J. Math. Anal. Appl. 515 (2022) 126378

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Regular Articles

Generalized circular projections

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ARTICLE INFO

Article history:
Received 20 February 2021
Available online 1 June 2022
Submitted by R. Curto

Keywords:
Isometry
Norm
Circular projection

ABSTRACT

We study r-circular projections of matrix norms with some special properties, including the unitarily invariant norms, the unitary congruence invariant norms, and the unitary similarity invariant norms. In each case, we determine all values r for the existence of r-circular projections corresponding to isometries of a certain form.

1. Introduction

The study of isometries and projections is at the core of the study and understanding of the structure of a Banach space. The concept of a bicircular projection unites both. Although it arises from complex analysis, more precisely from the study of continuous Reinhardt domains [19], it is a purely Banach space notion. A projection P on a complex Banach space is said to be bicircular if $P + \lambda(I - P)$ is an isometry for every unimodular $\lambda \in \mathbb{C}$, [20]. It turns out that bicircular projections are precisely those projections that are (norm) Hermitian operators [15], thus the term Hermitian projections is also used. While early results on bicircular projections were obtained for Banach spaces with nice algebraic structure [21], their connection with Hermitian operators enables description of bicircular projections on other important Banach spaces [15]. Further, the knowledge of the isometry group of a given norm (the symmetry group of the norm function) allows description of projections P such that $P + \lambda(I - P)$ is an isometry for not necessarily all, but *some* unimodular $\lambda \in \mathbb{C}$, [11]. Such projections are now known as generalized bicircular projections. They decompose a Banach space into two complementary subspaces. At the same time, they can be considered as eigenprojections of a two-point spectrum isometry with one eigenvalue 1 (which can be always assumed without loss of generality). With this in mind, the results of [18] can be interpreted as follows: Eigenprojections of a two-point spectrum isometry with an eigenvalue 1 are both Hermitian projections, or the other

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https://doi.org/10.1016/j.jmaa.2022.126378

eigenvalue is some n-th root of unity. Furthermore, for any n there is a complex Banach space allowing an isometry with nonhermitian eigenprojections and spectrum $\{1, e^{i2\pi/n}\}$. However, it turns out that in many classical Banach spaces the other eigenvalue is always -1, see e.g. a recent paper [6] for a list of examples and references. In particular, it was pointed out in [6] that such Banach spaces act as a suitable framework for studying the spectrum of isometries with an arbitrary finite number of eigenvalues.

It is natural to consider the decomposition of a given space into more subspaces based on the norm structure, as well as to consider possible spectra of finite spectrum isometries. In this way this problem can be regarded as a problem within the class of so-called inverse eigenvalue problems which are known to have various applications, see e.g. [7]. It provides a link between the geometry of the norm, the corresponding decomposition of a space into subspaces and also the spectral properties of the isometries. With this motivation, we consider generalized circular projections.

Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space and $\mathcal{B}(\mathcal{X})$ the algebra of all bounded linear operators on $\mathcal{X}$. Generalized circular projections were introduced in [14] (although some similar but less general concepts appeared in earlier papers [2], [5] and [3]) and defined as follows.

Definition 1.1. A collection of nonzero projections $\{P_1, \ldots, P_r\} \subseteq \mathcal{B}(\mathcal{X})$, $r \geq 2$, is said to be a family of generalized circular projections (with respect to $\|\cdot\|$) if

1. $P_1 \oplus \cdots \oplus P_r = I$ (that is, $P_iP_j = \delta_{ij}P_i$ and $\sum_{j=1}^r P_j = I$), and
2. there exist distinct complex numbers $\lambda_1, \ldots, \lambda_r$ (necessarily of modulus one) such that

$$T = \lambda_1 P_1 + \cdots + \lambda_r P_r$$

is an isometry of $\|\cdot\|$, that is, $\|Tx\| = \|x\|$ for all $x \in \mathcal{X}$.

To be more precise, we say that $\{P_1, \ldots, P_r\}$ is a family of r-circular projections corresponding to the isometry T and each P_i is called an r-circular projection (with respect to $\|\cdot\|$).

Existence of a family $\{P_1, \ldots, P_r\} \subseteq \mathcal{B}(\mathcal{X})$ of r-circular projections is equivalent to the existence of an isometry T of $(\mathcal{X}, \|\cdot\|)$ with precisely r elements in its spectrum, $\sigma(T) = \{\lambda_1, \ldots, \lambda_r\}$, that is, to the existence of an isometry with spectral decomposition $\lambda_1 P_1 + \cdots + \lambda_r P_r$.

Such r-circular projections are always polynomials in the corresponding isometry T. More precisely, we have the following result; see [14, Proposition 2.4].

Proposition 1.2. *Let $P_j \in \mathcal{B}(\mathcal{X})$ for $j = 1, \ldots, r$ ($r \geq 2$) and let $\lambda_1, \ldots, \lambda_r$ be distinct complex numbers. The following are equivalent.*

1. *$T = \sum_{j=1}^r \lambda_j P_j$ and $P_1, \ldots, P_r$ are projections satisfying $P_1 \oplus \cdots \oplus P_r = I$.*
2. *$\prod_{j=1}^r (T - \lambda_j I) = 0$ and*

$$P_\ell = \frac{\prod_{j\neq\ell}(T - \lambda_j I)}{\prod_{j\neq\ell}(\lambda_\ell - \lambda_j)} \quad \text{for } \ell = 1, \ldots, r. \tag{1.1}$$

The case $r = 2$ has been studied extensively in different settings, in particular on various matrix spaces in [11]. For 2-circular projections the term "generalized bicircular projections" has been used.

Extending the ideas and techniques in the previous study [11], we utilize the structure of the isometry group of the norm to help study our problem. However, instead of treating just a particular normed space, we solve the problem for whole families of finite dimensional Banach spaces, where each family consists of norms with a certain group of symmetries. A key feature of our proofs involves using the knowledge of the common isometries of such norms; as such, special techniques arise naturally, and the analysis is necessarily more technical and involved, requiring some intricate linear algebraic and combinatorial arguments. Thus, although it might be nice find purely analytic tools to prove our results, our research becomes an interesting example of a successful interaction of (linear) algebra and analysis.

Initial research of generalized bicircular projections [11] was also conducted in the finite dimensional setting, but it led to the study of these mappings in various specific infinite dimensional Banach spaces and initiated research in other directions that are interesting from the point of view of Banach space theory. Similarly, we hope that our results on general classes of norms lead to further study in the infinite dimensional case; perhaps more elegant proofs can be found. Since the study of an arbitrary number of projections (instead of only two for the bicircular case) is more natural, we hope that this paper will also be a good starting point for those interested in isometries of Banach spaces, in particular their spectra and eigenprojections. Description of the isometries between various specific Banach spaces (including matrix spaces) can be found in the two volumes [9] and [10].

Let G be the isometry group of a norm $\|\cdot\|$, that is, $G = \{T \in \mathcal{B}(\mathcal{X}) : \|Tx\| = \|x\| \; \forall x \in \mathcal{X}\}$. Then the $P_1, \ldots, P_r$ in Proposition 1.2 are the eigenprojections of $T \in G$ with eigenvalues $\lambda_1, \ldots, \lambda_r$. In the finite dimensional case, it is known that for every bounded group G in M_n (the set of complex $n \times n$ matrices), there is an invertible matrix $S \in M_n$ such that $S^{-1}GS = \{S^{-1}TS : T \in G\}$ is a subgroup of the unitary group in M_n (by the Auerbach Theorem [4]; see also [8]). If G is the isometry group of the norm $\|\cdot\|$ then $S^{-1}GS$ will be the isometry group of the norm $\|\cdot\|_S$ defined by $\|v\|_S = \|Sv\|$. Moreover, $\{P_1, \ldots, P_r\}$ is a family of r-circular projections for $(\mathbb{C}^n, \|\cdot\|)$ if and only if $\{S^{-1}P_1S, \ldots, S^{-1}P_rS\}$ is a family of r-circular projections for $(\mathbb{C}^n, \|\cdot\|_S)$. Thus, for a finite dimensional Banach space, we may focus on a norm $\|\cdot\|$ whose isometry group G is a subgroup of the unitary group. The study of r-circular projections reduces to the study of the eigenprojections of operators in certain subgroups of the unitary operators.

Similar to early study of bicircular projections, the study is easy if the Banach space under consideration has nice structure. For our problem, the situation is easy if the norm is an inner product norm so that we can assume that the isometry group is the full unitary group. We have the following.

Proposition 1.3. *Let $\|\cdot\|$ be the norm on $\mathbb{C}^n$ induced by the standard inner product so that the isometry group is the unitary group in M_n.*

(a) *Any set $\{P_1, \ldots, P_r\} \subseteq M_n$ of non-trivial orthogonal projections satisfying $P_1 \oplus \cdots \oplus P_n = I$ is a family of r-circular projections, and $\mu_1P_1 + \cdots + \mu_rP_r$ is an isometry for any r distinct complex units $\mu_1, \ldots, \mu_r$.*
(b) *For any complex units $\mu_1, \ldots, \mu_r$ with $2 \le r \le n$, there is a family of r-circular projections $\{P_1, \ldots, P_r\}$ such that $\mu_1P_1 + \cdots + \mu_rP_r$ is an isometry for the norm $\|\cdot\|$.*

As mentioned before, it is helpful to know the isometries of a given norm in order to determine its r-circular projections. A class of norms may have the same isometry group if they share some common properties, see [16]. In such a case, the class of norms will have the same r-circular projections.

For example, consider the ℓ_p-norm defined by $\ell_p(x) = (\sum_{j=1}^n |x_j|^p)^{1/p}$ for $x = (x_1, \ldots, x_n)^t \in \mathbb{C}^n$ for $p \in [1, \infty]$, where $\ell_\infty(x) = \max\{|x_1|, \ldots, |x_n|\}$. If $p \ne 2$, then the isometry group of the ℓ_p-norm is the group of generalized permutation matrices, i.e., matrices equal to the product of a permutation matrix and a diagonal unitary matrix. So, all these norms will have the same r-circular projections. In fact, ℓ_p-norms are examples of symmetric norms (a.k.a. symmetric gauge functions) on $\mathbb{C}^n$. Recall that $\|\cdot\|$ is a symmetric norm on $\mathbb{C}^n$ if $\|Px\| = \|x\|$ for any generalized permutation matrix P. It is known that if $\|\cdot\|$ is a symmetric norm on $\mathbb{C}^n$ not equal to an inner product norm, then the isometry group of $\|\cdot\|$ is the group of generalized permutation matrices; e.g., see [16, Theorem 2.5]. Thus, all such norms have the same r-circular projections. In such a case, Proposition 1.3(b) still holds. However, Proposition 1.3(a) fails. Instead, we have the following.

Proposition 1.4. *Let $\|\cdot\|$ be a symmetric norm on $\mathbb{C}^n$ which is not a multiple of the ℓ_2-norm. A set $\{P_1,\ldots,P_r\}\subseteq M_n$ of non-trivial orthogonal projections satisfying $P_1\oplus\cdots\oplus P_r = I$ is a family of r-circular projections if and only if there are r distinct complex units $\mu_1,\ldots,\mu_r$ such that $\mu_1P_1+\cdots+\mu_rP_r$ is a generalized permutation matrix.*

Given the simple structure of generalized permutation matrices, it is easy to check whether there are distinct complex units $\mu_1,\ldots,\mu_r$ such that $\mu_1P_1+\cdots+\mu_rP_r$ is a generalized permutation matrix for a given family $\{P_1,\ldots,P_r\}$ of non-trivial orthogonal projections satisfying $P_1\oplus\cdots\oplus P_n = I$. For any distinct complex units $\mu_1,\ldots,\mu_r$ with $2\le r\le n$, there are orthogonal projections $P_1,\ldots,P_r$ such that $\mu_1P_1+\cdots+\mu_rP_r$ is a generalized permutation matrix. Also, one may theoretically generate all r-circular projections by considering the eigenvalues and eigenprojections of all generalized permutation matrices (see e.g. [11] for the case $r = 2$ and [3] for the case $r = 3$).

The situation may be more intricate for other classes of norms. In the next three sections, we consider several classes of norms on matrices including the unitarily invariant norms, the unitary congruence invariant norms, and the unitary similarity invariant norms. In each case, we determine the structure of r-circular projections, and determine all possible values r for the existence of a family of r-circular projections $\{P_1,\ldots,P_r\}$ such that $\mu_1P_1+\cdots+\mu_rP_r$ is an isometry of a certain form. For instance, in Section 2, the following is proved in Theorem 2.3.

(a) For a unitarily invariant norm on $M_{m,n}$, there is a family of r-circular projections corresponding to an isometry of the form $A\mapsto UAV$ if and only if $2\le r\le mn$.

(b) For a unitarily invariant norm on M_n, there is a family of r-circular projections corresponding to an isometry of the form $A\mapsto UA^tV$ if and only if $r\in\{2,\ldots,n^2\}\setminus J_n$, where $J_4=\{3,7,11\}$ and $J_n=\{3,7\}$ for other n.

More complex results for unitary congruence invariant norms and unitary similarity invariant norms are proved in Sections 3 and 4. Some remarks and future research directions will be mentioned in Section 5.

2. Unitarily invariant norms

Let $M_{m,n}$ denote the set of complex $m\times n$ matrices (write M_n instead if $m = n$). A norm $\|\cdot\|$ on $M_{m,n}$ is unitarily invariant (ui) if $\|UAV\| = \|A\|$ for any unitary $U\in M_m$ and $V\in M_n$. This is an important class of norms comprising the operator norm, the trace norm, and the Schatten p-norms. In this section we consider r-circular projections with respect to a unitarily invariant norm $\|\cdot\|$ on $M_{m,n}$ that is not a multiple of the Frobenius norm (so $m,n\ge 2$). (This restriction is made because the Frobenius norm is induced by an inner product.) It is known (e.g., see [17]) that the isometries of $\|\cdot\|$ have the form

(1) $A\mapsto UAV$ for some unitary $U\in M_m$ and $V\in M_n$, or
(2) $A\mapsto UA^tV$ for some unitary $U,V\in M_n$ in the case $m = n$.

Thus, we can generate all the r-circular projections by studying the eigenvalues and eigenprojections of isometries of the form (1) or (2).

On the other hand, given a family of non-trivial projections $\{P_1,\ldots,P_r\}$ acting on $M_{m,n}$ such that $P_1\oplus\cdots\oplus P_r = I$, we may check whether it is a family of r-circular projections as follows.

Let

$$\mathcal{B} = \{E_{11},E_{21},\ldots,E_{m1},E_{12},E_{22},\ldots,E_{mn}\}$$

be the standard basis for $M_{m,n}$, and $\mathcal{P}_1, \ldots, \mathcal{P}_r, \mathcal{T} \in M_{mn}$ be the matrix representation of $P_1, \ldots, P_r$ and the transposition map $A \mapsto A^t$, respectively. Consider

(1) $\mathcal{P}(z_1, \ldots, z_r) = z_1\mathcal{P}_1 + \cdots + z_r\mathcal{P}_r$, and

(2) $\tilde{\mathcal{P}}(z_1, \ldots, z_r) = (z_1\mathcal{P}_1 + \cdots + z_r\mathcal{P}_r)\mathcal{T}$ if $m = n$.

Then $\{P_1, \ldots, P_r\}$ is a family of r-circular projections if and only if there are distinct complex units $z_1, \ldots, z_r$ such that

(1') $\mathcal{P}(z_1, \ldots, z_r)$ has the form $U \otimes V$ for some unitary $U \in M_m, V \in M_n$, or

(2') $m = n$ and $\tilde{\mathcal{P}}(z_1, \ldots, z_r)$ has the form $U \otimes V$ for some unitary $U, V \in M_n$.

In the following, we will take a closer look at the structure of r-circular projections for unitarily invariant norms. We can then determine all possible values $r \in [2, mn]$ for the existence of r-circular projections, where $[n_1, n_2]$ denotes the set of integers ℓ such that $n_1 \leq \ell \leq n_2$.

It will be convenient to write $x \otimes y$ for xy^t; then the map $A \mapsto UAV$ is represented by $U \otimes V^t \in M_{mn}$, and if $m = n$, the transposition map $A \mapsto A^t$ is represented by the 'swap' operation $x \otimes y \mapsto y \otimes x$ for $x, y \in \mathbb{C}^n$. Using this notation we record the following observations.

Lemma 2.1. *Let $m, n \geq 2$ be positive integers. Let $U \in M_m, V \in M_n$ be unitary matrices.*

(a) *Define $T: M_{m,n} \to M_{m,n}$ by $T(A) = UAV$. Suppose U has orthonormal eigenvectors $u_1, \ldots, u_m$ with corresponding eigenvalues $\mu_1, \ldots, \mu_m$; suppose V^t has orthonormal eigenvectors $v_1, \ldots, v_n$ with corresponding eigenvalues $\nu_1, \ldots, \nu_n$. Then T has orthonormal eigenvectors $u_i \otimes v_j$ with corresponding eigenvalues $\mu_i\nu_j$, $1 \leq i \leq m, 1 \leq j \leq n$.*

(b) *Suppose $m = n$ and define $T: M_n \to M_n$ by $T(A) = UA^tV$. Suppose UV^t has orthonormal eigenvectors $x_1, \ldots, x_n$ with corresponding eigenvalues $\mu_1, \ldots, \mu_n$. Then T has orthonormal eigenvectors*

$$v_{jj} = x_j \otimes U^*x_j, \; j = 1, \ldots, n,$$

(with corresponding eigenvalues μ_j) and

$$v_{jk}^{\pm} = \frac{1}{\sqrt{2}}(\sqrt{\mu_k}x_j \otimes U^*x_k \pm \sqrt{\mu_j}\, x_k \otimes U^*x_j), \; 1 \leq j < k \leq n$$

(with corresponding eigenvalues $\pm\sqrt{\mu_j}\sqrt{\mu_k}$). Our convention for the square roots is that if $\mu_j = e^{i\theta_j}$ with $-\pi < \theta_j \leq \pi$, then $\sqrt{\mu_j} = e^{i\theta_j/2}$.

Proof. (a) Since $T = U \otimes V^t$, the assertion follows immediately.

(b) Assume the hypotheses. Then $T(x \otimes y) = Uy \otimes V^tx$ for all $x, y \in \mathbb{C}^n$. In particular,

$$T(x_j \otimes U^*x_k) = \mu_j x_k \otimes U^*x_j,$$

so a straightforward computation verifies that

$$x_j \otimes U^*x_j, \; j = 1, \ldots, n,$$

(with corresponding eigenvalues μ_j) and

$$\frac{1}{\sqrt{2}}(\sqrt{\mu_k}x_j \otimes U^*x_k \pm \sqrt{\mu_j}\, x_k \otimes U^*x_j), \; 1 \leq j < k \leq n$$

(with corresponding eigenvalues $\pm\sqrt{\mu_j}\sqrt{\mu_k}$) are orthonormal eigenvectors for T. Since there are n^2 such eigenvectors they form a basis for M_n and the result follows. □

Theorem 2.2. *Let $\|\cdot\|$ be a unitarily invariant norm on $M_{m,n}$ not equal to a multiple of the Frobenius norm. Then $\{P_1,\ldots,P_r\}$ is a family of generalized circular projections on $M_{m,n}$ with respect to $\|\cdot\|$ if and only if one of the following holds:*

(a) *There exist complex units $\mu_1,\ldots,\mu_p,\nu_1,\ldots,\nu_q$ with $\{\mu_i\nu_j : 1\le i\le p, 1\le j\le q\} = \{\lambda_1,\ldots,\lambda_r\}$, orthogonal projections $U_1,\ldots,U_p\in M_m$ with $\sum_{j=1}^p U_j = I_m$, orthogonal projections $V_1,\ldots,V_q\in M_n$ with $\sum_{j=1}^q V_j = I_n$, such that for $j=1,\ldots,r$,*

$$P_j(A) = \sum_{\mu_i\nu_k=\lambda_j} U_iAV_k \quad \textit{for all } A\in M_{m,n}.$$

(b) *$m=n$ and there exist complex units $\xi_1,\ldots,\xi_n$ (possibly with repetition) with*

$$\{\lambda_1,\ldots,\lambda_r\} = \{\xi_j\xi_k : 1\le j\le k\le n\}\cup\{-\xi_k\xi_j : 1\le j<k\le n\},$$

an orthonormal basis $x_1,\ldots,x_n$, and a unitary $U\in M_n$ such that for $j=1,\ldots,r$,

$$P_j = \sum_{\xi_k^2=\lambda_j} v_{kk}v_{kk}^* + \sum_{\xi_i\xi_k=\lambda_j} v_{ik}^+v_{ik}^{+*} + \sum_{-\xi_i\xi_k=\lambda_j} v_{ik}^-v_{ik}^{-*}$$

where

$$v_{jj} = x_j\otimes U^*x_j,\ j=1,\ldots,n,$$
$$v_{jk}^{\pm} = \frac{1}{\sqrt{2}}(\xi_kx_j\otimes U^*x_k \pm \xi_jx_k\otimes U^*x_j),\ 1\le j<k\le n.$$

Proof. Apply Lemma 2.1 and set $\xi_j=\sqrt{\mu_j}$ in part (b). □

Next, we consider the range of r values for the existence of r-circular projections for a ui norm. Since an isometry $T\colon M_{mn}\to M_{mn}$ has at most mn distinct eigenvalues, we see that $r\in[2,mn]$. As shown in part (a) of the following theorem, we can get r-circular projections corresponding to isometries of the form $A\mapsto UAV$ for any $r\in[2,mn]$. The situation is different if $m=n$ and we are interested in r-circular projections corresponding to isometries of the form $A\mapsto UA^tV$. It turns out that in this case r cannot be 3 or 7 for any n, and r cannot be 11 when $n=4$ as shown in part (b) of the following theorem.

Theorem 2.3. *Let $\|\cdot\|$ be a unitarily invariant norm on $M_{m,n}$ not equal to a multiple of the Frobenius norm.*

(a) *There is a family of r-circular projections corresponding to an isometry of the form $A\mapsto UAV$ if and only if $r\in[2,mn]$.*
(b) *Suppose $m=n$. Let $J_4=\{3,7,11\}$ and $J_n=\{3,7\}$ for all other $n\ge 2$. There is a family of r-circular projections corresponding to an isometry of the form $A\mapsto UA^tV$ if and only if $r\in[2,n^2]\setminus J_n$.*

Proof. (a) Since $\dim M_{m,n}=mn$, necessity is clear. To prove sufficiency, let $r\in[2,mn]$. Define $\xi = e^{2\pi i/r}$, $U=\sum_{j=1}^m \xi^{(j-1)n}E_{jj}\in M_m$ and $V=\sum_{j=1}^n \xi^jE_{jj}\in M_n$. Then the map $A\mapsto UAV$ has spectral decomposition $\xi P_1+\xi^2P_2+\cdots+\xi^rP_r$.

(b) From Theorem 2.2(b) r-circular projections exist if and only if there exists a set of the form

$$\mathcal{S} = \{\xi_1^2, \ldots, \xi_n^2, \pm\xi_j\xi_k : 1 \leq j \leq k \leq n, j \neq k\}, \tag{2.2}$$

with r distinct elements.

Given $v = [\xi_1, \xi_2, \ldots, \xi_n] \in \mathbb{C}^{1\times n}$, we define a matrix $\mathcal{S}_v$ with diagonal entries $\xi_1^2, \ldots, \xi_n^2$, and (j,k)-entry $\xi_j\xi_k$ if $j < k$ and $-\xi_j\xi_k$ if $j > k$. It will be convenient to abuse notation by writing $|\mathcal{S}_v|$ for the number of distinct entries in $\mathcal{S}_v$, and $a \in \mathcal{S}_v$ to mean that a is an entry of $\mathcal{S}_v$. As our only concern is $|\mathcal{S}_v|$ we may without loss of generality assume $\xi_1 = 1$ and $\xi_j = e^{i\varphi_j}$ with $\varphi_j \in [0,\pi)$ for $j = 2,\ldots,n$.

We divide the proof into several steps, starting with the cases $n = 2, 3$ and the necessity.

Step 1. Suppose $n = 2$. Let $u_2 = [1,1]$, $u_4 = [1,i]$. Then $|\mathcal{S}_{u_j}| = j$ for $j = 2,4$. In fact, only $\mathcal{S}_{u_2}$ has 2 distinct entries and $\mathcal{S}_{u_4}$ has 4 distinct entries for any $v = [1, e^{i\varphi}]$ with $e^{i\varphi} \neq 1$.

Step 2. Let $n \geq 3$. We show that $|S_v| \neq 3$. If $v = [1,1,\ldots,1]$ then $\mathcal{S}_v$ has 2 distinct entries. If there is some $\xi_j \neq 1$ then we may assume that $\xi_2 = e^{i\varphi} \neq 1$ and the upper left 2-by-2 submatrix of $\mathcal{S}_v$ already has 4 distinct entries. Therefore $|\mathcal{S}_v| \neq 3$.

Step 3. Suppose $n = 3$. Then $|\mathcal{S}_{v_k}| = k$ for $k = 2,4,5,6,8,9$, and each $\mathcal{S}_{v_k}$ contains -1 as an entry if

$$v_2 = [1,1,1], \quad v_4 = [1,1,i], \quad v_5 = [1,1,e^{i\pi/4}], \quad v_6 = [1, e^{i\pi/3}, e^{i2\pi/3}],$$
$$v_8 = [1, e^{i\pi/4}, e^{i\pi/2}], \quad v_9 = [1, i, ie^{i\pi/6}].$$

Let us remark that $\mathcal{S}_{v_4}$ is the only 4 distinct entries case, while the 5 distinct entries cases are induced by every $[1,1,e^{i\varphi}]$ with $e^{i\varphi} \neq i$.

Furthermore, $\mathcal{S}_{v_6}$ is the only 6 distinct entries case and all 8 distinct entries cases are induced by $[1, e^{i\varphi}, e^{2i\varphi}]$ or $[1, e^{i\varphi}, -e^{2i\varphi}]$ or $[1, e^{i\varphi}, -e^{-i\varphi}]$ with $e^{6i\varphi} \neq 1$. Let us elaborate the last conclusion. If $\mathcal{S}_v$ has at least 6 distinct entries then $v = [1, e^{i\varphi_1}, e^{i\varphi_2}]$ with $1 \neq e^{i\varphi_1} \neq e^{i\varphi_2} \neq 1$. Then the off-diagonal entries of S_v are all distinct (recall that $\varphi_1, \varphi_2 \in [0,\pi)$), so if S_v has less than 9 distinct entries then $1 = -e^{i(\varphi_j+\varphi_k)}$ or $e^{2i\varphi_j} = e^{i\varphi_k}$ or $e^{2i\varphi_j} = -e^{i\varphi_k}$ for $\{j,k\} = \{1,2\}$. If any two of these three equations hold (for example if $e^{i\varphi_1} = e^{2i\varphi_2} = (-e^{2i\varphi_1})^2$) then we get v_6; otherwise we get an 8 distinct entries case.

Step 4. Let $n \geq 4$. We show that $|S_v| \neq 7$. If v has at most one entry different from 1 then $\mathcal{S}_v$ has at most 5 distinct entries. Suppose that v has at least two distinct entries not equal to 1, which are not both sixth roots of unity. We may assume that they are ξ_2 and ξ_3. Then the upper left 3-by-3 submatrix of $\mathcal{S}_v$ has at least 8 distinct entries. Now suppose that ξ_2 and ξ_3 are distinct sixth roots of unity. If all entries of v are sixth roots of unity then $\mathcal{S}_v$ has 6 distinct entries. Otherwise we may assume that ξ_4 is not a sixth root of unity. Then the upper left 4-by-4 submatrix has at least 6 more distinct entries than the upper left 3-by-3 submatrix does; that is, the upper left 4-by-4 submatrix has at least 12 distinct entries. Therefore $|\mathcal{S}_v| \neq 7$.

Step 5. Suppose $n = 4$. We show that $|S_v| \neq 11$. Suppose there exists $u_{11} = [z_1, z_2, z_3, z_4]$ such that $\mathcal{S}_{u_{11}}$ has 11 distinct entries. Then the set $\{z_jz_k : 1 \leq j < k \leq 4\} \cup \{-z_jz_k : 1 \leq j < k \leq 4\}$ has at most 11 elements. So $z_jz_k = z_pz_q$ or $z_jz_k = -z_pz_q$; since the elements of $S_{u_{11}}$ are unchanged by replacing z_j with $-z_j$, we may assume the former. We must have $\{j,k,p,q\} = \{1,2,3,4\}$ (otherwise we have $z_r = z_s$ for some $r \neq s$, giving at most 10 distinct entries for $S_{u_{11}}$). By permuting if necessary, we may assume that $z_1z_2 = z_3z_4$. Replace $[z_1, z_2, z_3, z_4]$ by $\frac{1}{\sqrt{z_1z_2}}[z_1, z_2, z_3, z_4]$ so that $z_1z_2 = z_3z_4 = 1$. Thus, $u_{11} = [\alpha, \bar{\alpha}, \beta, \bar{\beta}]$. Then $\pm 1 \in \mathcal{S}_{u_{11}}$ and $\mathcal{S}_{u_{11}}$ is invariant under complex conjugation, so $\mathcal{S}_{u_{11}}$ has an even number of distinct entries. Contradiction.

We have thus far proven the necessity of our statement, as well as sufficiency for $n = 2, 3$. To prove sufficiency for $n \geq 4$, we establish some auxiliary results first.

Step 6. Let $n \ge 3$. We show that for each integer r satisfying $4n-4 \le r \le 6n-10$, there exists a vector $x \in \mathbb{C}^n$ such that $x_1 = 1$, all other entries of x are distinct numbers in the upper half-plane, $-1 \notin \mathcal{S}_x$, and $|\mathcal{S}_x| = r$.

Let $\mu = e^{i\theta}$ with $\theta \in (0, \pi/2)$. Let $n \ge 3$ and $x = [1, \mu, \dots, \mu^{n-2}, \mu^{n-1+d}]$ with $0 \le d \le n-3$. If we choose μ so that $2\theta(n-1+d) < \pi$ then the entries of $\mathcal{S}_x$ are

$$\{1, \mu^{2(n-1+d)}\} \cup \pm\{\mu, \mu^2, \dots, \mu^{2n-3+d}\},$$

so $|\mathcal{S}_x| = 4n-4+2d$. If $d > 0$ and we choose μ so that $\mu^{2(n-1+d)} = -\mu$ then $|\mathcal{S}_x| = 4n-3+2d$. In both cases -1 is not an entry of $\mathcal{S}_x$. By varying d we get the conclusion.

Step 7. We claim that, for all integers $n \ge 3$ and $4n-4 \le r \le n^2$, there exists a vector $x \in \mathbb{C}^n$ such that $x_1 = 1$, all other entries of x are distinct numbers in the upper half-plane, $-1 \notin \mathcal{S}_x$, and $\mathcal{S}_x$ has r distinct entries.

The proof is by induction on n. Taking x equal to $[1, e^{i\pi/6}, e^{i\pi/3}]$ or $[1, e^{i\pi/6}, e^{i\pi/4}]$ shows that the claim holds for $n = 3$. Now suppose the claim holds when $n = m \ge 3$, so for each $r \in [4m-4, m^2]$ there exists a vector $x \in \mathbb{C}^m$ such that $x_1 = 1$, all other entries of x are distinct numbers in the upper half-plane, and $-1 \notin \mathcal{S}_x$. We may write $x_j = e^{i\pi a_j}$ with $0 = a_1 < a_2 < \cdots < a_m$. Choose $a_{m+1} \in (0, \pi/2)$ such that a_{m+1} is not a $\mathbb{Q}$-linear combination of $1, a_1, \dots, a_m$. Let $y = [x, e^{i\pi a_{m+1}}]$. Then $\mathcal{S}_y$ has $r+2m+1$ distinct entries and $-1 \notin \mathcal{S}_y$. Thus the assertion holds for $n = m+1$ and $6(m+1)-9 = 6m-3 \le r \le m^2+2m+1 = (m+1)^2$. Combining this with Step 6 we see that the claim holds for $n = m+1$, and result holds by induction.

Step 8. We prove the sufficiency of our statement for $n = 4$. Observe that if $x_1 = 1$ and $-1 \in \mathcal{S}_x$, then $|\mathcal{S}_{[x,1]}| = |\mathcal{S}_x|$; if $x_1 = 1$ and $-1 \notin \mathcal{S}_x$, then $|\mathcal{S}_{[x,1]}| = 1 + |\mathcal{S}_x|$ (the extra element in $\mathcal{S}_{[x,1]}$ is -1). Applying this observation to Step 3 we see that $r \in \{2, 4, 5, 6, 8, 9, 10\}$ can be achieved for all $n \ge 4$. Coupled with Step 7 we get the conclusion.

Step 9. We claim that, for all integers $n \ge 5$ and $2 \le r \le 4n-4$, $r \ne 3, 7$, there exists a vector $x \in \mathbb{C}^n$ such that $x_1 = 1$, $-1 \in \mathcal{S}_x$, and $|\mathcal{S}_x| = r$.

The proof is by induction on n. Let

$$u_{11} = [1, 1, e^{i\pi/10}, e^{i\pi/10}, e^{i\pi/4}], \quad u_{12} = [1, 1, e^{i\pi/6}, e^{i\pi/3}, i].$$

Then $|\mathcal{S}_{u_k}| = k$ and $-1 \in \mathcal{S}_{u_k}$ for $k = 11, 12$. Applying the observation in Step 8 to Step 7 shows that the assertion holds for $n = 5$, $13 \le r \le 17$. Thus the claim holds for $n = 5$.

Now suppose the claim holds for $n = m \ge 5$. Applying the observation in Step 8 the assertion holds for $n = m+1$ and $2 \le r \le 4m-4$. Applying the observation in Step 8 to Step 7 the assertion holds for $n = m+1$ and $4m-3 \le r \le m^2+1$. Note that $m^2+1 \ge 4(m+1)-4$ since $m \ge 5$. Thus the claim holds for $n = m+1$.

Together Step 7 and Step 9 show that the theorem holds for $n \ge 5$. □

3. Unitary congruence invariant norms

In this section, we consider unitary congruence invariant (uci) norms on square matrices, i.e., $\|A\| = \|UAU^t\|$ for any unitary matrix U. It is natural to consider such norms on the space of symmetric matrices and the space of skew-symmetric matrices (since these are reducing subspaces for the map $A \mapsto UAU^t$), and the space of general square matrices when a matrix $A \in M_n$ is viewed as a bilinear form $(x, y) \mapsto x^t Ay$. As we will see, the structure of r-circular projections for uci norms is more intricate. The case $r = 3$ was studied in [1].

To facilitate our discussion, let S_n (respectively K_n) denote the space of complex $n \times n$ symmetric (respectively skew-symmetric) matrices. To avoid the trivialities of one-dimensional spaces we assume $n \geq 2$ for S_n, M_n, and $n \geq 3$ for K_n. Let V be one of M_n, S_n, or K_n. Then a uci norm $\|\cdot\| : V \to \mathbb{R}$ is a norm for which the operator T_U, defined by $T_U(A) = UAU^t$, is an isometry whenever U is unitary.

As mentioned before, it is necessary to understand the isometries of a given norm to determine its r-circular projections. By [16, Theorem 2.8], an isometry for a uci norm on S_n that is not an inner product norm has the form

(S1) $A \mapsto UAU^t$ for some unitary $U \in M_n$.

By [16, Theorem 2.9], an isometry for a uci norm on K_n that is not an inner product norm has the form

(K1) $A \mapsto UAU^t$ for some unitary $U \in M_n$, or
(K2) $A \mapsto \psi(UAU^t)$ for some unitary $U \in M_4$ and $\psi(X)$ is obtained from X by interchanging its $(1,4)$ and $(2,3)$ entries, as well as its $(4,1)$ and $(3,2)$ entries, in the case $n = 4$.

By [12, Theorem 7.3], if a uci norm on M_n is not ui, then an isometry has the form

$$A \mapsto T_1(A + A^t)/2 + T_2(A - A^t)/2, \quad \text{where}$$

(M1) T_1 is a unitary operator on S_n, or T_1 has the form $A \mapsto VAV^t$ for some unitary $V \in M_n$,
(M2) T_2 is a unitary operator on K_n, or T_2 has the form (K1) or (K2) above.

Note that if T_1 has the form $X \mapsto UXU^t$ and T_2 has the form $Y \mapsto UYU^t$ for the same unitary $U \in M_n$, then T has the form $A \mapsto UAU^t$. If T_1 has the form $X \mapsto UXU^t$ for some unitary $U \in M_n$ and T_2 has the form $Y \mapsto (iU)Y(iU)^t = -UYU^t$, then T has the form $A \mapsto UA^tU^t$.

Similar to the situation for ui norms, one can generate "all" r-circular projections for a uci norm $\|\cdot\|$ on S_n, K_n, M_n, respectively, by studying all the eigenvalues and eigenprojections of the isometries of the norm $\|\cdot\|$.

On the other hand, given a family of projections $\{P_1, \ldots, P_r\}$, one may use the matrix representations of $P_1, \ldots, P_r$ with respect to the orthonormal basis

$$\mathcal{B}_1 = \{E_{jj} : 1 \leq j \leq n\} \cup \{\tfrac{1}{\sqrt{2}}(E_{ij} + E_{ji}) : 1 \leq i < j \leq n\},$$
$$\mathcal{B}_2 = \{\tfrac{1}{\sqrt{2}}(E_{ij} - E_{ji}) : 1 \leq i < j \leq n\}, \quad \text{and} \quad \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2,$$

for S_n, K_n, and M_n respectively. Similar to the discussion in Section 2, if $\mathcal{P}_1, \ldots, \mathcal{P}_r$ are the matrix representations of the projections $P_1, \ldots, P_r$ with respect to the orthonormal basis $\mathcal{B}_1, \mathcal{B}_2$ or $\mathcal{B}_1 \cup \mathcal{B}_2$ depending on the underlying space, one only needs to check whether there are complex units $\mu_1, \ldots, \mu_r$ such that $\mu_1\mathcal{P}_1 + \cdots + \mu_r\mathcal{P}_r$ corresponds to a matrix representation of an isometry T for the uci norm $\|\cdot\|$. It is worth noting that given a unitary $U \in M_n$, the matrix representation of the map $A \mapsto UAU^t$ with respect to the basis $\mathcal{B}_1 \cup \mathcal{B}_2$ has the form

$$\begin{pmatrix} U \otimes U|_{S_n} & \\ & U \otimes U|_{K_n} \end{pmatrix},$$

where $U \otimes U|_{S_n} \in M_{n(n+1)/2}$ and $U \otimes U|_{K_n} \in M_{n(n-1)/2}$ are the matrix representations of the restrictions of $U \otimes U$ on the subspace S_n and K_n with respect to the bases $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Moreover, the following statements hold.

(a) For a uci norm on S_n, $\{P_1, \dots, P_r\}$ is a family of r-circular projections if and only if there are complex units $\mu_1, \dots, \mu_r$ and unitary $U \in M_n$ such that $\mu_1 \mathcal{P}_1 + \cdots + \mu_r \mathcal{P}_r = U \otimes U|_{S_n}$.

(b) For a uci norm on K_n, $\{P_1, \dots, P_r\}$ is a family of r-circular projections if and only if there are complex units $\mu_1, \dots, \mu_r$ and unitary $U \in M_n$ such that

(b.1) $\mu_1 \mathcal{P}_1 + \cdots + \mu_r \mathcal{P}_r = U \otimes U|_{K_n}$, or

(b.2) $n = 4$ and $\mu_1 \mathcal{P}_1 + \cdots + \mu_r \mathcal{P}_r = (U \otimes U|_{K_4})Q$, where Q is the matrix representation of ψ with respect to the basis $\mathcal{B}_2$. Note that $Q \in M_6$ is the permutation matrix corresponding to the exchange of the basic vectors $E_{14} - E_{41}$ and $E_{23} - E_{32}$.

(c) For a uci norm on M_n which is not ui, $\{P_1, \dots, P_r\}$ is a family of r-circular projections if and only if there are complex units $\mu_1, \dots, \mu_r$ and unitary $U \in M_n$ such that $\mu_1 \mathcal{P}_1 + \cdots + \mu_r \mathcal{P}_r = \mathcal{T}_1 \oplus \mathcal{T}_2$ such that

(c.1) $\mathcal{T}_1 \in M_{n(n+1)/2}$ is a unitary operator or $\mathcal{T}_1 = U \otimes U|_{S_n}$ for some unitary $U \in M_n$, and

(c.2) $\mathcal{T}_2 \in M_{n(n-1)/2}$ is a unitary operator, $\mathcal{T}_2 = V \otimes V|_{K_n}$ for some unitary $V \in M_n$, or $n = 4$ and $\mathcal{T}_2$ has the form in (b.2).

In the following lemma, we obtain additional information about the structure of the eigenvalues and eigenprojections of maps of the form $A \mapsto UAU^t$ on S_n, K_n, and M_n. The results allow us to determine r for the existence of r-circular projections.

Lemma 3.1. *Let $U \in M_n$ be unitary with eigenvalues $\mu_1, \dots, \mu_n$ and corresponding orthonormal eigenvectors $u_1, \dots, u_n$. Then the map $T_U \colon A \mapsto UAU^t$ has*

1. *eigenvalues $\{\mu_i \mu_j : 1 \leq i, j \leq n\}$ (with corresponding eigenvectors $u_i u_j^t$) when T_U acts on M_n;*
2. *eigenvalues $\{\mu_i \mu_j : 1 \leq i \leq j \leq n\}$ (with corresponding eigenvectors $u_i u_j^t + u_j u_i^t$) when T_U acts on S_n; and*
3. *eigenvalues $\{\mu_i \mu_j : 1 \leq i < j \leq n, i \neq j\}$ (with corresponding eigenvectors $u_i u_j^t - u_j u_i^t$) when T_U acts on K_n.*

Proof. Direct verification. □

Thus there is an r-circular projection associated to an isometry T_U (with U unitary) acting on M_n or S_n (respectively K_n) if and only if there exist complex units $\mu_1, \dots, \mu_n$ such that

$$\mathcal{S}_S = \{\mu_i \mu_j : 1 \leq i, j \leq n\} \qquad \text{(respectively } \mathcal{S}_K = \{\mu_i \mu_j : 1 \leq i < j \leq n\}) \tag{3.3}$$

has exactly r elements.

Theorem 3.2. *Let $\|\cdot\|$ be a uci norm on S_n that is not an inner product norm, where $n \geq 2$. A family of operators $\{P_1, \dots, P_r\}$ is a family of generalized circular projections if and only if there exist complex units $\mu_1, \dots, \mu_p$ with $\{\mu_i \mu_j : 1 \leq i \leq j \leq p\} = \{\lambda_1, \dots, \lambda_r\}$, and orthogonal projections $U_1, \dots, U_p \in M_n$ with $\sum_{j=1}^p U_j = I_n$, such that for $j = 1, \dots, r$,*

$$P_j(A) = \sum_{\mu_i \mu_k = \lambda_j} \frac{1}{2}(U_i A U_k^t + U_k A U_i^t) \quad \textit{for all } A \in S_n.$$

Moreover, there is a family of r-circular projections corresponding to the norm $\|\cdot\|$ if and only if $r \in [2, n(n+1)/2]$.

Proof. Since every isometry for $\|\cdot\|$ has the form $A \mapsto UAU^t$ for some unitary $U \in M_n$, the first statement follows readily from Lemma 3.1.

For the last statement, the necessity is clear because $\dim S_n = n(n+1)/2$. To prove the sufficiency, we first establish the following.

Claim. *Let r, n be integers satisfying $n \geq 2$, $2n-1 \leq r \leq n(n+1)/2$. There exist distinct integers $0 = a_1 < \cdots < a_n$ such that the cardinality of $\{a_i + a_j : 1 \leq i, j \leq n\}$ equals r.*

The proof is by induction on n. Setting $a_1 = 0$, $a_2 = 1$ proves the claim for $n = 2$.

Suppose the claim is true for $n = k$. Choose integers $0 = a_1 < \cdots < a_k$ so that the cardinality of $\{a_i + a_j : 1 \leq i, j \leq k\}$ is c. Choose $a_{k+1} = 2a_k + 1$. Then the cardinality of $\{a_i + a_j : 1 \leq i, j \leq k+1\}$ is $c + k + 1$. By the induction hypothesis $\{a_i + a_j : 1 \leq i, j \leq k+1\}$ can have cardinalities ranging from $3k$ to $k(k+1)/2 + k + 1 = (k+1)(k+2)/2$. But we can also attain cardinalities from $2(k+1) - 1$ to $3k$ by choosing $a_j = j - 1$ for $j = 1, \ldots, k$ and $a_{k+1} = k + d$ and letting d range from 0 to $k-1$, so by induction the claim is proven.

To finish the proof of the theorem, we exhibit a set of the form $\mathcal{S}_S$ in (3.3) of size r. First let $2n-1 \leq r \leq n(n+1)/2$. By the claim above, we can choose integers $0 = a_1 < \cdots < a_n$ so that $\{a_i + a_j : 1 \leq i, j \leq n\}$ has r distinct elements. Let $\mu_j = e^{i\pi a_j/3a_n}$. Then $\{\mu_i\mu_j : 1 \leq i, j \leq n\}$ has size r.

Now let $2 \leq r < 2n - 1$. Let $\mu_j = e^{i2\pi(j-1)/r}$. Then $\{\mu_i\mu_j : 1 \leq i, j \leq n\}$ consists of the complete set of rth roots of unity, and hence has size r. □

Theorem 3.3. *Let $\|\cdot\|$ be a uci norm on K_n that is not an inner product norm, where $n > 2$. A family of operators $\{P_1, \ldots, P_r\}$ is a family of generalized circular projections if and only if one of the following holds.*

(a) *There exist complex units $\mu_1, \ldots, \mu_p$ with $\{\mu_i\mu_j : 1 \leq i < j \leq p\} = \{\lambda_1, \ldots, \lambda_r\}$, and orthogonal projections $U_1, \ldots, U_p \in M_n$ with $\sum_{j=1}^{p} U_j = I_n$, such that for $j = 1, \ldots, r$,*

$$P_j(A) = \sum_{\mu_i\mu_k = \lambda_j} \frac{1}{2}(U_i A U_k^t + U_k A U_i^t) \quad \text{for all } A \in K_n.$$

Moreover, there is a family of r-circular projections corresponding to an isometry of the form $A \mapsto UAU^t$ on K_n if and only if $r \in [2, n(n-1)/2]$.

(b) *$n = 4$ and $\{P_1, \ldots, P_r\}$ correspond to the eigenprojections of a map of the form $A \mapsto \psi(UAU^t)$ for some unitary $U \in M_4$. Moreover, there is a family of r-circular projections corresponding to an isometry of the form $A \mapsto \psi(UAU^t)$ if and only if $r \in \{2, 4, 6\}$.*

Proof. Since every isometry for $\|\cdot\|$ has the form (K1) or (K2), the first statements of (a) and (b) follow readily from Lemma 3.1.

For the second statement of (a), the necessity is clear because $\dim K_n = n(n-1)/2$. To prove the sufficiency, we establish the following.

Claim. *Let r, n be integers satisfying $n \geq 3$, $2n-3 \leq r \leq n(n-1)/2$. There exist distinct integers $0 = a_1 < \cdots < a_n$ such that the cardinality of $\{a_i + a_j : 1 \leq i, j \leq n, i \neq j\}$ equals r.*

We prove the claim by induction on n. Setting $a_1 = 0$, $a_2 = 1$, $a_3 = 2$ proves the claim for $n = 3$.

Suppose the claim is true for $n = k$. Choose integers $0 = a_1 < \cdots < a_k$ so that the cardinality of $\{a_i + a_j : 1 \leq i, j \leq k, i \neq j\}$ is c. Choose $a_{k+1} = 2a_k$. Then the cardinality of $\{a_i + a_j : 1 \leq i, j \leq k+1, i \neq j\}$

is $c+k$. By the induction hypothesis $\{a_i + a_j : 1 \le i, j \le k+1, i \ne j\}$ can have cardinalities ranging from $3k-3$ to $k(k-1)/2 + k = k(k+1)/2$. But we can also attain cardinalities from $2(k+1)-3$ to $3k-3$ by choosing $a_j = j-1$ for $j = 1, \ldots, k$ and $a_{k+1} = k+d$ and letting d range from 0 to $k-2$, so by induction the claim is proven.

Now, we finish the proof of the second statement of (a). We will exhibit a set of the form $\mathcal{S}_K$ in (3.3) of size r. First let $2n-3 \le r \le n(n-1)/2$. By the above claim, we can choose integers $0 = a_1 < \cdots < a_n$ so that $\{a_i + a_j : 1 \le i, j \le n, i \ne j\}$ has r distinct elements. Let $\mu_j = e^{i\pi a_j/3a_n}$. Then $\{\mu_i\mu_j : 1 \le i, j \le n, i \ne j\}$ has size r.

Now let $2 \le r < 2n-3$. Let $\mu_j = e^{i2\pi(j-1)/r}$. Then $\{\mu_i\mu_j : 1 \le i, j \le n, i \ne j\}$ consists of the complete set of rth roots of unity, and hence has size r.

Next, we turn to the last statement of (b). For $U \in M_4$ unitary we define T on K_4 by $T(A) = \psi(UAU^t)$. If $U = \mathrm{diag}(\mu_1, \mu_2, \mu_3, \mu_4) \in M_4$, then T has eigenvalues

$$\mathcal{S} = \{\mu_1\mu_2, \mu_1\mu_3, \sqrt{\det(U)}, -\sqrt{\det(U)}, \mu_2\mu_4, \mu_3\mu_4\}$$

and corresponding eigenvectors

$$\{e_1 \wedge e_2, e_1 \wedge e_3, \tfrac{1}{\sqrt{2}}(\sqrt{\mu_2\mu_3}\, e_1 \wedge e_4 + \sqrt{\mu_1\mu_4}\, e_2 \wedge e_3), \tfrac{1}{\sqrt{2}}(\sqrt{\mu_2\mu_3}\, e_1 \wedge e_4 - \sqrt{\mu_1\mu_4}\, e_2 \wedge e_3), e_2 \wedge e_4, e_3 \wedge e_4\},$$

where $e_j \wedge e_k = e_j e_k^t - e_k e_j^t$. Now, for $U = \mathrm{diag}(1,1,1,1)$ the set $\mathcal{S}$ has 2 distinct values; for $U = \mathrm{diag}(1, i, -1, -i\}$ the set $\mathcal{S}$ has 4 distinct values; for $U = \mathrm{diag}(1, w, w^2, w^3)$ with $w = e^{i2\pi/6}$ the set $\mathcal{S}$ has 6 distinct values.

It remains to show that $r = 3, 5$ are impossible. We claim that, up to a unit multiple, T has eigenvalues of the form $1, -1, \alpha, \bar{\alpha}, \beta, \bar{\beta}$, where α, β may be real or complex, whence T cannot have 3 or 5 distinct eigenvalues. To prove our claim, assume $U \in M_4$ is unitary. We may replace U by $U/\det(U)^{1/4}$ and assume that $\det(U) = 1$. Let $P, R \in M_6$ be the matrices representing the transformations $A \mapsto \psi(A)$ and $A \mapsto UAU^t$, respectively, acting on K_4 with respect to the lexicographically ordered basis $\{e_r \wedge e_s = e_r e_s^t - e_s e_r^t : 1 \le r < s \le 4\}$. Then P is obtained from I_6 by interchanging columns 3 and 4, and $R = C_2(U)$ is the 2-compound matrix of U. Note that the entries of R are given by $\det(U(\alpha, \beta))$, where $U(\alpha, \beta) \in M_2$ is the submatrix of U with rows and columns indexed by the entries of α and β, where $\alpha, \beta \in \{(j_1, j_2) : 1 \le j_1 < j_2 \le 4\}$.

By Jacobi's theorem on complementary minors (e.g. see [13, Section 0.8.4]), if $U(\alpha', \beta')$ is the complementary submatrix of $U(\alpha, \beta)$, then $\overline{\det(U(\alpha, \beta))} = (-1)^{\alpha_1+\alpha_2+\beta_1+\beta_2} \det(\alpha', \beta')$. Consequently, we have $\overline{R} = DRD$ with $D = E_{16} + E_{61} + E_{34} + E_{43} - E_{25} - E_{52}$, and thus $\overline{PR} = (DPD)(DRD) = DPRD$. Hence $\overline{PR}$ is unitarily similar to PR, so the complex eigenvalues of PR occur in conjugate pairs. Since $\det(PR) = -1$, we see that $T = PR$ has eigenvalues $1, -1, \alpha, \bar{\alpha}, \beta, \bar{\beta}$. Our claim follows. □

Note that we do not give a general description of the structure of $P_1, \ldots, P_r$ in (b). Nevertheless, if T has the form $A \mapsto \psi(UAU^*)$, it is easy to write down the matrix representation of T as a unitary matrix in M_6, and determine its eigenvalues and eigenvectors.

Finally, we turn to the generalized circular projections for a uci norm on M_n that is not ui. To put our result into perspective, we note that for a ui norm on M_n, its isometry group consists of operators of the form $A \mapsto UAV$ or $A \mapsto UA^tV$ for some unitary matrices $U, V \in M_n$. So, in Section 2 we have determined all values r for the existence of a family of r-circular projections according to these two types of isometries. The situation for a uci norm on M_n is different; we have the following three possibilities for the isometry group (see [12,16] and references therein).

(a) The isometry group consists of operators of the form $A \mapsto UAU^t$ for a unitary matrix $U \in M_n$.

(b) The isometry group consists of operators of the form $A \mapsto UAU^t$ or $A \mapsto UA^tU^t$ for some unitary $U \in M_n$.

(c) The isometry group contains an element of the form $A \mapsto [(A + A^t) + W(A - A^t)W^t]/2$ for some non-scalar unitary $W \in M_n$, in which case the isometry group contains all the maps of the form $A \mapsto [U(A + A^t)U^t + V(A - A^t)V^t]/2$ for any unitary $U, V \in M_n$.

In case (c) it is possible that the isometry group contains many other elements, such as operators of the form $A \mapsto [T_1(A + A^t) + T_2(A - A^t)]/2$ for any unitary operators T_1 and T_2 on S_n and K_n, respectively. In these cases one can easily construct r-circular projections with r within a certain desired range as noted in Proposition 1.3. Hence, we will focus on r-circular projections corresponding to isometries of the forms in (a)–(c).

Note also that not much is said about the structure of r-circular projections in Theorem 3.4 for the following reason. Suppose that a family of r-circular projections $\{P_1, \dots, P_r\}$ corresponds to the isometry $A \mapsto T_1(A + A^t)/2 + T_2(A - A^t)/2$. We can always assume that $P_j(S_n) \subseteq S_n$ and $P_j(K_n) \subseteq K_n$ for $j = 1, \dots, r$. If T_j is a unitary operator for $j \in \{1, 2\}$, then the structure of $P_1, \dots, P_r$ is quite liberal as shown in Proposition 1.3. Otherwise, Theorems 3.2 and 3.3 can be used to determine the structure of $P_1, \dots, P_r$. We are now ready to state and prove the following.

Theorem 3.4. *Let $\|\cdot\|$ be a uci norm on M_n that is not ui. Then $\{P_1, \dots, P_r\}$ is a family of generalized circular projections if and only if $P_1, \dots, P_r$ are the eigenprojections of a map of the form $A \mapsto T_1(A + A^t)/2 + T_2(A - A^t)/2$, corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_r$, where T_1, T_2 are described in* (M1), (M2) *at the beginning of this section.*

(a) *There is a family of r-circular projections corresponding to an isometry of the form $A \mapsto UAU^t$ for a unitary $U \in M_n$ if and only if $r \in [2, n(n + 1)/2]$.*

(b) *There is a family of r-circular projections corresponding to an isometry of the form $A \mapsto UA^tU^t$ for a unitary $U \in M_n$ if and only if $r \in [2, n^2] \setminus J_n$, where $J_4 = \{3, 7, 11\}$ and $J_n = \{3, 7\}$ for all other $n \geq 2$.*

(c) *There is a family of r-circular projections corresponding to an isometry of the form $A \mapsto U(A + A^t)U^t/2 + V(A - A^t)V^t/2$ for some unitary $U, V \in M_n$ if and only if $r \in [2, n^2]$.*

Proof. The first statement is clear. We turn to the proof of statements (a)–(c).

(a) From Lemma 3.1, an isometry of the form $A \mapsto UAU^t$ has at most $n(n + 1)/2$ eigenvalues (namely $\mathcal{S} = \{\mu_i\mu_j : 1 \leq i \leq j \leq n\}$, where $\mu_1, \dots, \mu_n$ are the eigenvalues of U), proving the necessity. For the sufficiency, Theorem 3.2 shows that for any $r \in [2, n(n + 1)/2]$ there exists a unitary U with $|\mathcal{S}| = r$, so the result holds.

(b) The eigenvalues of a map T of the form $T(A) = UA^tU^t$ are characterized by Lemma 2.1(b), with $V = U^t$. Note that the eigenvalues of T depend only on the eigenvalues of $UV^t = U^2$, so the restriction $V = U^t$ poses no additional constraints. Thus we may apply Theorem 2.3(b) to obtain the desired conclusion.

(c) If $r \in [2, n(n + 1)/2]$ we can, by part (a), find a unitary U such that the map $A \mapsto UAU^t$ has r distinct eigenvalues; the result follows by taking $V = U$. For larger r, we may write $r = r_1 + r_2$, with $r_1 \in [2, n(n + 1)/2]$ and $r_2 \in [2, n(n - 1)/2]$. By Theorem 3.2 there exists a unitary U such that the map $T_1 : S_n \to S_n$, defined by $T_1(X) = UXU^t$, has r_1 distinct eigenvalues; by Theorem 3.3 there exists a unitary V such that the map $T_2 : K_n \to K_n$, defined by $T_2(Y) = VYV^t$, has r_2 distinct eigenvalues. By replacing V with $e^{i\theta}V$ for an appropriate choice of θ we may assume the eigenvalues of T_1 and T_2 are disjoint. Then the map $T_1 \oplus T_2 : M_n \to M_n$ has r distinct eigenvalues and the result follows. □

4. Unitary similarity invariant norms

A norm $\|\cdot\|$ on M_n is unitary similarity invariant if $\|UAU^*\| = \|A\|$ for all unitary $U \in M_n$. This is a broad family of norms which includes the unitarily invariant norms, but also other important norms like the numerical radius, as well as hybrids like the sum of the spectral radius with the operator norm. For unitary similarity invariant (usi) norms which are not unitarily invariant (ui), let G be the group of maps on M_n of the form $A \mapsto UAU^*$, where U is unitary. There are two irreducible subspaces of the group action: the set of scalar matrices $\mathbb{C}I_n$, and the set of trace-zero matrices $M_n^0 = \{A \in M_n : \operatorname{tr} A = 0\}$. We can also regard G as a group of real orthogonal operators acting on H_n, the real linear space of Hermitian matrices. By [12, Theorem 6.2] (see also [16, Theorem 2.7]), we have the following.

- If $\|\cdot\|$ is a usi norm on H_n that is not induced by an inner product, then its isometries have the form

$$A \mapsto \pm(\operatorname{tr} A)I/n \pm T_0(A - (\operatorname{tr} A)I/n),$$

  where T_0 is an orthogonal operator on $\{A \in H_n : \operatorname{tr} A = 0\}$, or T_0 has one of the forms

$$A \mapsto UAU^* \qquad \text{or} \qquad A \mapsto UA^tU^*.$$

- If $\|\cdot\|$ is a usi norm on M_n that is not ui, then its isometries have the form

$$A \mapsto \alpha(\operatorname{tr} A)I/n + \beta T_0(A - (\operatorname{tr} A)I/n),$$

  where α, β are complex units, and one of the following holds for T_0.
  (1) T_0 is a unitary operator on $M_n^0 = \{A \in M_n : \operatorname{tr} A = 0\}$.
  (2) T_0 has the form

$$A \mapsto UAU^* \qquad \text{or} \qquad A \mapsto UA^tU^*.$$

  (3) $n = 4$ and $T_0 \in \Lambda$, the group of operators acting on $M_4^0 = \{A \in M_4 : \operatorname{tr} A = 0\}$ that is isomorphic to $U(6)$ acting on K_6 through the action $A \mapsto UAU^t$ via the following identification (e.g., see [22]). Let

$$\sigma_0 = I_2, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  Let $\sigma^{ij} = \sigma_i \otimes \sigma_j$ for $0 \le i, j \le 3$, and consider the orthonormal basis

$$\mathcal{C} = \{\sigma^{ij} = \sigma_i \otimes \sigma_j : 0 \le i, j \le 3\} \setminus \{\sigma^{00}\}$$

  of the real linear space $H_4^0 = \{A \in H_4 : \operatorname{tr} A = 0\}$. Denote by $A_{ij} = E_{ij} - E_{ji} \in K_6$, and define a linear map L from H_4^0 to K_6 by sending

$$\begin{aligned} &(\sigma^{01}, \sigma^{02}, \sigma^{03}, \sigma^{10}, \sigma^{11}, \sigma^{12}, \sigma^{13}, \sigma^{20}, \sigma^{21}, \sigma^{22}, \sigma^{23}, \sigma^{30}, \sigma^{31}, \sigma^{32}, \sigma^{33}) \\ \text{to} \quad &(A_{24}, A_{46}, A_{26}, A_{15}, A_{36}, A_{32}, A_{43}, A_{13}, A_{65}, A_{25}, A_{54}, A_{35}, A_{61}, A_{21}, A_{14}). \end{aligned}$$

  Then L is a real orthogonal map preserving the Lie product, i.e., $L(AB - BA) = L(A)L(B) - L(B)L(A)$. Moreover, a linear map of the form $A \mapsto UAU^*$ on H_4^0 corresponds to a linear map $\tilde{U}L(A)\tilde{U}^t$ on the real linear span of $\{A_{ij} : 1 \le i < j \le 6\} \subseteq K_6$. As a result, a linear map of the form $A \mapsto UAU^*$ on M_4^0 corresponds to a linear map $\tilde{U}L(A)\tilde{U}^t$ on K_6, and is an element in Λ. Of course, Λ contains many other elements.

Once we know the isometry group of a given usi norm $\|\cdot\|$ on M_n, we can generate all the r-circular projections by studying the eigenvalues and eigenprojections of its isometries as before. Given a family of projections $\{P_1, \dots, P_r\}$ we can use their matrix representations $\mathcal{P}_1, \dots, \mathcal{P}_r$ to check whether it is a family of r-circular projections. The trickiest case is to check whether $\{P_1, \dots, P_r\}$ is associated with an isometry $T \in \Lambda$. In such a case, we have to embed the projections as maps on K_6 and then determine whether it corresponds to a map $A \mapsto UAU^t$ for a unitary $U \in M_6$ with r distinct eigenvalues.

In the following, we investigate the structure of r-circular projections for usi norms, and determine all values r such that r-circular projections exist for a given usi norm. We start with a simple observation.

Lemma 4.1. *Let $U \in M_n$ be unitary.*

(a) *Define $T\colon M_n \to M_n$ by $T(A) = UAU^*$. Suppose U has orthonormal eigenvectors $u_1, \dots, u_n$ with corresponding eigenvalues $\mu_1, \dots, \mu_n$. Then T has orthonormal eigenvectors $u_i \otimes \overline{u}_j = u_i u_j^*$ with corresponding eigenvalues $\mu_i \overline{\mu}_j$ for $1 \le i, j \le n$.*

(b) *Define $T\colon M_n \to M_n$ by $T(A) = UA^tU^*$. Suppose $U\overline{U}$ has orthonormal eigenvectors $x_1, \dots, x_n$ with corresponding eigenvalues $\xi_1^2, \dots, \xi_n^2$. Then T has orthonormal eigenvectors*

$$v_{jj} = x_j \otimes U^* x_j, \ j = 1, \dots, n,$$

(with corresponding eigenvalues ξ_j^2) and

$$v_{jk}^{\pm} = \frac{1}{\sqrt{2}}(\xi_k x_j \otimes U^* x_k \pm \xi_j \, x_k \otimes U^* x_j), \ 1 \le j < k \le n$$

(with corresponding eigenvalues $\pm \xi_j \xi_k$).

Proof. Apply Lemma 2.1 with $V = U^*$ and write ξ_j for $\sqrt{\mu_j}$ in part (b). □

Theorem 4.2. *Suppose $\|\cdot\|$ is a usi norm on M_n that is not induced by an inner product. Then $\{P_1, \dots, P_r\}$ is a family of generalized circular projections if and only if they are the eigenprojections of a map of the form*

$$A \mapsto \alpha(\operatorname{tr} A)I/n + \beta T_0(A - (\operatorname{tr} A)I/n),$$

where α, β are complex units and T_0 is either a unitary map on $M_n^0 = \{A \in M_n : \operatorname{tr} A = 0\}$, $T_0 \in \Lambda$, or T_0 is a map of the form

$$A \mapsto UAU^* \qquad \text{or} \qquad A \mapsto UA^tU^*.$$

(a) *There is a family of r-circular projections corresponding to an isometry of the form $A \mapsto UAU^*$ if and only if $r \in [2, n^2 - n + 1]$.*

(b) *There is a family of r-circular projections corresponding to an isometry of the form $A \mapsto UA^tU^*$ if and only if r is even and $r \in [2, n^2 - n + 2]$.*

(c) *There is a family of r-circular projections corresponding to an isometry of the form $A \mapsto \alpha(\operatorname{tr} A)I/n + T_0(A - (\operatorname{tr} A)I/n)$ for a complex unit α and a unitary operator T_0 on M_n^0 if and only if $r \in [2, n^2]$. Here, for the sufficiency, one may choose T_0 to be a real orthogonal map leaving H_n^0 invariant.*

(d) *There is a family of r-circular projections corresponding to an isometry of the form $A \mapsto \alpha(\operatorname{tr} A)I/4 + T_0(A - (\operatorname{tr} A)I/4)$ on M_4 with $T_0 \in \Lambda$ if and only if $r \in [2, 16]$, which equals $[2, n^2]$ with $n = 4$.*

We do not say much about the structure of the r-circular projections. In fact, if one sees that the r-circular projections correspond to a certain type of isometry, it is not difficult to determine the structure of the r-circular projections with the help of Proposition 1.2 and Lemma 4.1. Our proof will focus on the possible values of r in (a)–(d). To obtain the result in (a), we need the following.

Lemma 4.3. *Let r, n be integers satisfying $n \geq 2$, r is odd, and $2n-1 \leq r \leq n(n-1)+1$. There exist distinct integers $0 = a_1 < \cdots < a_n$ such that $\{a_i - a_j : 1 \leq i, j \leq n\}$ has r elements.*

Proof. Setting $a_1 = 0$, $a_2 = 1$ proves the lemma for $n = 2$.

Suppose the lemma is true for $n = k$. If $0 = a_1 < \cdots < a_k$ and $\{a_i - a_j : 1 \leq i, j \leq k\}$ has c elements, then choosing $a_{k+1} = 2a_k + 1$ results in $\{a_i - a_j : 1 \leq i, j \leq k+1\}$ having $c + 2k$ elements. By the induction hypothesis, the lemma holds when $n = k+1$ and $4k - 1 \leq r \leq (k+1)k + 1$ with r odd. On the other hand, one may choose $a_j = j - 1$ for $j = 1, \ldots, k$ and $a_{k+1} = k + d$, where $0 \leq d \leq k - 1$. Then

$$\{a_i - a_j : 1 \leq i, j \leq k+1\} = \{0, \pm 1, \ldots, \pm(k+d)\}$$

has $2(k+d)+1$ elements, so the lemma holds for $n = k+1$ and $2k+1 \leq r \leq 4k-1$ with r odd, and the result holds by induction. □

Proof of Theorem 4.2. (a) Note that 1 is an eigenvalue of multiplicity at least n for the map $A \mapsto UAU^*$, so $r \leq n^2 - n + 1$. To prove sufficiency, we consider two cases.

i) Suppose $2n-1 \leq r \leq n(n-1)+1$. If r is odd then, by Lemma 4.3, there exist integers $0 = a_1 < \cdots < a_n$ such that $\Omega = \{a_i - a_j : 1 \leq i, j \leq n\}$ has r elements. Let $\mu_j = e^{i\pi a_j / 2a_n}$. Then $\{\mu_i \bar{\mu}_j : 1 \leq i, j \leq n\}$ has r elements. If r is even let $s = r + 1$ and choose integers $0 = a_1 < \cdots < a_n$ such that Ω has s elements. Let $\mu_j = e^{i\pi a_j / a_n}$. Then $\{\mu_i \bar{\mu}_j : 1 \leq i, j \leq n\}$ has r elements. Let $D = \operatorname{diag}(\mu_1, \ldots, \mu_n)$. By Lemma 4.1(a) the map $A \mapsto DAD^*$ has r distinct eigenvalues.

ii) Suppose $2 \leq r \leq 2n - 1$. Let $\mu_j = e^{2\pi i (j-1)/r}$. Then $\{\mu_i \bar{\mu}_j : 1 \leq i, j \leq n\}$ consists of the complete set of rth roots of unity, and thus has size r. Let $D = \operatorname{diag}(\mu_1, \ldots, \mu_n)$. By Lemma 4.1(a) the map $A \mapsto DAD^*$ has r distinct eigenvalues.

(b) By Lemma 4.1(b) there exists a family of r-circular projections corresponding to an isometry of the form $A \mapsto UA^tU^*$ if and only if there exists a unitary U such that $\{\xi_1^2, \ldots, \xi_n^2\}$ are the eigenvalues of $U\overline{U}$ and

$$\mathcal{S} = \{\xi_1^2, \ldots, \xi_n^2, \pm \xi_j \xi_k : 1 \leq j < k \leq n\}$$

has exactly r elements.

Since $\sigma(U\overline{U}) = \sigma(\overline{U}U) = \overline{\sigma(U\overline{U})}$, the spectrum of $U\overline{U}$ is symmetric about the real axis; moreover, $\det U\overline{U} = 1$. Conversely, any finite set of unimodular complex numbers that is invariant under complex conjugation and has product equal to 1 can be realized as the spectrum of $U\bar{U}$; for example, take U to be a direct sum of $\begin{bmatrix} 0 & e^{it} \\ 1 & 0 \end{bmatrix}$ for appropriate values of $t \in [0, \pi]$, and a copy of $[1]$ if n is odd. Thus r-circular projections exist in this case if and only if there exists a set of unimodular complex numbers $\{\xi_1^2, \ldots, \xi_n^2\}$ that is invariant under complex conjugation, has product equal to 1, and is such that $\mathcal{S}$ has exactly r elements.

In particular we may assume that $\{\xi_1, \ldots, \xi_n\}$ may be divided into pairs of complex conjugates, with an extra 1 if n is odd. Thus $\mathcal{S}$ is invariant under complex conjugation and always contains $\{-1, 1\}$, so $\mathcal{S}$ always has an even number of elements.

If $n = 2m$ we may write $\{\xi_1, \ldots, \xi_n\} = \{\lambda_1, \ldots, \lambda_m, \overline{\lambda}_1, \ldots, \overline{\lambda}_m\}$, so

$$\mathcal{S} \subseteq \{\lambda_j^2, \overline{\lambda}_j^2 : 1 \leq j \leq m\} \cup \{\pm 1, \pm\lambda_j\lambda_k, \pm 1/(\lambda_j\lambda_k), \pm\lambda_j/\lambda_k : 1 \leq j, k \leq m, j \neq k\}.$$

The sets on the right-hand side have at most $2m + (2 + 4m(m-1)) = 4m^2 - 2m + 2 = n^2 - n + 2$ distinct elements. If $N = 2m + 1$ we simply adjoin $\xi_{n+1} = 1$ to the preceding case; then $\mathcal{S}$ gains at most $2n$ more distinct elements (from $\{\pm\xi_j : 1 \leq j \leq n\}$), to give a maximum of $n^2+n+2 = (n+1)^2-(n+1)+2 = N^2-N+2$ distinct elements. This proves that r is restricted to even values between 2 and $n^2 - n + 2$.

To show that all such values of r are possible, we divide the proof into two main cases: n even or n odd. As in the proof of Theorem 2.3, given $v = [\xi_1, \ldots, \xi_n] \in \mathbb{C}^{1\times n}$ we define $\mathcal{S}_v \in M_n$ to be the matrix with diagonal entries $\xi_1^2, \ldots, \xi_n^2$ and (j,k)-entry $\xi_j\xi_k$ if $j < k$ and $-\xi_j\xi_k$ if $j > k$. It will be convenient to write $\mathcal{T}_x = \mathcal{S}_{[x,\overline{x}]}$, and let $|\mathcal{T}_x|$ be the number of distinct entries in $\mathcal{T}_x$.

We begin by considering the case when $n = 2m$ is even.

Step 1. We show that the result holds for $n = 2$. If $v_2 = [i]$ then $\mathcal{T}_{v_2}$ has 2 distinct entries, namely ± 1. If $v_4 = [e^{i\varphi}]$ for $0 < \varphi < \pi$, $\varphi \neq \pi/2$, then $\mathcal{T}_{v_4}$ has 4 distinct entries.

Step 2. We show that the result holds for $n = 4$. Let v be one of the following:

$$v_2 = [1, 1], \quad v_4 = [1, i], \quad v_6 = [e^{i\pi/3}, e^{i2\pi/3}], \quad v_8 = [e^{i\pi/4}, e^{i\pi/2}],$$
$$v_{10} = [e^{i\pi/5}, e^{i2\pi/5}], \quad v_{12} = [e^{i\pi/6}, e^{i\pi/3}], \quad v_{14} = [e^{i\pi/8}, e^{i\pi/4}].$$

Then $\mathcal{T}_{v_k}$ has k distinct entries.

Step 3. Suppose $n = 2m$ with $m \geq 3$. Let $x = [\mu, \ldots, \mu^m]$. Then the distinct entries of $\mathcal{T}_x$ are $\{\mu^n, \mu^{-n}\} \cup \{\pm\mu^j : -(n-1) \leq j \leq n-1\}$. Let r be an even integer in $[2, 4n]$ and let $\mu = e^{2\pi i/r}$. Then $\operatorname{Arg} \mu^n \geq \pi/2$, so $\mathcal{T}_x$ contains all the rth roots of 1 in the first quadrant, as well as the negatives of these roots. Since the set of entries of $\mathcal{T}_x$ is invariant under conjugation, $\mathcal{T}_x$ consists of the complete set of rth roots of unity, hence $|\mathcal{T}_x| = r$.

Step 4. We show that the result holds for $n = 6$. Let

$$v_{26} = [e^{i\pi/15}, e^{i2\pi/15}, e^{i4\pi/15}], \quad v_{28} = [e^{i\pi/15}, e^{i3\pi/15}, e^{i4\pi/15}],$$
$$v_{30} = [e^{i\pi/24}, e^{i3\pi/24}, e^{i6\pi/24}], \quad v_{32} = [e^{i\pi/20}, e^{i4\pi/20}, e^{i5\pi/20}].$$

Then $|\mathcal{T}_{v_k}| = k$. Combine this with Step 3 to see that the result holds for $n = 6$.

Step 5. Suppose $n = 2m$ and $m \geq 4$. We show that there is v such that $|\mathcal{T}_v| = r$ for any even number r satisfying $4n - 2 \leq r \leq 8n - 14$.

Let $x = [\mu, \ldots, \mu^{m-1}, \mu^d]$ with $\mu = e^{ia}$, $a > 0$. Suppose $m \leq d \leq 2m-3$. If we choose μ so that $2ad < \pi/2$ then the entries of $\mathcal{T}_x$ in the first quadrant are $\{\mu, \mu^2, \ldots, \mu^{d+m-1}, \mu^{2d}\}$; except for μ^{2d} the negatives of these elements are also in $\mathcal{T}_x$. Then $\mathcal{T}_x$ has $4(d + m - 1) + 2 + 2 = 4(d + m)$ distinct entries. If one chooses μ so that $(3d + m - 1)a = \pi$ (in which case $\mu^{2d} = -\mu^{-(d+m-1)}$) then $\mathcal{T}_x$ has $4(d + m - 1) + 2 = 4(d + m) - 2$ distinct entries. Thus we can achieve $4n - 2 \leq r \leq 6n - 12$, r even.

Now let $2m - 2 \leq d \leq 3m - 3$ and choose μ so that $2ad < \pi/2$. As before, the entries of $\mathcal{T}_x$ in the first quadrant are $\{\mu, \mu^2, \ldots, \mu^{d+m-1}, \mu^{2d}\}$; except for μ^d and μ^{2d} the negatives of these entries are also in $\mathcal{T}_x$, so $\mathcal{T}_x$ has $4(d + m) - 2$ distinct entries. If one chooses μ so that $(3d + m - 1)a = \pi$ (in which case $\mu^{2d} = -\mu^{-(d+m-1)}$) then $\mathcal{T}_x$ has $4(d + m - 1)$ distinct entries. Thus we can achieve $6n - 12 \leq r \leq 8n - 14$, r even.

Step 6. We claim that for all even $n \geq 8$ and even $r \in [4n-2, n^2-n+2]$ there exists $v_r \in \mathbb{C}^{n/2}$ such that $\mathcal{T}_{v_r}$ has r distinct entries, and the entries of v_r have distinct arguments in $(0, \pi/2)$. Coupled with Step 3 this proves the result for $n \geq 8$.

We prove the claim by induction on n. Suppose $v = [e^{i\pi b_1}, \dots, e^{i\pi b_m}]$ with $b_j \in (0, 1/2)$ distinct. Choose $b \in (0, 1/2)$ such that b is not in the $\mathbb{Q}$-linear span of $\{1, b_1, \dots, b_m\}$. Let $x = [v, e^{i\pi b}]$. Observe that $|\mathcal{T}_x| = |\mathcal{T}_v| + 8m + 2$. (Reason: Due to the $\mathbb{Q}$-linear independence of b, the entries of $\mathcal{T}_x$ that are different from those of $\mathcal{T}_v$ are

$$\{e^{i\pi(\pm b \pm b_j)}, e^{i\pi(1 \pm b \pm b_j)} : 1 \leq j \leq m\} \cup \{e^{\pm i\pi 2b}\}.$$

These $8m+2$ entries are all distinct because of the way b is chosen and because $\pm b_1, \dots, \pm b_m$ are distinct modulo 1.)

Apply this observation to $v = v_{26}, v_{28}, v_{30}, v_{32}$ in Step 4 to see that the claim holds for $n = 8$ and $52 \leq r \leq 58$. Coupled with Step 5 the claim holds for $n = 8$.

Suppose the claim holds when $n = 2m$. Applying the preceding observation, we conclude the claim holds when $n = 2(m+1)$ and $r \in [(4n-2)+(4n+2), n^2-n+2+(4n+2)] = [8(n+2)-16, (n+2)^2-(n+2)+2]$ is even. Coupled with Step 5 the claim holds for $n = 2(m+1)$.

We now consider the case when $n = 2m+1$ is odd. It will be convenient to write $\widetilde{\mathcal{T}}_x = \mathcal{S}_{[x,\bar{x},1]}$.

Step I. Note that the entries of $\widetilde{\mathcal{T}}_{[e^{2\pi i/r}]}$ are precisely the rth roots of unity for $r = 2, 4, 6, 8$ so the result holds for $n = 3$.

Step II(a). Let $\mu = e^{2\pi i/r}$ (with r even), $m \geq 2$, and $x = [\mu, \dots, \mu^m]$ (that is, $x_j = \mu^j$). Then the entries of $\widetilde{\mathcal{T}}_x$ are

$$\{\pm\mu^j : -(2m-1) \leq j \leq 2m-1\} \cup \{\mu^{\pm 2m}\}.$$

When $2m(2\pi/r) \geq \pi/2$ this set contains all the rth roots of 1 in the first quadrant, as well as the negatives of these roots. Since the elements of $\widetilde{\mathcal{T}}_x$ are invariant under conjugation, $\widetilde{\mathcal{T}}_x$ has r distinct elements, namely the rth roots of unity, when $n = 2m+1$ and $r \in [2, 8m] = [2, 4n-4]$ is even.

Step II(b). Let $m \geq 3$ and $x = [\mu, \mu^2, \dots, \mu^{m-1}, \mu^d] \in \mathbb{C}^m$ with $m \leq d \leq 3m-3$. Then the entries of $\widetilde{\mathcal{T}}_x$ are

$$\{\pm\mu^j : -(d+m-1) \leq j \leq d+m-1\} \cup \{\mu^{\pm 2d}\}.$$

If $\mu = e^{ia}$ with $a > 0$ sufficiently small (for example, if $2ad < \pi/2$) then all these entries are distinct, so $|\widetilde{\mathcal{T}}_x| = 4(d+m-1)+4 = 4(d+m)$. If one chooses μ so that $(3d+m-1)a = \pi$ (in which case $\mu^{2d} = -\mu^{-(d+m-1)}$) then $\widetilde{\mathcal{T}}_x$ has $4(d+m)-2$ distinct elements. By varying d we can thus achieve even $r \in [8m-2, 16m-12] = [4n-6, 8n-20]$ if $n = 2m+1$. Note that all entries of x lie in the first quadrant.

Step III(a). Let $n = 5$. By Step II(a), the result holds for even $r \in [2, 16]$. Let

$$v_{18} = [e^{i\pi/9}, e^{i\pi 3/9}], \quad v_{20} = [e^{i\pi/11}, e^{i4\pi/11}], \quad v_{22} = [e^{i\pi/17}, e^{i4\pi/17}].$$

Then $|\widetilde{\mathcal{T}}_{v_k}| = k$ so the result holds for $n = 5$.

Step III(b). Let $n = 7$. By Steps II(a) and II(b), the result holds for even $r \in [2, 36]$. Let

$$v_{38} = [e^{i\pi/24}, e^{i\pi 4/24}, e^{i\pi 6/24}], \qquad v_{40} = [e^{i\pi/27}, e^{i\pi 3/27}, e^{i\pi 8/27}],$$
$$v_{42} = [e^{i\pi/34}, e^{i\pi 4/34}, e^{i\pi 10/34}], \qquad v_{44} = [e^{i\pi/40}, e^{i\pi 4/40}, e^{i\pi 10/40}].$$

Then $\widetilde{\mathcal{T}}_{v_k}$ has k distinct entries, so the result holds for $n = 7$.

Step IV. We claim that for all odd $n \geq 7$ and r even in $[4n - 6, n^2 - n + 2]$ there exists $v_r \in \mathbb{C}^{(n-1)/2}$ such that $\widetilde{\mathcal{T}}_{v_r}$ has r distinct entries, and the entries of v_r have distinct arguments in $(0, \pi/2)$.

We prove the claim by induction on n. By Steps II(b) and III(b) the claim holds when $n = 7$.

Suppose the claim holds when $n = 2m + 1 \geq 7$. For each even $r \in [4n - 6, n^2 - n + 2]$ there exists $v = [e^{i\pi b_1}, \ldots, e^{i\pi b_m}]$ with $b_j \in (0, 1/2)$ distinct such that $|\widetilde{\mathcal{T}}_v| = r$. Choose $b \in (0, 1/2)$ such that b is not in the $\mathbb{Q}$-linear span of $\{1, b_1, \ldots, b_m\}$. Let $x = [v, e^{i\pi b}]$. Then $|\widetilde{\mathcal{T}}_x| = |\widetilde{\mathcal{T}}_v| + 4n + 2$ (by the same reasons as for Step 6 in part (a)). Thus the claim holds when $n = 2m+3$ and $r \in [(4n-6)+(4n+2), n^2-n+2+(4n+2)] = [8(n+2) - 20, (n+2)^2 - (n+2) + 2]$ is even. Coupled with Step II(b) the claim holds for $n = 2m + 3$.

Step V. By Steps II(a) and V the result holds for all odd $n \geq 7$. This concludes the proof of part (b).

(c) Note that for any $r \in [2, n^2 - 1]$, one can always construct a unitary operator T_0 on M_n^0 with r distinct eigenvalues. In fact, one can construct a real orthogonal operator acting on the real linear space $H_n^0 = H_n \cap M_n^0$ and extend it to a complex linear map by $T_0(H + iG) = T_0(H) + iT_0(G)$ for $H, G \in H_n^0$. If we choose a complex unit α from the spectrum of T_0, then the map $A \mapsto \alpha(\operatorname{tr} A)I/n + T_0(A - (\operatorname{tr} A)I/n)$ will also have r distinct eigenvalues; otherwise, the map will have $r + 1$ eigenvalues.

(d) By Theorem 3.3(a), the map $A \mapsto UAU^t$ on K_6 can generate r-circular projections for $r \in [2, 15]$, and hence so does $T_0 \in \Lambda$. We can add one more distinct eigenvalue by varying α in the map $A \mapsto \alpha(\operatorname{tr} A)I/4 + T_0(A - (\operatorname{tr} A)I/4)$. $\square$

5. Concluding remarks and further research

There are other interesting open questions, and some of them can be tackled using our techniques.

1. One may extend our techniques to study r-circular projections for infinite dimensional operators. For example, if one considers the operator norm on the algebra of bounded linear operators acting on a Hilbert space $\mathcal{H}$, its isometries have the form (a) $A \mapsto UAV$ or (b) $A \mapsto UA^tV$, where U, V are unitary operators and A^t is the transpose of A with respect to an orthonormal basis. In this case, for any integer $r \geq 2$, one can construct r-circular projections corresponding to an isometry of the form (a), and for any positive integer $r \notin \{1, 3, 7\}$, one can construct r-circular projections corresponding to an isometry of the form (b).
2. Given a real Banach space $\mathcal{X}$ of finite dimension with isometry group G, we may identify $\mathcal{X}$ with $\mathbb{R}^n$ and assume that G is a subgroup of the real orthogonal group in $M_n(\mathbb{R})$. One can adapt the idea of r-circular projections to this real setting as follows: in this case, let us say that $P_1, \ldots, P_r \in M_n$ are r-circular projections if $P_1 \oplus \cdots \oplus P_r = I$ and there exists an isometry of the form

$$\sum_{j=1}^{r} X_j R_j X_j^*,$$

with $X_j X_j^* = P_j$ for $j = 1, \ldots, r$ and $R_j \in M_{n_j}$ (with $n_j = \operatorname{rank} P_j$) of the following form:

$$I_{n_j}, \quad -I_{n_j}, \quad \text{or} \quad \begin{pmatrix} a_j I_{n_j/2} & b_j I_{n_j/2} \\ -b_j I_{n_j}/2 & a_j I_{n_j/2} \end{pmatrix} \quad \text{with} \quad a_j^2 + b_j^2 = 1,$$

such that R_i and R_j have no common eigenvalues in $\mathbb{C}$ whenever $i \neq j$. One may use our techniques to study such r-circular projections on real Banach spaces.

Declaration of competing interest

No competing interest.

Acknowledgments

We thank Professor Robert Guralnick for some correspondence about the group Λ in Section 4. We also thank the referee for some helpful comments.

Dijana Ilišević has been fully supported by the Croatian Science Foundation [project number IP-2016-06-1046].

Chi-Kwong Li is an affiliate member of the Institute for Quantum Computing, University of Waterloo; his research was partially supported by the Simons Foundation Grant 851334.

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