Minimum number of non-zero-entries in a stable matrix exhibiting Turing instability*

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Abstract

It is shown that for any positive integer $n \geq 3$, there is a stable irreducible $n \times n$ matrix A with $2n+1-\lfloor \frac{n}{3} \rfloor$ nonzero entries exhibiting Turing instability. Moreover, when $n = 3$, the result is best possible, i.e., every 3×3 stable matrix with five or fewer nonzero entries will not exhibit Turing instability. Furthermore, we determine all possible 3×3 irreducible sign pattern matrices with 6 nonzero entries which can be realized by a matrix A that exhibits Turing instability.

1 Introduction

Reaction-diffusion partial differential equation models have been used to describe the formation of spatiotemporal patterns in biology, chemistry and physics. Alan Turing [16] proposed that different diffusion coefficients of a pair of chemicals in a biochemical system are responsible for the generation of spatially inhomogeneous patterns, and this diffusion-induced instability (Turing instability) has been credited as one of the most important driving mechanisms of pattern formations [9].

The Turing instability is caused by the destabilization of a constant equilibrium solution $U = U_0$ of a spatially homogeneous reaction-diffusion system $U_t = P\Delta U + g(U)$ with n (≥ 2) variables and coupled with proper boundary conditions, where $U = U(x,t)$ with $t > 0$, x belongs to a spatial domain, P is a diagonal $n \times n$ matrix with non-negative diagonal entries (diffusion coefficients), and g is a smooth nonlinear vector function satisfying $g(U_0) = 0$. Through the techniques of linearization, the stability of the equilibrium $U = U_0$ is reduced to a linear system of diffusion equations $V_t = P\Delta V + AV$, where $A = g'(U_0)$ is a real-valued $n \times n$ Jacobian matrix. The constant equilibrium U_0 is asymptotically stable if each solution V of the linearized diffusion system converges to zero uniformly as $t \to \infty$. From the theory of linear differential equations, this is equivalent to the condition that each eigenvalue of the matrix $A - \mu_j P$ has negative real part, where μ_j

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($j = 0, 1, 2, \cdots$) are the eigenvalues of the Laplace operator with compatible boundary conditions, and μ_j satisfy $0 = \mu_0 < \mu_1 \le \mu_2 \le \cdots$ and $\lim_{j\to\infty} \mu_j = \infty$ [13, 16].

Because of the wide applicability of Turing instability, there has been considerable interest in the study of stable matrices and stable matrices exhibiting Turing instability [11, 15]. Many realistic biological reaction mechanisms involve a large number of chemical reactants and a complex biological regulatory network. It is important to identify the key components of the biological network that is capable of generating desired patterns, and it is also important to classify minimal biological network for pattern formation [17].

To capture the behavior of the model relating to or describing the network connection of the different components, we need the following definitions. Let M_n be the set of all $n \times n$ matrices with real-valued entries. A matrix $A \in M_n$ is said to be *stable* if for each of its eigenvalues λ_j ($j = 0, 1, 2, \cdots, n$), $\mathrm{Re}(\lambda_i) < 0$. We define the *sign pattern* of a matrix $A = [a_{jk}]$ to be an $n \times n$ matrix $S(A) = [s_{jk}]$ such that, for $j, k \in \{1, \cdots, n\}$, $s_{jk} = 0$ when $a_{jk} = 0$, $s_{jk} = -$ when $a_{jk} < 0$, and $s_{jk} = +$ when $a_{jk} > 0$. We also define the *non-zero pattern* of A to be an $n \times n$ matrix $N(A) = [n_{jk}]$ such that, for $j, k \in \{1, \cdots, n\}$, $n_{jk} = 0$ when $a_{jk} = 0$, and $n_{jk} = *$ when $a_{jk} \ne 0$. A non-zero pattern of A can be assigned $\pm$ signs so it becomes a sign pattern. If some matrix $A \in M_n$ is found to be stable, then the sign pattern $S(A)$ is said to be *potentially stable.* For a stable $n \times n$ matrix A, if there is a nonnegative $n \times n$ diagonal matrix P such that the matrix $A - tP$ is unstable for some positive $t > 0$, then A is said to exhibit *Turing Instability.* We are interested in the minimum number of nonzero entries of a stable matrix in M_n, which will exhibit Turing instability.

If the system is modeled by $A \in M_n$ which is reducible, i.e., there is a permutation matrix Q such that

$$QAQ^T = \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{bmatrix}, \quad A_{11} \in M_k, A_{22} \in M_{n-k}$$

with $1 < k < n$, then the eigenvalues of A are the eigenvalues of A_{11} and A_{22}. Furthermore, for any diagonal matrix P, if $QPQ^T = P_1 \oplus P_2$ (the direct sum of diagonal matrices $P_1 \in M_k$ and $P_2 \in M_{n-k}$), then the eigenvalues of $Q(A - tP)Q^T$ are those of $A_{11} - tP_1$ and $A_{22} - tP_2$. Thus, the stability and Turing stability behavior of A are determined by A_{11} and A_{22}. In view of these, we will focus on *irreducible* matrices, i.e., matrices that are not reducible. We will consider the minimal number S_n of nonzero entries that an $n \times n$ irreducible matrix A must have in order for it to exhibit Turing instability.

An $n \times n$ sign pattern $S(A)$ with only S_n nonzero entries can be considered as a minimal network topology generating Turing instability. Turing's original work on the subject [16] implies that $S_2 = 4$. Indeed it is well-known that up to a permutation or transpose, the only 2×2 sign pattern that can possibly generate Turing instability is $\begin{bmatrix} - & + \\ - & + \end{bmatrix}$.

In this paper we prove the following result:

Theorem 1. *Let S_n be the minimal number of nonzero entries that an $n \times n$ irreducible matrix A must have in order for it to exhibit Turing instability. If $n \ge 3$, then*

$$S_n \le 2n + 1 - \lfloor \frac{n}{3} \rfloor.$$

In particular the equality holds when $n = 3$ and $S_3 = 6$.

In the 2014 paper by Raspopovic et.al. [14], it was claimed that in order for an irreducible 3×3 matrix to exhibit Turing instability, it must have at least 6 nonzero entries. But the claim was not proved in the paper. Theorem 1 provides the justification for that claim. We also classify all distinct irreducible 3×3 non-zero patterns with 6 non-zero entries (up to a permutation or transpose) so that Turing instability can possibly occur (see Table 2). Note that a list of 19 3×3 sign patterns with 6 non-zero entries for Turing instability were identified in [14], and our list has 4 non-zero patterns corresponding to these sign patterns. In [14], the diagonal matrix P is assumed to be $\mathrm{diag}(p_1, p_2, 0)$ while our results hold for any nonnegative (including positive) diagonal matrix P. The 3×3 Turing instability was also studied in [1, 19], and graph-theoretical methods to analyze network topologies for Turing instability were also used in [10, 12].

A related index is the minimum number of nonzero entries required for an $n \times n$ irreducible sign pattern to be potentially stable, and it is denoted by m_n. Note that, trivially, $m_n \le S_n$ for any n since A is assumed to be stable. The following has been proved in [4] (for $n \le 6$ and $n \ge 9$), [5] (for $n = 7$) and [3] (for $n = 8$).

$$\begin{cases} m_n = 2n-1, & n = 2,3, \\ m_n = 2n-2, & n = 4,5, \\ m_n = 2n-3, & n = 6,7, \\ m_n = 2n-4, & n = 8, \\ m_n \le 2n - 1 - \lfloor \frac{n}{3} \rfloor, & n \ge 9. \end{cases} \tag{1.1}$$

Note that $m_2 = 3 < 4 = S_2$, and by our result $m_3 = 5 < 6 = S_3$. It is interesting to obtain the exact value of S_n for $n \ge 4$ and m_n for $n \ge 9$. We conjecture that $m_n < S_n$ for any $n \in \mathbb{N}$.

In Section 2, we give some preliminary results, and obtain an auxiliary result for extending a stable matrix exhibiting Turing instability to matrices of larger sizes. The proof of Theorem 1 will be done in Section 3. In particular, we prove all 3×3 potentially stable sign pattern with only 5 nonzero entries cannot exhibit Turing instability. In Section 4, we find all 3×3 potentially stable sign pattern matrices with 6 nonzero entries which can be realized by a matrix exhibiting Turing instability.

2 Preliminaries and an auxiliary result

Given an $n \times n$ matrix $A = [a_{jk}]$, we define the *digraph* of A to be the directed graph with vertex set $\{1, \ldots, n\}$ and having an edge from vertex j to vertex k if and only if $a_{jk} \neq 0$. For a digraph, we define a *path* as an ordered set of edges, where the terminal vertex of the m^{th} edge is the initial vertex of the $(m+1)^{th}$ edge. We define the *length* of a path as the number of edges in the path. In particular, if the entries $a_{j_0,j_1}, a_{j_1,j_2}, \cdots a_{j_{\ell-2},j_{\ell-1}}, a_{j_{\ell-1},j_\ell}$ of A are all nonzero, then the digraph of A contains a path of length ℓ from vertex j_0 to vertex j_ℓ, where the m^{th} edge is (j_{m-1}, j_m). We say that a digraph is *strongly connected* if for each pair of distinct vertices p and q in its vertex set, there exists a path which begins at p and ends at q. It is the case that for any $A \in M_n$, A is irreducible if and only if the digraph of A is strongly connected [2]. We define a *cycle* to be a path which begins and ends at the same point, and which only intersects itself at this point. We refer to a cycle of length 1 as a *loop*.

To study the stability of matrix A, we use the standard way to obtain the characteristic polynomial of A:

$$p(A) = \det(\lambda I - A) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-2}\lambda^2 + c_{n-1}\lambda + c_n, \tag{2.1}$$

where $c_k \in \mathbb{R}$ such that $c_k = (-1)^k C_k$ is the sum of the $k \times k$ principal minors of the matrix A. By Vieta's formula, C_k is the k-th elementary symmetric polynomial $E_k(\lambda_1, \cdots, \lambda_n)$ where λ_j ($1 \le j \le n$) are the eigenvalues of A, or the roots of $p(A) = 0$. The stability of A can be determined using the well-known Routh-Hurwitz stability criterion for polynomial:

Lemma 2. *Suppose that f is a degree-n polynomial in form $f(z) = \sum_{k=0}^{n} c_k z^{n-k}$ where $c_k \in \mathbb{R}$ and $c_0 = 1$. Then all the zeros of $f(z)$ have negative real parts if and only if the leading $k \times k$ principal minors Δ_k is positive for the following $n \times n$ matrix:*

$$H_n = \begin{bmatrix} c_1 & c_3 & c_5 & \cdots & \cdots \\ 1 & c_2 & c_4 & \cdots & \cdots \\ & c_1 & c_3 & c_5 & \cdots \\ & 1 & c_2 & c_4 & \cdots \\ & & \ddots & \ddots & \ddots \end{bmatrix}. \tag{2.2}$$

As pointed out by a referee, one may use the Liénard-Chipart criterion [8], which requires less computation, to determine the stability of a matrix. In any event, for $n = 3$, Lemma 2 implies the following conditions for stability of A:

$$H_3 = \begin{bmatrix} c_1 & c_3 & 0 \\ 1 & c_2 & 0 \\ 0 & c_1 & c_3 \end{bmatrix}, \qquad \begin{aligned} \Delta_1 &= c_1 > 0, \\ \Delta_2 &= c_1c_2 - c_3 > 0, \\ \Delta_3 &= c_3(c_1c_2 - c_3) > 0. \end{aligned} \tag{2.3}$$

That is, $c_1, c_2, c_3 > 0$ and $c_1c_2 > c_3$. Note that for 3×3 matrix $A = (a_{ij})$, we have

$$\begin{aligned} c_1 =& - E_1(A) = -\mathrm{Tr}(A) = a_{11} + a_{22} + a_{33}, \\ c_2 =& E_2(A) = a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32}, \\ c_3 =& - E_3(A) = -\det(A) = -a_{11}a_{22}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{32}a_{21} \\ & \qquad + a_{12}a_{21}a_{33} + a_{23}a_{32}a_{11} + a_{13}a_{31}a_{22}. \end{aligned} \tag{2.4}$$

The following examples are useful for our subsequent discussion. In particular, they show that one can extend a matrix $B \in M_2$ which exhibits Turing stability to a larger matrix

$$A = \begin{pmatrix} B & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

which also exhibits Turning stability by a suitable choice of A_{12}, A_{21}, A_{22}. This idea will be used in the poofs presented in the next section.

Example 3. Suppose $P = \mathrm{diag}(2,0)$, $P_1 = \mathrm{diag}(2,0,0)$, $P_2 = \mathrm{diag}(2,0,0,0)$,

$$B = \begin{bmatrix} -2 & 1 \\ -3 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 1 & 1 \\ 0 & -0.1 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 1 & 1 \\ -0.1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ -3 & 1 & 1 & 0 \\ -0.1 & 0 & 0 & 1 \\ 0 & -0.01 & 0 & 0 \end{bmatrix}.$$

Then the eigenvalues of B and $B-P$ are: $-0.5000+0.8660i$, $-0.5000-0.8660i$, and $-3.3028, 0.3028$. Thus, B is a stable matrix exhibiting Turing instability. The eigenvalues of A and $A - P_1$ are:

$$-0.3926 + 0.8816i, -0.3926 - 0.8816i, -0.2147,$$

$$\text{and} \quad -3.3086, 0.1543 + 0.3116i, 0.1543 - 0.3116i;$$

the eigenvalues of A_1 and $A_1 - P_1$ are

$$-0.4445 + 0.8389i, -0.4445 - 0.8389i, -0.1109,$$

$$\text{and} \quad -3.3111, 0.1556 + 0.0775i, 0.1556 - 0.0775i;$$

the eigenvalues of A_2 and $A_2 - P_2$ are

$$-0.4494 + 0.8274i, -0.4494 - 0.8274i, -0.0506 + 0.1414i - 0.0506 - 0.1414i,$$

$$\text{and} \quad -3.3110, -0.1348, 0.2229 + 0.1999i, 0.2229 - 0.1999i.$$

So, all A, A_1, A_2 have B as the leading principal submatrix, and exhibit Turing instability.

3 Proof of Theorem 1

3.1 Proof of the case when $n = 3$

First, we prove that for a 3×3 matrix A to exhibit Turing instability (there exists a positive diagonal matrix P, such that $A - tP$ is unstable for some $t > 0$) it must have at least 6 nonzero entries. From the fact that $m_3 = 5$, in order for a 3×3 irreducible matrix to be stable it must have at least 5 nonzero entries. So in order to prove the statement regarding Turing instability, we consider all possible 3×3 stable irreducible nonzero patterns (up to permutation similarity and transposition) containing only 5 nonzero entries and show that any stable matrix realizing such a pattern cannot exhibit Turing instability.

In order for a 3×3 matrix to be irreducible, its digraph must be strongly connected, thus the digraph either contains (a) two connected 2-cycles, or (b) one 3-cycle. Then, in order for such a matrix to be potentially stable, in both cases a loop is required [6, 18], then in case (a) there is either a 2-cycle which can intersect the loop or be separate from it, or an additional loop, and in case (b) you can have the loop either on the end of one of the 2-cycles, or at the intersection of the two 2-cycles. Thus, we have the list of digraphs in Table 1 to consider. Note that the determinant of any matrix corresponding to the fifth digraph is always zero, which means the matrix cannot be stable. Thus, we only need to consider the first 4 graphs. Note also that patterns 1-4 in Table

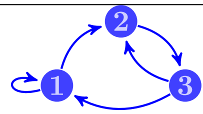
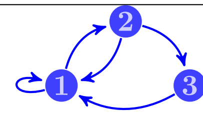
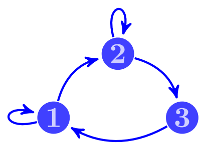
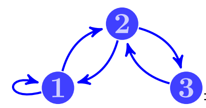
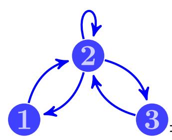
1.  $\Longrightarrow \begin{bmatrix} * & * & 0 \\ 0 & 0 & * \\ * & * & 0 \end{bmatrix}$	2.  $\Longrightarrow \begin{bmatrix} * & * & 0 \\ * & 0 & * \\ * & 0 & 0 \end{bmatrix}$
3.  $\Longrightarrow \begin{bmatrix} * & * & 0 \\ 0 & * & * \\ * & 0 & 0 \end{bmatrix}$	4.  $\Longrightarrow \begin{bmatrix} * & * & 0 \\ * & 0 & * \\ 0 & * & 0 \end{bmatrix}$
5.  $\Longrightarrow \begin{bmatrix} 0 & * & 0 \\ * & * & * \\ 0 & * & 0 \end{bmatrix}$	

Table 1: List of potential digraphs with 3 vertices and 5 edges.

1 were also identified in [3, Theorem 5.2] as the only minimally potentially stable 3×3 nonzero patterns.

We assume the potentially stable patterns 1-4 in Table 1 to be realized by a stable matrix A, then we use the stability conditions in the Routh-Hurwitz criterion (Lemma 2) to show that the matrix $A - tP$ is still stable for $t \geq 0$ and non-negative diagonal matrix $P = \mathrm{diag}(p_1, p_2, p_3)$. For that purpose, we recall that the characteristic polynomial of $A - tP$ is given by

$$p(A - tP) = \lambda^3 + c_1(t)\lambda^2 + c_2(t)\lambda + c_3(t),$$

and $c_j(t)$ $(1 \leq j \leq 3)$ are polynomials of t. Then A is stable implies that $c_j(0) > 0$ $(1 \leq j \leq 3)$ and $c_1(0)c_2(0) - c_3(0) > 0$, and from the Routh-Hurwitz criterion we shall show that $c_j(t) > 0$ $(1 \leq j \leq 3)$ and $c_1(t)c_2(t) - c_3(t) > 0$ for all $t > 0$. We show that for patterns 1-4 in Table 1.

Case 1:

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix}, \quad A - tP = \begin{bmatrix} a_{11} - tp_1 & a_{12} & 0 \\ 0 & -tp_2 & a_{23} \\ a_{31} & a_{32} & -tp_3 \end{bmatrix}.$$

Then for any $t \geq 0$, from $c_1(0) = -a_{11} > 0$, $c_2(0) = -a_{23}a_{32} > 0$, $c_3(0) = a_{11}a_{23}a_{32} - a_{12}a_{23}a_{31} > 0$

and $c_1(0)c_2(0) - c_3(0) = a_{12}a_{23}a_{31} > 0$, we obtain that

$$
\begin{aligned}
c_1(t) =& (p_1 + p_2 + p_3)t - a_{11} > 0, \\
c_2(t) =& (p_1p_2 + p_1p_3 + p_2p_3)t^2 - a_{11}(p_2 + p_3)t - a_{23}a_{32} > 0, \\
c_3(t) =& p_1p_2p_3t^3 - a_{11}p_2p_3t^2 - a_{23}a_{32}p_1t + (a_{11}a_{23}a_{32} - a_{12}a_{23}a_{31}) > 0, \\
c_1(t)c_2(t) - c_3(t) =& (p_1 + p_2)(p_1 + p_3)(p_2 + p_3)t^3 - a_{11}(p_2 + p_3)(2p_1 + p_2 + p_3)t^2 \\
&+ (p_2 + p_3)(a_{11}^2 - a_{23}a_{32})t + a_{12}a_{23}a_{31} > 0.
\end{aligned}
$$

Therefore $A - tP$ is stable for all $t \geq 0$.

Case 2:

$$
A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix}, \quad A - tP = \begin{bmatrix} a_{11} - tp_1 & a_{12} & 0 \\ a_{21} & -tp_2 & a_{23} \\ a_{31} & 0 & -tp_3 \end{bmatrix}.
$$

Then for any $t \geq 0$, from $c_1(0) = -a_{11} > 0$, $c_2(0) = -a_{12}a_{21} > 0$, $c_3(0) = -a_{12}a_{23}a_{31} > 0$ and $c_1(0)c_2(0) - c_3(0) = a_{12}a_{11}a_{21} + a_{12}a_{23}a_{31} > 0$, we obtain

$$
\begin{aligned}
c_1(t) =& t(p_1 + p_2 + p_3) - a_{11} > 0, \\
c_2(t) =& (p_1p_2 + p_1p_3 + p_2p_3)t^2 - a_{11}(p_2 + p_3)t - a_{12}a_{21} > 0, \\
c_3(t) =& p_1p_2p_3t^3 - a_{11}p_2p_3t^2 - a_{12}a_{21}p_1t - a_{12}a_{23}a_{31} > 0, \\
c_1(t)c_2(t) - c_3(t) =& (p_1 + p_2)(p_1 + p_3)(p_2 + p_3)t^3 - a_{11}(p_2 + p_3)(2p_1 + p_2 + p_3)t^2 \\
&+ [(p_2 + p_3)a_{11}^2 - (p_1 + p_2)a_{12}a_{21}]t + a_{12}a_{11}a_{21} + a_{12}a_{23}a_{31} > 0.
\end{aligned}
$$

Therefore $A - tP$ is stable for all $t \geq 0$.

Case 3:

$$
A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix}, \quad A - tP = \begin{bmatrix} a_{11} - tp_1 & a_{12} & 0 \\ 0 & a_{22} - tp_2 & a_{23} \\ a_{31} & 0 & -tp_3 \end{bmatrix}.
$$

Then for any $t \geq 0$, from $c_1(0) = -(a_{11} + a_{22}) > 0$, $c_2(0) = a_{11}a_{22} > 0$, $c_3(0) = -a_{12}a_{23}a_{31} > 0$ and $c_1(0)c_2(0) - c_3(0) = -a_{11}a_{22}(a_{11} + a_{22}) + a_{12}a_{23}a_{31} > 0$, we obtain

$$
\begin{aligned}
c_1(t) =& (p_1 + p_2 + p_3)t - (a_{11} + a_{22}) > 0, \\
c_2(t) =& (p_1p_2 + p_1p_3 + p_2p_3)t^2 - (a_{11}p_2 + a_{22}p_1 + a_{11}p_3 + a_{22}p_3)t + a_{11}a_{22} > 0, \\
c_3(t) =& p_1p_2p_3t^3 - (a_{11}p_2p_3 + a_{22}p_1p_3)t^2 + a_{11}a_{22}p_3t - a_{12}a_{23}a_{31} > 0, \\
c_1(t)c_2(t) - c_3(t) =& [(p_1p_2 + p_1p_3 + p_2p_3)(p_1 + p_2 + p_3) - p_1p_2p_3]t^3 \\
&+ [(a_{11}p_2 + a_{22}p_1)p_3 - (p_1p_2 + p_1p_3 + p_2p_3)(a_{11} + a_{22}) \\
&- (p_1 + p_2 + p_3)(a_{11}p_2 + a_{11}p_3 + a_{22}p_1 + a_{22}p_3)]t^2 \\
&+ [(a_{11} + a_{22})(a_{11}p_2 + a_{11}p_3 + a_{22}p_1 + a_{22}p_3) \\
&- a_{11}a_{22}p_3 + a_{11}a_{22}(p_1 + p_2 + p_3)]t \\
&- a_{11}a_{22}(a_{11} + a_{22}) + a_{12}a_{23}a_{31} > 0.
\end{aligned}
$$

Note that here the coefficients of t^2 and t terms in $c_1(t)c_2(t) - c_3(t)$ can be reduced to $-a_{11}f_1(p_i) - a_{22}f_2(p_i)$, where f_1 and f_2 are positive, so we can conclude that the coefficients of t^2 and t terms are positive. Therefore $A - tP$ is stable for all $t \geq 0$.

Case 4:

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix}, \quad A - tP = \begin{bmatrix} a_{11} - tp_1 & a_{12} & 0 \\ a_{21} & -tp_2 & a_{23} \\ 0 & a_{32} & -tp_3 \end{bmatrix}.$$

Then for any $t \geq 0$, from $c_1(0) = -a_{11} > 0$, $c_2(0) = -(a_{12}a_{21} + a_{23}a_{32}) > 0$, $c_3(0) = a_{11}a_{23}a_{32} > 0$ and $c_1(0)c_2(0) - c_3(0) = a_{11}a_{12}a_{21} > 0$, we obtain

$$\begin{aligned} c_1(t) =& t(p_1 + p_2 + p_3) - a_{11} > 0, \\ c_2(t) =& (p_1p_2 + p_1p_3 + p_2p_3)t^2 - a_{11}(p_2 + p_3)t - (a_{12}a_{21} + a_{23}a_{32}) > 0, \\ c_3(t) =& p_1p_2p_3t^3 - a_{11}p_2p_3t^2 - (a_{12}a_{21}p_3 + a_{23}a_{32}p_1)t + a_{11}a_{23}a_{32} > 0, \\ c_1(t)c_2(t) - c_3(t) =& (p_1 + p_2)(p_1 + p_3)(p_2 + p_3)t^3 - a_{11}(p_2 + p_3)(2p_1 + p_2 + p_3)t^2 \\ & + [a_{11}^2(p_2 + p_3) - a_{12}a_{21}(p_1 + p_2) \\ & - a_{23}a_{32}(p_2 + p_3)]t + a_{11}a_{12}a_{21} > 0. \end{aligned}$$

Therefore $A - tP$ is stable for all $t \geq 0$.

From the four cases above, we see that for any matrix A whose nonzero pattern is given by one of the first four patterns in Table 1, we have $A - tP$ is stable for all $t \geq 0$ and nonnegative diagonal matrix P. Hence, there is no 3×3 irreducible, stable matrix with only 5 entries which can exhibit Turing instability.

Next, we show that some irreducible 3×3 matrix A with 6 nonzero entries could exhibit Turing instability. That is, there exists a positive diagonal matrix P, such that $A - tP$ is unstable for some $t > 0$. We identify all 3×3 sign patterns with 6 nonzero entries (or equivalently digraphs with 3 vertices and 6 edges) which exhibit Turing instability.

We assume that A is an irreducible 3×3 stable matrix with 6 nonzero entries. Similar to the approach in previous analysis, in order for a 3×3 matrix to be irreducible, its digraph must be strongly connected, so the digraph either contains (a) two connected 2-cycles, or (b) one 3-cycle. Also since A is stable, the digraph always contains at least one loop. We consider four cases: (i) the digraph contains one 3-cycle and exactly one 2-cycle, then the digraph must be in form of 1 or 2 in Table 2; (ii) the digraph contains two connected 2-cycles but not one 3-cycle, then the digraph must be in form of 3 or 4 in Table 2; (iii) the digraph contains both two connected 2-cycles and one 3-cycle, then the digraph must be in form of 5 or 6 in Table 2; and (iv) the digraph contains one 3-cycle and no 2-cycle, then the digraph must be in form of 7 in Table 2. Note that here we only consider topologically distinct zero-nonzero patterns, i.e. ones that cannot be obtained from another via permutation similarity or transposition.

Each of the seven non-zero patterns in Table 2 can be realized into a sign pattern to exhibit the Turing instability. For pattern 2 and 3, the stable matrices A_1 and A in Example 3 have the nonzero patterns exhibiting Turing instability.

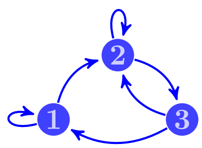
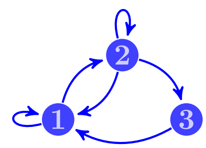
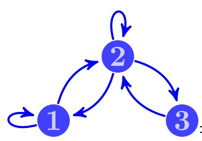
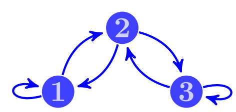
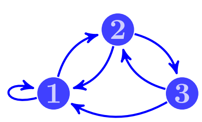
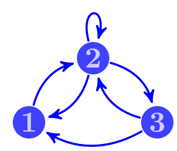
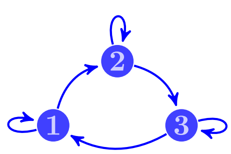
1.  $\Longrightarrow \begin{bmatrix} \ast & \ast & 0 \\ 0 & \ast & \ast \\ \ast & \ast & 0 \end{bmatrix}$	2.  $\Longrightarrow \begin{bmatrix} \ast & \ast & 0 \\ \ast & \ast & \ast \\ \ast & 0 & 0 \end{bmatrix}$
3.  $\Longrightarrow \begin{bmatrix} \ast & \ast & 0 \\ \ast & \ast & \ast \\ 0 & \ast & 0 \end{bmatrix}$	4.  $\Longrightarrow \begin{bmatrix} \ast & \ast & 0 \\ \ast & 0 & \ast \\ 0 & \ast & \ast \end{bmatrix}$
5.  $\Longrightarrow \begin{bmatrix} \ast & \ast & 0 \\ \ast & 0 & \ast \\ \ast & \ast & 0 \end{bmatrix}$	6.  $\Longrightarrow \begin{bmatrix} 0 & \ast & 0 \\ \ast & \ast & \ast \\ \ast & \ast & 0 \end{bmatrix}$
7.  $\Longrightarrow \begin{bmatrix} \ast & \ast & 0 \\ 0 & \ast & \ast \\ \ast & 0 & \ast \end{bmatrix}$	

Table 2: List of potential digraphs with 3 vertices and 6 edges.

All (except one) remaining patterns (1,4,5 and 6) in Table 2 have appeared in the list given in Figure 1 and 9 of the Supplementary Materials (SM) of [14]. The 19 digraphs in Figure 1 and 2 of [14]-SM can all be categorized into pattern 1,4,5 and 6 in Table 2. In particular, pattern 1 corresponds to T_9, T_{10} and T_{11} in Figure 2 of [14]-SM, pattern 4 corresponds to T_1 and T_2, pattern 5 corresponds to T_3, T_4, T_5, T_6 and T_7, and pattern 6 corresponds to T_8. The pattern 7 in Table 2 seems to be missing from the classification in [14]-SM.

It is not difficult to determine all the sign patters (up to permutation similarity and transposition) of the matrices in Table 2 that give rise to stable matrices with Turing stability, and we determine all these sign patterns in Section 4. We also remark that the first four digraphs in Table 1 and all diagraphs in Table 2 satisfy a known necessary condition for Turing instability: the digraph has an l-subgraph for $l = 1, 2, 3$, where the l-subgraph is a set of one or more disjoint cycles with total number of nodes being l [7, 12]. For Turing instability to occur, one of these subgraphs must be destabilizing.

3.2 Proof of the result when $n \geq 4$

Suppose $A \in M_3$ has pattern 1, 4, 5, or 6 exhibits Turing instability. Note that all these patterns correspond to irreducible matrices in upper Hessenberg form, i.e., $A = (a_{ij}) \in M_3$ is irreducible with $a_{12}a_{23} \neq 0 = a_{13}$. One can use the idea of Example 3 and modify the proof of Corollary 3.3.2 and Theorem 2.2.6 in [4] to construct a $3n \times 3n$ irreducible stable matrix R_n with A as the leading 3×3 submatrix A_1 inductively as follows. Let $R_1 = A_1$ and P_1 be defined as in Example 3 so that $c_2(R_1 - P_1) = -1$ is not stable. Let $P_n = P_1 \oplus 0_{3n-3}$ for any $n \in \mathbb{N}$. Assume that R_k has been constructed. Let

$$R_{k+1} = \begin{bmatrix} R_k & E \\ F & Y \end{bmatrix} \tag{3.1}$$

such that E has only one nonzero entry equal to 1 at the left bottom corner, F has only one nonzero entry equal -1 at the left bottom corner, and $Y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -(1/3)^k & 0 \end{bmatrix}$. Then R_{k+1} will be stable by Corollary 3.3.2 in [4]. Moreover,

$$c_2(R_{k+1} - P_{k+1}) = c_2(R_k - P_k) + (1/3)^k = \cdots = c_2(R_1 - P_1) = -1 + \sum_{j=1}^{k} (1/3)^j < 0.$$

Thus, $R_n \in M_{3n}$ is stable and $R_n - P_n$ is not stable for all n.

We may use the the construction (3.1) in the preceding case by setting $R_1 = A_2$ in Example 3, $P_n = [2] \oplus 0_{3n}$ for any $n \in \mathbb{N}$. Then the construction in (3.1) will yield stable matrices $R_n \in M_{3n+1}$ such that $c_2(R_n - P_n) < 0$ so that $R_n - P_n$ is not stable.

Finally, we can let

$$R_1 = \begin{bmatrix} -1 & 3 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 \\ -0.1 & 0 & 0 & 1 & 0 \\ 0 & -0.01 & 0 & 0 & 1 \\ 0 & 0 & -0.001 & 0 & 0 \end{bmatrix}, \quad P_n = [2] \oplus 0_{3n+1} \text{ for } n \in \mathbb{N}.$$

Then R_1 is stable and has eigenvalues

$$-0.4495 + 0.8274i, -0.4495 - 0.8274i, -0.0224 + 0.1397i, -0.0224 - 0.1397i, -0.0563.$$

We have $c_2(R_1 - P_1) = -1 < 0$ so that $R_1 - P_1$ is not stable. Then the construction in (3.1) will yield stable matrices $R_n \in M_{3n+2}$ such that $c_2(R_n - P_n) < 0$ so that $R_n - P_n$ is not stable.

One can readily check that in all the above constructions, the matrix has number of nonzero entries equal to $2n + 1 - \lfloor \frac{n}{3} \rfloor$. The result follows.

4 Sign patterns of 3×3 matrices exhibiting Turing instability

In this section, we determine all 3×3 sign pattern matrices, up to equivalence (diagonal similarity, transposition and permutation similarity), with exactly 6 nonzero entries that are realizable as matrices exhibiting Turing instability. Since the eigenvalues of a matrix depend continuously on its entries, if a matrix have sign patterns containing a subpattern of a matrix that exhibits Turing instability, then one can choose entries with sufficiently small magnitude for other nonzero entries so that the resulting matrix will also exhibit Turing instability.

For a matrix $A = [a_{ij}]$ realizing a pattern in Table 2, we can always apply a diagonal similarity so that the $a_{12} = a_{23} = 1$. By the Routh-Hurwitz criterion, we need only to look at the functions $c_1(t), c_2(t), c_3(t), h(t) = c_2(t)c_1(t) - c_3(t)$ and determine the signs of the entries of the matrix to ensure that $c_1(0), c_2(0), c_3(0), h(0) > 0$ but that there exists a positive t (usually assumed as $t = 1$ in the examples below) and nonnegative p_1, p_2, p_3 such that at least one of $c_1(t), c_2(t), c_3(t), h(t)$ is not positive. For brevity, we say that a sign pattern is PETI (for *potentially exhibiting Turing instability*) if it has a matrix realization that exhibits Turing instability.

We will prove the following.

Theorem 4. *Each of the seven non-zero patterns listed in Table 2 can be realized by one or more sign patterns and matrices exhibiting Turing instability. All non-equivalent sign patterns are listed in the following Table 3.*

Proof. Suppose $A = [a_{ij}]$. Let $t, p_1, p_2, p_3 \geq 0$ and $P = \text{diag}(p_1, p_2, p_3)$. Without loss of generality, assume $a_{12} = a_{23} = 1$. For each of the seven nonzero patterns in Table 2, we will look at the polynomials $c_1(t), c_2(t), c_3(t)$ and $h(t)$ arising from $p(A - tP)$. Assuming that $c_1(0), c_2(0), c_3(0)$ and $h(0)$ are all positive, $t \geq 0$ and $P = \text{diag}(p_1, p_2, p_3)$ is nonnegative, we indicate the expressions that may change signs depending on the signs of the entries of A by placing them in a box. From this, we eliminate sign patterns for A that make it impossible to exhibit Turing instability. From the

$$\begin{bmatrix} - & + & 0 \\ 0 & - & + \\ - & + & 0 \end{bmatrix} \quad \begin{bmatrix} - & + & 0 \\ 0 & + & + \\ + & - & 0 \end{bmatrix} \quad \begin{bmatrix} + & + & 0 \\ 0 & - & + \\ - & - & 0 \end{bmatrix} \quad \begin{bmatrix} - & + & 0 \\ - & + & + \\ - & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} - & + & 0 \\ + & - & + \\ 0 & - & 0 \end{bmatrix} \quad \begin{bmatrix} - & + & 0 \\ - & + & + \\ 0 & - & 0 \end{bmatrix} \quad \begin{bmatrix} + & + & 0 \\ - & - & + \\ 0 & + & 0 \end{bmatrix}$$

(a) (b) (c) (d) (e) (f) (g)

$$\begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & + & - \end{bmatrix} \quad \begin{bmatrix} + & + & 0 \\ - & 0 & + \\ 0 & - & - \end{bmatrix} \quad \begin{bmatrix} - & + & 0 \\ + & 0 & + \\ + & - & 0 \end{bmatrix} \quad \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ - & + & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & + & 0 \\ - & - & + \\ - & + & 0 \end{bmatrix} \quad \begin{bmatrix} - & + & 0 \\ 0 & + & + \\ - & 0 & - \end{bmatrix}$$

(h) (i) (j) (k) (l) (m)

Table 3: Nonequivalent sign patterns that are PETI (potentially exhibiting Turing Instability)

boxed expressions, we will also be able to construct specific values for the entries of A so that it exhibits Turing instability.

If A has nonzero pattern 1 in Table 2, then

$$\begin{aligned}
c_1(t) &= (p_1+p_2+p_3)t-(a_{11}+a_{22}) \\
c_2(t) &= (p_1p_2+p_1p_3+p_2p_3)t^2-(a_{11}+a_{22})p_3t-\boxed{(a_{11}p_2+a_{22}p_1)}\,t+a_{11}a_{22}-a_{32} \\
c_3(t) &= p_1p_2p_3t^3-\boxed{(a_{11}p_2+a_{22}p_1)}\,p_3t^2+\boxed{a_{11}a_{22}p_3-a_{32}p_1}\,t+a_{11}a_{32}-a_{31} \\
h(t) &= \Big[(p_1p_2+p_1p_3+p_2p_3)(p_1+p_2+p_3)-p_1p_2p_3\Big]t^3 \\
&\quad -\Big[(a_{11}+a_{22})(2p_1p_2+2p_1p_3+2p_2p_3+p_3^2)+\boxed{a_{11}p_2^2+a_{22}p_1^2}\,\Big]t^2 \\
&\quad +\Big[(a_{11}+a_{22})^2p_3+a_{11}^2p_2+a_{22}^2p_1+\boxed{2a_{11}a_{22}(p_1+p_2)-a_{32}(p_2+p_3)}\,\Big]t \\
&\quad -\Big[(a_{11}+a_{22})(a_{11}a_{22}-a_{32})+a_{11}a_{32}-a_{31}\Big]
\end{aligned}$$

If $a_{11}, a_{22} < 0$ and $a_{32} < 0$, then $c_1(t), c_2(t), c_3(t), h(t) > 0$ for any $t \geq 0$ and so in this case, A cannot exhibit Turing instability. Otherwise, we have the following.

- If $a_{11}, a_{22} < 0$ but $a_{32} > 0$, then $a_{31} > 0$ so that $c_3(0) > 0$. and we have the sign pattern in Table 3(a). This sign pattern is PETI using the matrix A, $t = 1$ and P in Table 4(a).

- If $a_{11} < 0 < a_{22}$, then $a_{32} < 0$ so that $c_2(0) > 0$. Note also that $h(0) = -a_{22}c_2(0) - a_{11}^2a_{22} + a_{31}$. Hence $a_{31} > 0$ so that $h(0) > 0$. Thus, we have the sign pattern in Table 3(b), which is PETI using the matrix in Table 4(b).

- If $a_{22} < 0 < a_{11}$, then $a_{32} < 0$ so that $c_2(0) > 0$ and $a_{31} < 0$ so that $c_3(0) > 0$. Thus, we have the sign pattern shown in Table 3(c). The matrix in Table 4(c) can be used to show that this sign pattern is PETI.

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 1 \\ -3 & 1 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & 1 & 1 \\ 7 & -4 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ -5 & -4 & 0 \end{bmatrix}$$

(a) $P = \mathrm{diag}(1,0,0)$ $\quad$ (b) $P = \mathrm{diag}(1,0,0)$ $\quad$ (c) $P = \mathrm{diag}(0,2,0)$

Table 4

If A has nonzero pattern 2 in Table 2, then

$$\begin{aligned}
c_1(t) &= (p_1+p_2+p_3)t - (a_{11}+a_{22}) \\
c_2(t) &= (p_1p_2+p_1p_3+p_2p_3)t^2 - (a_{11}+a_{22})p_3t - \boxed{(a_{11}p_2+a_{22}p_1)}\, t + a_{11}a_{22} - a_{21} \\
c_3(t) &= p_1p_2p_3t^3 - \boxed{(a_{11}p_2+a_{22}p_1)}\, p_3t^2 + (a_{11}a_{22}-a_{21})p_3t - a_{31} \\
h(t) &= \Big[(p_1p_2+p_1p_3+p_2p_3)(p_1+p_2+p_3) - p_1p_2p_3\Big]t^3 \\
&\quad -\Big[(a_{11}+a_{22})(2p_1p_2+2p_1p_3+2p_2p_3+p_3^2) + \boxed{a_{11}p_2^2+a_{22}p_1^2}\,\Big]t^2 \\
&\quad +\Big[(a_{11}+a_{22})^2p_3 + (a_{11}+a_{22})\,\boxed{(a_{11}p_2+a_{22}p_1)} + (a_{11}a_{22}-a_{21})(p_1+p_2)\Big]t \\
&\quad -\Big[(a_{11}+a_{22})(a_{11}a_{22}-a_{21}) - a_{31}\Big]
\end{aligned}$$

If $a_{11}, a_{22} < 0$, then $c_1(t), c_2(t), c_3(t), h(t) > 0$ for any $t \geq 0$ and so in this case, A cannot exhibit Turing instability. On the other hand, if $a_{11}a_{22} < 0$ then $a_{21} < 0$ so that $c_2(0) > 0$ and $a_{31} < 0$ so that $c_3(0) > 0$. Then up to permutation similarity, transposition and signature similarity, the sign pattern of the stable matrix is shown in Table 3(d), which is PETI using the following matrices.

$$A = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 1 & 1 \\ -0.1 & 0 & 0 \end{bmatrix}, \qquad P = \mathrm{diag}(1,0,0)$$

If A has nonzero pattern 3 in Table 2, then

$$\begin{aligned}
c_1(t) &= (p_1+p_2+p_3)t - (a_{11}+a_{22}) \\
c_2(t) &= (p_1p_2+p_1p_3+p_2p_3)t^2 - (a_{11}+a_{22})p_3t - \boxed{(a_{11}p_2+a_{22}p_1)}\, t + a_{11}a_{22} - a_{21} - a_{32} \\
c_3(t) &= p_1p_2p_3t^3 - \boxed{(a_{11}p_2+a_{22}p_1)}\, p_3t^2 + \boxed{a_{11}a_{22}p_3 - a_{21}p_3 - a_{32}p_1}\, t + a_{11}a_{32} \\
h(t) &= \Big[(p_1p_2+p_1p_3+p_2p_3)(p_1+p_2+p_3) - p_1p_2p_3\Big]t^3 \\
&\quad -\Big[(a_{11}+a_{22})(2p_1p_2+2p_1p_3+2p_2p_3+p_3^2) + \boxed{a_{11}p_2^2+a_{22}p_1^2}\,\Big]t^2 \\
&\quad +\Big[a_{11}^2(p_2+p_3) + a_{22}^2(p_1+p_3) + \boxed{2a_{11}a_{22}(p_1+p_2+p_3) - a_{21}(p_1+p_2) - a_{32}(p_2+p_3)}\,\Big]t \\
&\quad -(a_{11}+a_{22})(a_{11}a_{22}-a_{21}-a_{32}) + a_{11}a_{32}
\end{aligned}$$

If $a_{11}, a_{22} < 0$ then $a_{32} < 0$ (since $c_3(0) > 0$). If we assume further that $a_{21} < 0$, then $c_1(t), c_2(t), c_3(t), h(t) > 0$ for any $t \geq 0$. In this case, A cannot exhibit Turing instability. Meanwhile,

- if $a_{11}, a_{22} < 0$ and $a_{21} > 0$, then we have the sign pattern shown in Table 3(e), which is PETI using the example in Table 5(a).

- If $a_{11} < 0 < a_{22}$, then $a_{32} < 0$ so that $c_3(0) > 0$ and $a_{21} < 0$ since $h(0) = -a_{22}c_2(0) - a_{11}^2 a_{22} + a_{11}a_{21} > 0$. Hence, we have the sign pattern shown in Table 3(f). This is PETI using the example in Table 5(b).

- If $a_{22} < 0 < a_{11}$, then $a_{32} > 0$ so that $c_3(0) > 0$ and $a_{21} < 0$ so that $c_2(0) > 0$. This gives us the sign pattern shown in Table 3(g). The example Table 5(c) shows this sign pattern is PETI.

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 3 & -2 & 1 \\ 0 & -2 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} -2 & 1 & 0 \\ -4 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ -5 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(a) $P = \text{diag}(0,0,3)$ $\qquad$ (b) $P = \text{diag}(2,0,0)$ $\qquad$ (c) $P = \text{diag}(2,0,0)$

Table 5

If A has nonzero pattern 4 in Table 2, then

$$\begin{aligned}
c_1(t) &= (p_1 + p_2 + p_3)t - (a_{11} + a_{33}) \\
c_2(t) &= (p_1p_2 + p_1p_3 + p_2p_3)t^2 - (a_{11} + a_{33})p_2 t - \boxed{(a_{11}p_3 + a_{33}p_1)}\ t + a_{11}a_{33} - a_{21} - a_{32} \\
c_3(t) &= p_1p_2p_3t^3 - \boxed{(a_{11}p_3 + a_{33}p_1)}\ p_2t^2 + \boxed{(a_{11}a_{33}p_2 - a_{21}p_3 - a_{32}p_1)}\ t + a_{11}a_{32} + a_{21}a_{33} \\
h(t) &= ((p_1p_2 + p_1p_3 + p_2p_3)(p_1 + p_2 + p_3) - p_1p_2p_3)t^3 \\
&\quad - \left[\ \boxed{(a_{11}p_3^2 + a_{33}p_1^2)}\ + (a_{11} + a_{33})(2p_1p_2 + 2p_2p_3 + 2p_1p_3 + p_2^2)\right]t^2 \\
&\quad + \left[a_{11}^2(p_2 + p_3) + a_{33}(p_1 + p_2) + \boxed{2a_{11}a_{33}(p_1 + p_2 + p_3) - a_{21}(p_1 + p_2) - a_{32}(p_2 + p_3)}\ \right]t \\
&\quad - \left[(a_{11} + a_{33})a_{11}a_{33} - a_{11}a_{21} - a_{33}a_{32}\right]
\end{aligned}$$

If $a_{11}, a_{33} < 0$, then at least one of a_{21} or $_{32}$ must be negative for $c_3(0) > 0$. If both a_{21}, a_{32} are negative, then $c_1(t), c_2(t), c_3(t), h(t) > 0$ for any $t \geq 0$. In this case, the matrices having the said sign pattern cannot exhibit Turing instability. Otherwise,

- if $a_{11}, a_{33} < 0$ and exactly one of a_{21} or a_{32} is negative, then up to permutation similarity, transposition and signature similarity, the sign pattern of the stable matrix is shown in Table 3(h). Using the example in Table 6(a), we illustrate that this sign pattern is PETI.

- If $a_{11}a_{33} < 0$, then at least one of a_{32} or a_{21} must be negative for $c_2(0) > 0$. If both are negative, then the sign pattern of the matrix is equivalent to Table 3(i). This sign pattern is PETI using the example in Table 6(b).

We also consider the case when $a_{11}a_{33} < 0$ and $a_{32}a_{21} < 0$. Note that in this case, we must have $a_{11}a_{32} > 0$ and $a_{33}a_{21} > 0$ for $c_3(0) > 0$. In this case, the sign pattern of the matrix is equivalent

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & -3 & -2 \end{bmatrix}$$

$(a)\ P = \mathrm{diag}(2,0,0) \quad (b)\ P = \mathrm{diag}(0,0,2)$

Table 6

to the following.

$$\begin{bmatrix} + & + & 0 \\ - & 0 & + \\ 0 & + & - \end{bmatrix}$$

Note however, that this sign pattern is not potentially stable since the equations $c_2(0) > 0$ and $h(0) > 0$ will imply the following impossible inequality

$$0 < a_{32} < a_{11}a_{33} - a_{21} < a_{32}\frac{a_{33}}{a_{11}} - a_{33}^2 < 0.$$

If A has nonzero pattern 5 in Table 2, then

$$\begin{aligned} c_1(t) &= (p_1+p_2+p_3)t - a_{11} \\ c_2(t) &= (p_1p_2+p_1p_3+p_2p_3)t^2 - a_{11}(p_2+p_3)t - a_{21} - a_{32} \\ c_3(t) &= p_1p_2p_3t^3 - a_{11}p_2p_3t^2 - \boxed{(a_{21}p_3 + a_{32}p_1)}\ t - a_{31} + a_{11}a_{32} \\ h(t) &= (p_1+p_2)(p_1+p_3)(p_2+p_3))t^3 - a_{11}(p_2+p_3)(2p_1+p_2+p_3)t^2 \\ &\quad + \left[a_{11}^2(p_2+p_3) - (a_{21}+a_{32})p_2 - \boxed{(a_{21}p_1 + a_{32}p_3)}\ \right]t + a_{31} + a_{11}a_{21} \end{aligned}$$

Note that $a_{11} < 0$ and at least one of a_{21} and a_{32} is negative for $c_1(0)$ and $c_2(0)$ to be positive. If a_{21} and a_{32} are both negative, then the matrix cannot exhibit Turing instability.

- Suppose $a_{32} < 0 < a_{21}$. Then $a_{31} > 0$ for $h(0) > 0$. Thus, the sign pattern is as shown in Table 3(j). This sign pattern is PETI using the following example.

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & -3 & 0 \end{bmatrix}, \quad P = \mathrm{diag}(0,0,1)$$

- Suppose $a_{21} < 0 < a_{32}$. Then $a_{31} < 0$ for $c_3(0) > 0$. Thus, the sign pattern is as shown in Table 3(k). This sign pattern is PETI using the following example.

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -3 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}, \quad P = \mathrm{diag}(0,0,1)$$

If A has nonzero pattern 6 in Table 2, then

$$\begin{aligned}
c_1(t) &= (p_1+p_2+p_3)t - a_{22} \\
c_2(t) &= (p_1p_2+p_1p_3+p_2p_3)t^2 - a_{22}(p_1+p_3)t - a_{21} - a_{32} \\
c_3(t) &= p_1p_2p_3t^3 - a_{22}p_1p_3t^2 - \boxed{(a_{21}p_3+a_{32}p_1)}\; t - a_{31} \\
h(t) &= (p_1+p_2)(p_1+p_3)(p_2+p_3))t^3 - a_{22}(p_1+p_3)(p_1+2p_2+p_3)t^2 \\
&\quad + \left[a_{22}^2(p_1+p_3) - (a_{21}+a_{32})p_2 - \boxed{(a_{21}p_1+a_{32}p_3)}\;\right] t + a_{31} + a_{21}a_{22}
\end{aligned}$$

Note that a_{22}, a_{31} and a_{21} must all be negative for $c_1(0), c_3(0)$ and $h(0)$ to be positive. If $a_{32} < 0$, then the matrix cannot exhibit Turing instability. On the other hand, if $a_{32} > 0$, then we have the PETI sign pattern given in Table 3(l). The following matrix below is an example.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}, \quad P = \mathrm{diag}(1,0,0)$$

If A has nonzero pattern 7 in Table 2, then

$$\begin{aligned}
c_1(t) &= (p_1+p_2+p_3)t - (a_{11}+a_{22}+a_{33}) \\
c_2(t) &= (p_1p_2+p_1p_3+p_2p_3)t^2 - \left[\;\boxed{a_{11}(p_2+p_3)+a_{22}(p_1+p_3)+a_{33}(p_1+p_2)}\;\right] t \\
&\quad + a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} \\
c_3(t) &= p_1p_2p_3t^3 - \boxed{(a_{11}p_2p_3+a_{22}p_1p_3+a_{33}p_1p_2)}\; t^2 \\
&\quad + \boxed{(a_{11}a_{22}p_3+a_{11}a_{33}p_2+a_{22}a_{33}p_1)}\; t - a_{31} - a_{11}a_{22}a_{33} \\
h(t) &= \left[(p_1p_2+p_1p_3+p_2p_3)(p_1+p_2+p_3) - p_1p_2p_3\right] t^3 \\
&\quad - \left[(a_{11}+a_{22}+a_{33})(2p_1p_2+2p_2p_3+2p_1p_3) + \boxed{a_{11}(p_2^2+p_3^2)+a_{22}(p_1^2+p_3^2)+a_{33}(p_1^2+p_2^2)}\;\right] t^2 \\
&\quad + \left[a_{11}^2(p_2+p_3)+a_{22}^2(p_1+p_3)+a_{33}^2(p_1+p_2)+2(a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33})(p_1+p_2+p_3)\right] t \\
&\quad + a_{31} - (a_{11}+a_{22}+a_{33})(a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}) + a_{11}a_{22}a_{33}
\end{aligned}$$

Note that at least one of the diagonal entries of A must be negative. If a_{11}, a_{22}, a_{33} are all negative, then the matrix cannot exhibit Turing instability.

- Suppose that exactly one of a_{11}, a_{22} and a_{33} is negative. Then the matrix has one of the following sign patterns

$$\begin{bmatrix} - & + & 0 \\ 0 & + & + \\ + & 0 & + \end{bmatrix}, \begin{bmatrix} - & + & 0 \\ 0 & + & + \\ - & 0 & + \end{bmatrix}, \begin{bmatrix} + & + & 0 \\ 0 & - & + \\ + & 0 & + \end{bmatrix}, \begin{bmatrix} + & + & 0 \\ 0 & - & + \\ - & 0 & + \end{bmatrix}$$

  The first two sign patterns are not potentially stable since $c_1(0), c_2(0) > 0$ will imply that

$$a_{22} + a_{33} < -a_{11} < \frac{a_{22}a_{33}}{a_{22}+a_{33}} \Longrightarrow (a_{22}+a_{33})^2 < a_{22}a_{33} \Longrightarrow a_{22}^2 + a_{33}^2 + a_{22}a_{33} < 0.$$

Similarly, the latter two sign patterns are not potentially stable since $c_1(0), c_2(0) > 0$ will imply that

$$a_{11} + a_{33} < -a_{22} < \frac{a_{11}a_{33}}{a_{11}+a_{33}} \Longrightarrow (a_{11}+a_{33})^2 < a_{11}a_{33} \Longrightarrow a_{11}^2 + a_{33}^2 + a_{11}a_{33} < 0.$$

- Suppose that exactly two of a_{11}, a_{22} and a_{33} are negative. Then $a_{31} < 0$ so that $c_3(0) > 0$. Note that the matrix is equivalent to the sign pattern in Table 3(m), which is PETI using the following example.

$$\begin{bmatrix} -3 & 1 & 0 \\ 0 & 1 & 1 \\ -10 & 0 & -3 \end{bmatrix} \qquad P = \mathrm{diag}(1,0,0)$$

□

5 Conclusion and Further Research

In this paper, we show that for any positive integer $n \geq 3$, there is a stable irreducible $n \times n$ matrix A with $2n + 1 - \lfloor \frac{n}{3} \rfloor$ nonzero entries exhibiting Turing instability. When $n = 3$, the result is best possible, i.e., every 3×3 stable matrix with five or fewer nonzero entries will not exhibit Turing instability. Furthermore, we determine all possible sign patterns of 3×3 matrix A with 6 nonzero entries which exhibit Turning instability. There are many interesting problems worth studying.

1. Can we determine the exact value S_n, the smallest number of nonzero entries for the existence of a stable matrix A, which will exhibit Turing stability.

2. Determine the sign patterns of matrices A (with smallest number of nonzero entries) which exhibit Turing instability.

With more involved calculations, some of our techniques may be used to study 4×4 matrices. New techniques are needed to study the general problems.

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