n+1)$, has determinant $j+1$, n, or 0, respectively. Thus $2I_{n+1} - (S_{n+1} + S_{n+1}^* + T'_{n+1}) \ge 0$ by Sylvester's criterion [7, Theorem 7.2.5 (c)]. Hence $w(S_{n+1} + S_{n+1}^* + T'_{n+1}) \le 2$ and $w(F_{T'}) = 1$ by Proposition 3.1. It follows that $w(F_T) = 1$.

(b) Let $T = \operatorname{diag}(a, a, 0, 0, \ldots)$ with $|a| \le (3-\sqrt{5})/2$. Then, as in (a) above, we may assume that $a = (3-\sqrt{5})/2$ and infer that

$$
S_3 + S_3^* + T'_3 = \begin{bmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & 1 \end{bmatrix} \le 2I_3,
$$

where $T'_3 = \operatorname{diag}(a, a, 1)$. Hence $w(F_T) = 1$ by Proposition 3.1.

We remark that part (a) above also follows from [4, Proposition 2.7 and Theroem 2.8].

In the remaining part of this section, we consider the relationship between $w(F_T) = 1$ and the compactness of T. Recall that, for a compact T, it is known that $w(F_T) = 1 + (1/2)\|T\|$ if and only if $T = 0$. The next theorem says that, for a diagonal T, $w(F_T) = 1$ implies the compactness of T.

Theorem 3.4. *Let $T = \operatorname{diag}(a_1, a_2, \dots)$. If $w(F_T) = 1$, then T is compact.*

The following lemma is needed for its proof.

Lemma 3.5. *If $T = \operatorname{diag}(a_1, a_2, \dots)$, then $\|S + S^* + T\| \geq 2$.*

Proof. Note that $\|S + S^* + T\| = 2w(\operatorname{Re} F_{T'})$, where $T' = \operatorname{diag}(\overline{a}_1, a_2, \overline{a}_3, a_4, \dots)$, by [3, Proposition 3.2 (a)]. Since $W(F_{T'}) \supseteq \mathbb{D}$, we have $w(\operatorname{Re} F_{T'}) \geq 1$. Thus $\|S + S^* + T\| \geq 2$ as asserted. ■

Proof. [Proof of Theorem 3.4] We claim that if the a_n's are real and $\|S + S^* + T\| = 2$, then T is compact. For this, we first assume that $a_n \geq 0$ for all n. Let $T_n = \operatorname{diag}(0, \dots, 0, \underset{n\text{th}}{a_n}, 0, 0, \dots)$ for $n \geq 1$. Since $0 \preccurlyeq S + S^* + T_n \preccurlyeq S + S^* + T$, we have

$$2 \leq \|S + S^* + T_n\| = w(S + S^* + T_n) \leq w(S + S^* + T) = \|S + S^* + T\| = 2$$

by Lemma 3.5 and Proposition 1.1 (a). Thus the above inequalities become equalities throughout, which yield that $w(S + S^* + T_n) = 2$ for all $n \geq 1$. By Lemma 3.2 (b), we have $w(F_{T_n}) = 1$. Hence [4, Proposition 2.7 and Theorem 2.8] yields that $|a_n| \leq 1/n$ for all n. The compactness of T for this case follows.

Now assume that the a_n's are all real. Let $C = S + S^* + T$ and, for each $n \geq 2$, let

$$C_{2n} = \begin{bmatrix} a_1 & 1 & & \\ 1 & a_2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & a_{2n} \end{bmatrix}$$

be the $(2n)$-by-$(2n)$ leading principal submatrix of C. Note that $\|C\| = 2$ by our assumption. We rearrange the standard basis $\{e_j\}_{j=1}^{2n}$ of $\mathbb{C}^{2n}$ via the permutation $(1, 2, \dots, 2n) \to (1, 3, \dots, 2n-1, 2, 4, \dots, 2n)$. Then C_{2n} is permutationally similar to the matrix

$$C'_{2n} = \begin{bmatrix} D'_{2n} & I_n + S_n \\ I_n + S_n^* & D''_{2n} \end{bmatrix},$$

where $D'_{2n} = \operatorname{diag}(a_1, a_3, \ldots, a_{2n-1})$ and $D''_{2n} = \operatorname{diag}(a_2, a_4, \ldots, a_{2n})$. Then $C'^2_{2n} = \begin{bmatrix} E_n & * \\ * & F_n \end{bmatrix}$, where

$$E_n = D'^2_{2n} + (I_n + S_n)(I_n + S_n^*) = \begin{bmatrix} 1 + a_1^2 & 1 & & \\ 1 & 2 + a_3^2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2 + a_{2n-1}^2 \end{bmatrix}$$

and

$$F_n = D''^2_{2n} + (I_n + S_n^*)(I_n + S_n) = \begin{bmatrix} 2 + a_2^2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & 2 + a_{2n-2}^2 & 1 \\ & & 1 & 1 + a_{2n}^2 \end{bmatrix}.$$

Let

$$E'_n = \begin{bmatrix} 2 + a_3^2 & 1 & & \\ 1 & 2 + a_5^2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2 + a_{2n-1}^2 \end{bmatrix}.$$

Then

$$w(E'_n) = \|E'_n\| \le \|E_n\| \le \|C'^2_{2n}\| = \|C'_{2n}\|^2 = \|C_{2n}\|^2 \le \|C\|^2 = 4$$

for all $n \ge 2$. Let $T' = \operatorname{diag}(a_3^2, a_5^2, a_7^2, \ldots)$ and $C' = S + S^* + T'$. Since E'_n is the $(n-1)$-by-$(n-1)$ leading principal submatrix of $C' + 2I$, we have $w(C' + 2I) = \lim_n w(E'_n) \le 4$. Note that $C' \not\ge 0$ implies that $w(C') = \|C'\|$ is in $\overline{W(C')}$ by Proposition 1.1 (b). Hence $w(C' + 2I) = w(C') + 2$ by [1, Theorem 2.1]. It follows that $w(C') \le 2$. On the other hand, we also have $w(C') = \|C'\| \ge 2$ by Lemma 3.5. This shows that $w(C') = 2$. As T' has nonnegative diagonals, the first paragraph of our proof yields that $\lim_n a_{2n+1} = 0$. In a similar fashion, considering

$$F'_n = \begin{bmatrix} 2 + a_2^2 & 1 & & \\ 1 & 2 + a_4^2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2 + a_{2n-2}^2 \end{bmatrix}$$

instead of E'_n and following the arguments as above, we also obtain $\lim_n a_{2n} = 0$. These together prove our claim of the compactness of T.

Finally, for the general case of complex a_n's, we have

$$2 \le \|S + S^* + \operatorname{Re} T\| = w(S + S^* + \operatorname{Re} T) \le 2w(F_T) = 2$$

by Lemmas 3.5 and 3.2 (a). This shows that $\|S + S^* + \operatorname{Re} T\| = 2$. From our claim in the beginning of the proof, we obtain $\lim_n \operatorname{Re} a_n = 0$. Similarly, as $\operatorname{Im} T = \operatorname{Re}(-iT)$, the above arguments also result in $\lim_n \operatorname{Im} a_n = 0$. Hence $\lim_n a_n = 0$ and T is compact. ■

The next proposition is in contrast to the known result that if $T = \operatorname{diag}(1, a, a^2, \dots)$ with $|a| = 1$, then $w(F_T) = 3/2$ and $W(F_T)$ is open but not a circular disc (cf. [3, Corollary 3.6]).

Proposition 3.6. *Let $T = \operatorname{diag}(1, a, a^2, \dots)$ with $|a| < 1$. Then the following are equivalent:*

(a) $w(F_T) = 1$,

(b) $a = 0$,

(c) $W(F_T) = \mathbb{D}$,

(d) $W(F_T)$ *is open, and*

(e) $\|C_\lambda\| = 2$ *for all* λ, $|\lambda| = 1$, *where* $C_\lambda = S + S^* + T_\lambda$ *with* $T_\lambda = \operatorname{diag}(1, \lambda^2 a, \overline{\lambda}^4 \overline{a}^2, \dots)$.

The proofs of some equivalences here need the following two lemmas. In the first one, $W_e(A)$ denotes the essential numerical range of operator A on an infinite-dimensional space (cf. [8, Section 4.2]).

Lemma 3.7. *Let T be a compact operator on ℓ^2. Then $W(F_T)$ is open if and only if $W(F_T) = \mathbb{D}$.*

Proof. If $W(F_T)$ is open, then $\overline{W(F_T)} = W_e(F_T) = W_e(S^* \oplus S) = \overline{\mathbb{D}}$, where the first equality is by [8, Corollary 4.5.5]. Since $W(F_T)$ already contains $\mathbb{D}$, we obtain $W(F_T) = \mathbb{D}$. ■

Lemma 3.8. *Let $T = \operatorname{diag}(a_1, a_2, \dots)$. Then $w(F_T) = 1$ if and only if $\|C_\lambda\| = 2$ for all λ, $|\lambda| = 1$, where $C_\lambda = S + S^* + T_\lambda$ with $T_\lambda = \operatorname{diag}(\overline{a}_1, \lambda^2 a_2, \overline{\lambda}^4 \overline{a}_3, \dots)$.*

Proof. It was proved in [3, Proposition 3.2 (a)] that $w(\operatorname{Re}(\lambda F_T)) = (1/2)\|C_\lambda\|$ for any λ, $|\lambda| = 1$, and hence $w(F_T) = (1/2)\max\{\|C_\lambda\| : |\lambda| = 1\}$. If $w(F_T) = 1$, then, as $W(F_T) \supseteq \mathbb{D}$, we have $\overline{W(F_T)} = \overline{\mathbb{D}}$. Thus $1 = w(\operatorname{Re}(\lambda F_T)) = (1/2)\|C_\lambda\|$ or $\|C_\lambda\| = 2$ for all λ, $|\lambda| = 1$. The converse follows from $w(F_T) = (1/2)\max\{\|C_\lambda\| : |\lambda| = 1\}$. ■

Proof. [Proof of Proposition 3.6] (a)⇒(b). Let $a = \lambda_0 |a|$ for some λ_0 with $|\lambda_0| = 1$. If λ is such that $\lambda^2 \lambda_0 = 1$, then λF_T is unitarily similar to $F_{|T|}$ by [3, Lemma 3.1 (a)]. Hence we may assume that $w(F_T) = 1$ with $T = \operatorname{diag}(1, a, a^2, \dots)$, $0 \le a < 1$. For any t, $0 < t < 1$, let $x_t = \sqrt{1 - t^2}(1, t, t^2, \dots)$ in ℓ^2. Then x_t is a unit vector, $\langle S x_t, x_t \rangle = \langle S^* x_t, x_t \rangle = t$, and $\langle T x_t, x_t \rangle = (1 - t^2)/(1 - a t^2)$. Assuming that $0 < a < 1$, we show that $\langle (S+S^*+T)x_{t_0}, x_{t_0}\rangle > 2$ for some t_0, $0 < t_0 < 1$. Indeed, this asserted inequality is the same as $2t_0 + (1-t_0^2)/(1-at_0^2) > 2$ or $2at_0^2 + t_0 - 1 > 0$. For $0 < a < 1$, we have $0 < (-1+\sqrt{1+8a})/(4a) < 1$. Thus if t_0 is such that $(-1+\sqrt{1+8a})/(4a) < t_0 < 1$, then $0 < t_0 < 1$ and $2at_0^2 + t_0 - 1 > 0$, which means that t_0 meets our requirement that $\langle (S+S^*+T)x_{t_0}, x_{t_0}\rangle > 2$. Hence $w(S+S^*+T) > 2$ and $w(F_T) = (1/2)\max\{\|C_\lambda\| : |\lambda| = 1\} > 1$. This contradicts our assumption that $w(F_T) = 1$ and thus a must be 0.

(b)⇒(c) was shown in [4, Proposition 2.7], (c)⇔(d) (resp., (a)⇔(e)) is by Lemma 3.7 (resp., Lemma 3.8), and (c)⇒(a) is trivial. Thus the proof is completed. ■

In the preceding proposition, the equivalence of (a) and (b) can also be proved by using [4, Lemma 2.3 (d)]. This is given below.

Proof. [Alternative proof of Proposition 3.6 (a)⇔(b)] As in the previous proof, we may assume that $0 \le a < 1$. Let $C = S + S^* + T$. As $T \not\ge 0$, (a) is equivalent to $w(C) = \|C\| = 2$ (cf. [3, Proposition 3.2 (c)]). Hence [4, Lemma 2.3 (d)] says that the latter is equivalent to the sequence $\{b_n\}_{n=1}^{\infty}$ defined by $b_1 = 1$ and $b_n = 1/(2 - a^{n-1} - b_{n-1})$ for $n \ge 2$ satisfying $1/2 \le b_n \le 1$ for all n. In particular, this latter condition implies that $1/2 \le b_2 = 1/(1-a) \le 1$ or that $a = 0$. This proves (a)⇒(b). For the converse, if $a = 0$, then $b_n = 1/(2 - b_{n-1})$ for $n \ge 2$. Hence $b_n = 1$ for all n by induction. Therefore, $w(F_T) = 1$ follows. ■

We end this paper with the question: Does $w(F_T) = 1$ imply the compactness of T? By Theorem 3.4, the answer is affirmative for a diagonal T.

Declaration of competing interest

No competing interest.

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