Linear Algebra and its Applications 736 (2026) 137–174

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Linear maps preserving matrices annihilated by a fixed polynomial, II

Chi-Kwong Li [a], Ming-Cheng Tsai [b,*], Ya-Shu Wang [c], Ngai-Ching Wong [d,e]

[a] *Department of Mathematics, The College of William & Mary, Williamsburg, VA 13185, USA*
[b] *General Education Center, National Taipei University of Technology, Taipei 10608, Taiwan*
[c] *Department of Applied Mathematics, National Chung Hsing University, Tainting 40227, Taiwan*
[d] *Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, 80424, Taiwan*
[e] *Department of Healthcare Administration and Medical Information, Kaohsiung Medical University, 80708 Kaohsiung, Taiwan*

A R T I C L E   I N F O

Article history:
Received 19 September 2024
Received in revised form 7 January 2026
Accepted 26 January 2026
Available online 29 January 2026
Submitted by P. Semrl

Dedicated to Professor Yiu-Tung Poon on the occasion of his 70th birthday

MSC:
08A35
15A86
46L40
47B49

A B S T R A C T

Let $\mathbf{V}_n(\mathbb{F}) = \mathbf{S}_n(\mathbb{F})$ or $\mathbf{M}_n(\mathbb{F})$ be the sets of symmetric matrices or general matrices over any field $\mathbb{F}$. Let $f(x)$ be a polynomial splitting in $\mathbb{F}$ with degree at least 2 and a simple zero, such that the zeroes do not form an additive coset. Suppose a unital linear map $\Phi\colon \mathbf{V}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ satisfies $f(\Phi(A)) = 0$ whenever $f(A) = 0$. Except when $\mathbf{V}_n(\mathbb{F}) = \mathbf{M}_2(\mathbb{Z}_2), \mathbf{S}_n(\mathbb{Z}_2), \mathbf{S}_3(\mathbb{Z}_3), \mathbf{S}_2(\mathbb{F})$ with $|\mathbb{F}| \le 5$, it is shown that Φ has the form

$$A \mapsto S \begin{pmatrix} A \otimes I_p & \\ & A^{\mathrm{t}} \otimes I_q \end{pmatrix} S^{-1}.$$

The unital assumption can be removed and a similar form of Φ is obtained, when either 0 is another simple zero of $f(x)$, or $f(0) \neq 0$ and $\Phi(I)$ commutes with every element in the

* Corresponding author.
E-mail addresses: ckli@math.wm.edu (C.-K. Li), mctsai2@mail.ntut.edu.tw (M.-C. Tsai), yashu@nchu.edu.tw (Y.-S. Wang), wong@math.nsysu.edu.tw (N.-C. Wong).

https://doi.org/10.1016/j.laa.2026.01.029

Keywords:
Polynomial zero preservers
Idempotent preservers
Symmetric matrices

range of Φ. Counterexamples are given to the assertion in the exceptional cases.

1. Introduction

Linear preserver problems have attracted the attention of many researchers; see for example [1,3,9,10,14–16,18–21], and the references therein. In [12] we characterize those linear maps $\Phi\colon \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ between matrix algebras over an arbitrary field $\mathbb{F}$ preserving matrices annihilated by a fixed polynomial $f(x) \in \mathbb{F}[x]$, that is, $f(\Phi(A)) = 0$ whenever $f(A) = 0$. We show that such a map Φ will send disjoint rank one idempotents to disjoint idempotents, provided that $\Phi(I_n) = I_r$, the zeroes of $f(x)$ are all simple, and they do not form an additive coset, i.e. $a - b + c$ is not always a zero of $f(x)$ when a, b, c are. Then we conclude that Φ has a nice form using the following

Theorem 1.1 *([12, Theorem 2.1]). Let $\mathbb{F}$ be an arbitrary field, $n \geq 2$ be a positive integer, and $\mathbf{M}_n(\mathbb{F}) \neq \mathbf{M}_2(\mathbb{Z}_2)$. Suppose a linear map $\Phi\colon \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ sends disjoint rank one idempotents to disjoint idempotents. Then there are nonnegative integers p, q with $s = n(p + q) \leq r$, and an invertible matrix S in $\mathbf{M}_r(\mathbb{F})$ such that Φ has the form*

$$\Phi(A) = S \begin{pmatrix} A \otimes I_p & & \\ & A^{\mathrm{t}} \otimes I_q & \\ & & 0_{r-s} \end{pmatrix} S^{-1} \quad \text{for all } A \in \mathbf{M}_n(\mathbb{F}). \tag{1.1}$$

One may see the formal definitions of disjoint idempotents, the symbol $A \otimes B$ and other notions in Section 2.

In this paper, we relax the conditions by assuming instead that $f(x)$ has at least one (rather than all) simple zero and Φ preserves *symmetric* (rather than all) matrices annihilated by $f(x)$. More precisely, consider a fixed polynomial $f(x) = (x - a_1)(x - a_2)(x - a_3) \cdots (x - a_m)$ splitting in $\mathbb{F}$. Let $\Phi\colon \mathbf{S}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ be a unital linear map from $n \times n$ symmetric matrices into $r \times r$ general matrices such that $f(\Phi(A)) = 0$ whenever $f(A) = 0$. We show in Theorem 3.4 that if $f(x)$ has a simple zero, and the set $Z(f)$ of zeroes is not an additive coset of $(\mathbb{F},+)$, then Φ sends disjoint rank one symmetric idempotents to disjoint idempotents. Then our new Theorem 1.2 below applies, where $|\mathbb{F}|$ is the cardinality of the underlying field $\mathbb{F}$.

Theorem 1.2. *Let $\mathbb{F}$ be an arbitrary field, $n \geq 2$ be a positive integer, and $\Phi\colon \mathbf{S}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ be a linear map. Suppose that $\mathbf{S}_n(\mathbb{F}) \neq \mathbf{S}_n(\mathbb{Z}_2), \mathbf{S}_3(\mathbb{Z}_3), \mathbf{S}_2(\mathbb{F})$ with $|\mathbb{F}| \leq 5$. Then Φ sends disjoint rank one symmetric idempotents to disjoint idempotents if and only if there is a nonnegative integer k and an invertible matrix S in $\mathbf{M}_r(\mathbb{F})$ such that Φ has the form*

$$\Phi(A) = S \begin{pmatrix} A \otimes I_k & 0 \\ 0 & 0_{r-nk} \end{pmatrix} S^{-1} \quad \textit{for all } A \in \mathbf{S}_n(\mathbb{F}). \tag{1.2}$$

The necessity may fail in the exceptional cases, as counter examples exist.

It is worth noting that our approach is different from [5,8,17], in which the preservers are assumed to be surjective and the underlying field is either the field $\mathbb{C}$ of complex numbers or algebraically closed. One may see other related works in [6,10,11]. A result similar to Theorem 1.2 was obtained in [21, Theorem 6.3.1] when the characteristic of $\mathbb{F}$ is not $2, 3$, or 5. In Theorem 1.2, we only exclude the cases $\mathbf{S}_n(\mathbb{F}) = \mathbf{S}_n(\mathbb{Z}_2), \mathbf{S}_3(\mathbb{Z}_3), \mathbf{S}_2(\mathbb{F})$ with $|\mathbb{F}| \leq 5$.

When $\mathbb{F}$ has characteristic 2, the linear map $\Phi\colon \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ defined by $A \mapsto \operatorname{trace}(A)E$ for any idempotent $E \in \mathbf{M}_r(\mathbb{F})$, sends idempotents to idempotents, as well as being a linear Jordan homomorphism. But Φ does not have the form (1.1) or (1.2). Therefore, we assume a stronger condition that Φ sends disjoint rank one idempotents to disjoint idempotents rather than that Φ sends idempotents to idempotents in Theorems 1.1 and 1.2. Note that when $\mathbb{F}$ has characteristic not 2, linear idempotent preservers do send disjoint rank one idempotents to disjoint idempotents (see Corollary 2.7).

For the exception cases, when $\mathbb{F} = \mathbb{Z}_2$, a symmetric idempotent might not be a sum of mutually disjoint rank one symmetric idempotents. For example, consider the rank two symmetric idempotent

$$E = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \in \mathbf{S}_3(\mathbb{Z}_2), \tag{1.3}$$

which cannot be written as a sum of disjoint symmetric rank one idempotents. In fact, a rank one symmetric idempotent in $\mathbf{S}_3(\mathbb{Z}_2)$ assumes the form ee^{t} with $e^{\mathrm{t}}e = 1$ in $\mathbb{Z}_2$ for some vector $e \in \mathbb{Z}^3$. Hence, e can only be $(1 \;\; 0 \;\; 0)^{\mathrm{t}}$, $(0 \;\; 1 \;\; 0)^{\mathrm{t}}$, $(0 \;\; 0 \;\; 1)^{\mathrm{t}}$ or $(1 \;\; 1 \;\; 1)^{\mathrm{t}}$. In matrix form, all symmetric rank one matrices ee^{t} in $\mathbf{S}_3(\mathbb{Z}_2)$ are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{1.4}$$

The first three are mutually disjoint, while the fourth is not disjoint from the others. Observing the diagonal entries, we see that E cannot be written as a sum of the first three rank one symmetric idempotents. Hence a linear map of $\mathbf{S}_n(\mathbb{Z}_2)$ might not send idempotents to idempotents even if it sends disjoint rank one idempotents to disjoint idempotents.

While the symmetric idempotent E is the sum of all four rank one symmetric idempotents in (1.4), we will see in Remark 2.8(1) that the symmetric matrix $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 0_{n-2} \in S_n(\mathbb{Z}_2)$ is linearly independent from rank one symmetric idempotents. Consequently, the behavior of a linear map of $\mathbf{S}_n(\mathbb{Z}_2)$ towards its action on rank one idempotents does not suffice to determine its action at Q. This explains why we need to exclude the case when $\mathbb{F} = \mathbb{Z}_2$ from Theorem 1.2.

For $\mathbf{S}_n(\mathbb{F}) = \mathbf{S}_3(\mathbb{Z}_3)$ or $\mathbf{S}_2(\mathbb{F})$ with $|\mathbb{F}| \leq 5$, the domain $\mathbf{S}_n(\mathbb{F})$ of Φ contains too few disjoint rank one idempotent pairs to ensure the form (1.2). See Remark 2.8 for more details.

In Section 2, we present the proof of Theorem 1.2 and the counterexamples for the exceptional cases when $\mathbf{S}_n(\mathbb{F}) = \mathbf{S}_n(\mathbb{Z}_2), \mathbf{S}_3(\mathbb{Z}_3)$ or $\mathbf{S}_2(\mathbb{F})$ with $|\mathbb{F}| \leq 5$. In Section 3, we apply Theorems 1.1 and 1.2 to study those linear maps Φ preserving general or symmetric matrices annihilated by a fixed polynomial $f(x)$. Under some mild conditions on $\Phi(I_n)$ and the zero set $Z(f)$ of $f(x)$, we see that Φ assumes the form (1.1) or (1.2), aside from the exceptional cases. In Section 4, we show that the matrix S in (1.1) and (1.2) can be chosen to be orthogonal when Φ preserves transposition.

In a forthcoming paper [13], we characterize those linear maps Φ between real symmetric matrix spaces or complex hermitian matrix spaces satisfying that $f(\Phi(A)) = 0$ whenever $\Phi(A) = 0$ for a fixed polynomial $f(x)$ with at least two distinct zeroes. We show that such a linear preserver has the form

$$A \mapsto S \begin{pmatrix} A \otimes I_{p_1} & & & & \\ & -A \otimes I_{p_2} & & & \\ & & A^{\mathrm{t}} \otimes I_{q_1} & & \\ & & & -A^{\mathrm{t}} \otimes I_{q_2} & \\ & & & & 0_{r-n(p_1+p_2+q_1+q_2)} \end{pmatrix} S^*,$$

in which $S^*S = I_r$ and p_1, p_2, q_1, q_2 are nonnegative integers with $r - n(p_1+p_2+q_1+q_2) \geq 0$, provided that $f(0) = 0$ or $\Phi(I_n)$ commutes with every element in the range of Φ.

2. Linear maps sending symmetric idempotents to idempotents

Let $\mathbb{F}$ be an arbitrary field, and $\mathbf{M}_{m\times n}(\mathbb{F})$ be the linear space of $m \times n$ matrices over $\mathbb{F}$. Let $\{e_1, \ldots, e_n\}$ be the standard basis for the (column) vector space $\mathbb{F}^n$ over $\mathbb{F}$, and let $\{E_{11}, E_{12}, \ldots, E_{mn}\}$ be the standard basis of matrix units for $\mathbf{M}_{m\times n}(\mathbb{F})$; namely, $E_{ij} = e_i e_j^{\mathrm{t}}$. Here, $A^{\mathrm{t}} = (a_{ji}) \in \mathbf{M}_{n\times m}$ is the transpose of a matrix $A = (a_{ij}) \in \mathbf{M}_{m\times n}(\mathbb{F})$.

Let $\mathbf{M}_n(\mathbb{F})$ (resp. $\mathbf{S}_n(\mathbb{F})$) be the algebra (resp. Jordan algebra) of $n \times n$ matrices (resp. symmetric matrices) over $\mathbb{F}$. Recall that a matrix $A = (a_{ij}) = \sum_{i,j=1}^n a_{ij}E_{ij} \in \mathbf{M}_n(\mathbb{F})$ is symmetric if its transpose $A^{\mathrm{t}} = (a_{ji}) = A$, and A is an idempotent if $A^2 = A$. Two idempotents E, F are said to be *disjoint* if $EF = FE = 0$. In particular, two symmetric idempotents E, F are disjoint if $EF = 0$.

The *Kronecker product* $A \otimes B$ of $A = (a_{ij}) \in \mathbf{M}_{m\times n}(\mathbb{F})$ and $B \in \mathbf{M}_{r\times s}(\mathbb{F})$ is the block matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{pmatrix} \in \mathbf{M}_{mr\times ns}(\mathbb{F}).$$

By abusing notations, the block matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix} = \sum_{\substack{i=1,\dots,m,\\ j=1,\dots,n}} E_{ij} \otimes A_{ij},$$

where $A_{11}, A_{12}, \dots, A_{mn}$ can have different but appropriate sizes.

We state some simple facts whose proofs are straightforward.

Lemma 2.1. *Let $\mathbb{F}$ be a field and $n \geq 2$ be a positive integer.*

(a) (i) *The linear space $\mathbf{M}_n(\mathbb{F})$ has the following basis consisting of rank one idempotents*

$$\{E_{jj} : 1 \leq j \leq n\} \cup \{E_{ii} + E_{ij} : 1 \leq i \leq n, i \neq j\}.$$

(ii) *For $i \neq j$, $E_{ii} + E_{ij}$ and $E_{jj} - E_{ij}$ are disjoint rank one idempotents.*

(b) *Suppose $\mathbb{F} \neq \mathbb{Z}_2$. Then there is $a \in \mathbb{F}$ such that $a(a^2 + 1) \neq 0$. For any such a,*

(i) *the linear space $\mathbf{S}_n(\mathbb{F})$ has the following basis consisting of symmetric rank one idempotents*

$$\{E_{jj} : 1 \leq j \leq n\} \cup \{(1 + a^2)^{-1}(a^2 E_{ii} + a(E_{ij} + E_{ji}) + E_{jj}) : 1 \leq i < j \leq n\};$$

(ii) $(1+a^2)^{-1}(a^2E_{ii}+a(E_{ij}+E_{ji})+E_{jj})$ *and* $(1+a^2)^{-1}(E_{ii}-a(E_{ij}+E_{ji})+a^2E_{jj})$ *are disjoint rank one idempotents whenever $i \neq j$.*

The proof of Theorem 1.2 is divided into several propositions. We first establish the main case.

Proposition 2.2. *The conclusion in Theorem 1.2 holds when $|\mathbb{F}| > 5$.*

Proof. The sufficiency is clear, and we are verifying the necessity. Assume that Φ sends disjoint rank one idempotents to disjoint idempotents. In particular, $\Phi(E_{11})$, $\Phi(E_{22})$, $\dots$, $\Phi(E_{nn})$ are disjoint idempotents in $\mathbf{M}_r(\mathbb{F})$. Arguing as in [12, Lemma 2.4], we see that there is an invertible $S \in \mathbf{M}_r(\mathbb{F})$ and nonnegative integers $k_1, \dots, k_n$ with $k_1 + \cdots + k_n = s \leq r$ such that

$$S^{-1}\Phi(E_{jj})S = 0_{k_1} \oplus \cdots \oplus 0_{k_{j-1}} \oplus I_{k_j} \oplus 0_{k_{j+1}} \oplus \cdots \oplus 0_{k_n} \oplus 0_{r-s}. \tag{2.1}$$

For simplicity of notation, we replace Φ by the map $A \mapsto S^{-1}\Phi(A)S$ and assume that (2.1) holds with $S = I_r$; namely,

$$\Phi(E_{jj}) = 0_{k_1} \oplus \cdots \oplus 0_{k_{j-1}} \oplus I_{k_j} \oplus 0_{k_{j+1}} \oplus \cdots \oplus 0_{k_n} \oplus 0_{r-s}. \tag{2.2}$$

Step 1. Since $\mathbb{F} \neq \mathbb{Z}_2$, Lemma 2.1 gives a nonzero $a \in \mathbb{F}$ such that $a^2 + 1 \neq 0$, and

$$X_1 = (1+a^2)^{-1}(E_{11} + a^2 E_{22} + a(E_{12} + E_{21})) = (1+a^2)^{-1}\begin{pmatrix} 1 & a \\ a & a^2 \end{pmatrix} \oplus 0_{n-2},$$

$$X_2 = (1+a^2)^{-1}(E_{11} + a^2 E_{22} - a(E_{12} + E_{21})) = (1+a^2)^{-1}\begin{pmatrix} a^2 & -a \\ -a & 1 \end{pmatrix} \oplus 0_{n-2}$$

are disjoint rank one symmetric idempotents in $\mathbf{S}_n(\mathbb{F})$.

Let

$$B = \Phi(E_{12} + E_{21}) = (B_{ij})_{1 \leq i,j \leq n+1}$$

with $B_{jj} \in \mathbf{M}_{k_j}(\mathbb{F})$ for $j = 1, \ldots, n$. For all $l \neq 1, 2$ and $i = 1, 2$, since $X_i E_{ll} = E_{ll} X_i = 0$, we see that

$$\Phi(X_i) = (1+a^2)^{-1}(\Phi(E_{11}) + a^2\Phi(E_{22}) + aB)$$

satisfies

$$0_r = \Phi(E_{ll})\Phi(X_i) = \Phi(X_i)\Phi(E_{ll}).$$

Thus, $B_{ij} = 0$ for all (i, j) except when

$$(i, j) = (1, 1), (1, 2), (2, 1), (2, 2), (1, n+1), (2, n+1), (n+1, 1), (n+1, 2), \text{ or } (n+1, n+1).$$

On the other hand, since $X_1 X_2 = X_2 X_1 = 0_n$, we have

$$0_r = (1+a^2)^2\Phi(X_1)\Phi(X_2) = (\Phi(E_{11}) + a^2\Phi(E_{22}) + aB)(a^2\Phi(E_{11}) + \Phi(E_{22}) - aB). \tag{2.3}$$

$$0_r = (1+a^2)^2\Phi(X_2)\Phi(X_1) = (a^2\Phi(E_{11}) + \Phi(E_{22}) - aB)(\Phi(E_{11}) + a^2\Phi(E_{22}) + aB). \tag{2.4}$$

By adding (2.3) and (2.4), we obtain that

$$a(a^2+1)(B\Phi(E_{11} + E_{22}) - \Phi(E_{11} + E_{22})B) = 0.$$

Hence, $B_{1,n+1}, B_{2,n+1}, B_{n+1,1}, B_{n+1,2}$ are all zero matrices. As a result,

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \oplus 0_{k_3} \oplus \cdots \oplus 0_{k_n} \oplus B_{n+1,n+1},$$

and thus by (2.2), (2.3) and (2.4),

$$\begin{aligned}
0_r &= \left[\begin{pmatrix} I_{k_1} & 0 \\ 0 & a^2 I_{k_2} \end{pmatrix} \oplus 0_{s'} + aB\right]\left[\begin{pmatrix} a^2 I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'} - aB\right] \\
&= \begin{pmatrix} a^2 I_{k_1} & 0 \\ 0 & a^2 I_{k_2} \end{pmatrix} \oplus 0_{s'} - a^2B^2 - a\begin{pmatrix} B_{11} & B_{12} \\ a^2B_{21} & a^2B_{22} \end{pmatrix} \oplus 0_{s'} + a\begin{pmatrix} a^2B_{11} & B_{12} \\ a^2B_{21} & B_{22} \end{pmatrix} \oplus 0_{s'} \\
&= a^2\begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'} - a^2B^2 - a\begin{pmatrix} (1-a^2)B_{11} & 0 \\ 0 & (a^2-1)B_{22} \end{pmatrix} \oplus 0_{s'},
\end{aligned}$$

where $s' = r - k_1 - k_2$. Therefore, $B^2_{n+1,n+1} = 0$, and

$$\begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'} - B^2 = \begin{pmatrix} \frac{1-a^2}{a}B_{11} & 0 \\ 0 & \frac{a^2-1}{a}B_{22} \end{pmatrix} \oplus 0_{s'}. \tag{2.5}$$

When $\mathbb{F}$ does not have characteristic 2, that is $1^2 + 1 \neq 0$, we can choose $a = 1$ in (2.5), and derive that

$$B^2 = \begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'}. \tag{2.6}$$

Suppose $\mathbb{F}$ has characteristic 2. Since $|\mathbb{F}| > 5$, we see that $\mathbb{F}$ contains at least 8 elements. Thus we can find distinct a_1, a_2, a_3 from $\mathbb{F}$ such that $a_i(a_i^2-1)(a_i^2+1) \neq 0$ for $i = 1, 2, 3$. We can also assume $a_1a_2 \neq -1$, for else we can replace a_2 with a_3. It follows from (2.5) that

$$\left(\frac{1-a_1^2}{a_1} - \frac{1-a_2^2}{a_2}\right)B_{11} = \frac{(a_2-a_1)(1+a_1a_2)}{a_1a_2}B_{11} = 0,$$

and thus $B_{11} = 0$. Similarly, $B_{22} = 0$, and (2.6) holds again.

Since $\Phi(X_1)$ is an idempotent, we have

$$\begin{aligned}
0_r &= \Phi(X_1) - \Phi(X_1)^2 \\
&= \frac{a}{1+a^2}B - \frac{a}{(1+a^2)^2}[(I_{k_1} \oplus a^2I_{k_2} \oplus 0_{s'})B + B(I_{k_1} \oplus a^2I_{k_2} \oplus 0_{s'})].
\end{aligned}$$

In particular, $B_{n+1,n+1} = 0$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \oplus 0_{s'}$.

Step 2. Since $|\mathbb{F}| > 5$, there is an element $a \in \mathbb{F}$ such that $a(a^2+1)(a^2-1) \neq 0$. Hence the equality (2.5) or (2.6) imply that

$$B_{11} = 0_{k_1}, \quad B_{22} = 0_{k_2}, \quad B_{12}B_{21} = I_{k_1} \quad \text{and} \quad B_{21}B_{12} = I_{k_2}.$$

Consequently, $k_1 = k_2$ and $B_{21} = B_{12}^{-1}$.

We can apply the arguments on $E_{11}, E_{22}, E_{12} + E_{21}$ to $E_{11}, E_{jj}, E_{1j} + E_{j1}$ to conclude that $k_1 = k_2 = \cdots = k_n$. Set this common value to be k. Furthermore, for $j = 2, \ldots, n$,

$$\Phi(E_{1j}+E_{j1}) = E_{1j}\otimes B_{1j} + E_{j1}\otimes B_{j1} \quad \text{with } B_{j1} = B_{1j}^{-1} \in \mathbf{M}_k(\mathbb{F}).$$

We may replace Φ with the map

$$X \mapsto (I_k \oplus B_{12} \oplus B_{13} \oplus \cdots \oplus B_{1n})\Phi(X)(I_k \oplus B_{12}^{-1} \oplus B_{13}^{-1} \oplus \cdots \oplus B_{1n}^{-1}),$$

so that

$$\Phi(E_{1j}+E_{j1}) = E_{1j}\otimes I_k + E_{j1}\otimes I_k \quad \text{for } j = 2,\dots,n.$$

If $n = 2$, then we are done.

Step 3. Suppose $n > 3$. We can apply the arguments on $E_{11}, E_{22}, E_{12}+E_{21}$ to $E_{ii}, E_{jj}, E_{ij}+E_{ji}$ to conclude that for any $2 \le i < j \le n$,

$$\Phi(E_{ij}+E_{ji}) = E_{ij}\otimes B_{ij} + E_{ji}\otimes B_{ji} \quad \text{with some } B_{ji} = B_{ij}^{-1} \in \mathbf{M}_k(\mathbb{F}). \tag{2.7}$$

We will show that $B_{ji} = B_{ij} = I_k$ for all $2 \le i < j \le n$.

Let $v = ce_1 + e_i + e_j$ for some nonzero $c \in \mathbb{F}$ such that $b = c^2 + 2 \neq 0$. Such c exists unless $\mathbb{F} = \mathbb{Z}_3$. Indeed, the polynomial $y^2 + 2 = 0$ has at most two nonzero solutions in $\mathbb{F}$, and thus such c exists in $\mathbb{F}$ if $\mathbb{F}$ contains at least four elements. On the other hand, $c = 1$ can be used if $\mathbb{F}$ has characteristic two. Consider the rank one symmetric matrix

$$A = b^{-1}vv^{\mathrm{t}} = b^{-1}\left(c^2E_{11} + c(E_{1i}+E_{i1}+E_{1j}+E_{j1}) + E_{ii} + E_{jj} + E_{ij} + E_{ji}\right)$$

in $\mathbf{S}_n(\mathbb{F})$. Observe that

$$A^2 = b^{-2}v(v^{\mathrm{t}}v)v^{\mathrm{t}} = b^{-1}vv^{\mathrm{t}} = A$$

is an idempotent. Now,

$$\Phi(A) = b^{-1}(c^2\Phi(E_{11}) + c\Phi(E_{1i}+E_{i1}+E_{1j}+E_{j1}) + \Phi(E_{ii}+E_{jj}+E_{ij}+E_{ij})).$$

Up to a permutation similarity, $0_r = \Phi(A)^2 - \Phi(A)$ is a direct sum of a zero matrix and $b^{-2}(C^2 - bC)$ with

$$C = \begin{pmatrix} c^2I_k & cI_k & cI_k \\ cI_k & I_k & B_{ij} \\ cI_k & B_{ji} & I_k \end{pmatrix}.$$

Looking at the $(1,3)$-th block of $C^2 - bC$, we see that

$$0_k = b^{-2}[(c^3I_k + cB_{ij} + cI_k) - (c^2+2)cI_k] = b^{-1}c(B_{ij} - I_k),$$

and hence $B_{ij} = I_k$. Thus, $B_{ji} = B_{ij}^{-1} = I_k$. Consequently, we have

$$\Phi(E_{ii}) = E_{ii} \otimes I_k \quad \text{and} \quad \Phi(E_{ij} + E_{ji}) = (E_{ij} + E_{ji}) \otimes I_k \quad \text{for } i, j = 1, \ldots, n \text{ with } i \neq j.$$

Since $\{E_{ii}, E_{ij} + E_{ji} : i, j = 1, \ldots, n, i \neq j\}$ is a basis for $\mathbf{S}_n(\mathbb{F})$, the result follows. □

Theorem 1.2 will be established after the following three additional propositions are verified.

Proposition 2.3. *The conclusion in Theorem 1.2 holds when $n \geq 4$ and $\mathbb{F} = \mathbb{Z}_3$.*

Proof. It suffices to check the necessity.
Step 1. Since $\mathbb{Z}_3$ contains a nonzero element $a = 1$ such that $a^2 + 1 \neq 0$, we can proceed as in Step 1 in the proof of Proposition 2.2 and conclude that, after a similarity transformation if necessary,

$$\Phi(E_{jj}) = 0_{k_1} \oplus \cdots \oplus 0_{k_{j-1}} \oplus I_{k_j} \oplus 0_{k_{j+1}} \oplus \cdots \oplus 0_{k_n} \oplus 0_{r-s} \quad \text{for } j = 1, \ldots, n,$$
$$B = \Phi(E_{12} + E_{21}) = \begin{pmatrix} B_{11}^{(12)} & B_{12}^{(12)} \\ B_{21}^{(12)} & B_{22}^{(12)} \end{pmatrix} \oplus 0_{s'}$$

with

$$B^2 = E_{11} \otimes I_{k_1} + E_{22} \otimes I_{k_2},$$

where $B_{ij}^{(12)}$ is an $k_i \times k_j$ matrix for $i, j = 1, 2$, $s = r - k_1 - k_2 - \cdots - k_n$ and $s' = r - k_1 - k_2$.

Applying the arguments on $E_{11}, E_{22}, E_{12} + E_{21}$ to $E_{ii}, E_{jj}, E_{ij} + E_{ji}$, we can conclude for any $1 \leq i < j \leq n$, with

$$U_{ij} = E_{ij} + E_{ji},$$

that

$$\Phi(U_{ij}) = \Phi(E_{ij} + E_{ji}) = E_{ii} \otimes B_{ii}^{(ij)} + E_{ij} \otimes B_{ij}^{(ij)} + E_{ji} \otimes B_{ji}^{(ij)} + E_{jj} \otimes B_{jj}^{(ij)}, \quad (2.8)$$

for some $B_{ii}^{(ij)} \in \mathbf{M}_{k_i}(\mathbb{Z}_3)$, $B_{ij}^{(ij)} \in \mathbf{M}_{k_i k_j}(\mathbb{Z}_3)$, $B_{ji}^{(ij)} \in \mathbf{M}_{k_j k_i}(\mathbb{Z}_3)$, $B_{jj}^{(ij)} \in \mathbf{M}_{k_j}(\mathbb{Z}_3)$, and

$$\Phi(U_{ij})^2 = E_{ii} \otimes I_{k_i} + E_{jj} \otimes I_{k_j}.$$

Consider the rank one symmetric idempotents

$$X_1 = -(e_1 + e_2)(e_1 + e_2)^{\mathrm{t}}, \quad X_2 = -(e_1 - e_2)(e_1 - e_2)^{\mathrm{t}},$$
$$X_3 = -(e_3 + e_4)(e_3 + e_4)^{\mathrm{t}}, \quad X_4 = -(e_3 - e_4)(e_3 - e_4)^{\mathrm{t}},$$

and

$$\begin{aligned}
A_1 &= (e_1 - e_2 + e_3 + e_4)(e_1 - e_2 + e_3 + e_4)^{\mathrm{t}}, \\
A_2 &= (e_1 - e_2 + e_3 - e_4)(e_1 - e_2 + e_3 - e_4)^{\mathrm{t}}, \\
B_1 &= (e_1 + e_2 + e_3 + e_4)(e_1 + e_2 + e_3 + e_4)^{\mathrm{t}}, \\
B_2 &= (e_1 + e_2 + e_3 - e_4)(e_1 + e_2 + e_3 - e_4)^{\mathrm{t}}.
\end{aligned}$$

Since

$$X_1A_1 = X_1A_2 = X_2B_1 = X_2B_2 = X_1X_2 = X_1X_3 = X_1X_4 = X_2X_3 = X_2X_4 = 0,$$

we have

$$\begin{aligned}
0 = \Phi(X_1)\Phi(A_1) &= \Phi(X_1)\left[\Phi(U_{13} - U_{23}) + \Phi(U_{14} - U_{24})\right], \\
0 = \Phi(X_1)\Phi(A_2) &= \Phi(X_1)\left[\Phi(U_{13} - U_{23}) - \Phi(U_{14} - U_{24})\right], \\
0 = \Phi(X_2)\Phi(B_1) &= \Phi(X_2)\left[\Phi(U_{13} + U_{23}) + \Phi(U_{14} + U_{24})\right], \\
0 = \Phi(X_2)\Phi(B_2) &= \Phi(X_2)\left[\Phi(U_{13} + U_{23}) - \Phi(U_{14} + U_{24})\right].
\end{aligned}$$

It follows

$$0 = \Phi(X_1)\Phi(U_{13} - U_{23}) = -\left[\begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'} + \Phi(U_{12})\right]\left[\Phi(U_{13}) - \Phi(U_{23})\right],$$

and

$$0 = \Phi(X_2)\Phi(U_{13} + U_{23}) = -\left[\begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'} - \Phi(U_{12})\right]\left[\Phi(U_{13}) + \Phi(U_{23})\right].$$

Hence,

$$\Phi(U_{12})\Phi(U_{13}) = \left[\begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'}\right]\Phi(U_{23})$$

and

$$\Phi(U_{12})\Phi(U_{23}) = \left[\begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'}\right]\Phi(U_{13}).$$

Step 2. With a direct calculation, we have

$$\begin{pmatrix} B_{11}^{(12)}B_{11}^{(13)} & 0 & B_{11}^{(12)}B_{13}^{(13)} & 0 \\ B_{21}^{(12)}B_{11}^{(13)} & 0 & B_{21}^{(12)}B_{13}^{(13)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & B_{22}^{(23)} & B_{23}^{(23)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.9}$$

and

$$\begin{pmatrix} 0 & B_{12}^{(12)}B_{22}^{(23)} & B_{12}^{(12)}B_{23}^{(23)} & 0 \\ 0 & B_{22}^{(12)}B_{22}^{(23)} & B_{22}^{(12)}B_{23}^{(23)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} B_{11}^{(13)} & 0 & B_{13}^{(13)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence,

$$B_{11}^{(13)} = 0, \quad B_{22}^{(23)} = 0, \quad B_{11}^{(12)}B_{13}^{(13)} = 0 \quad \text{and} \quad B_{22}^{(12)}B_{23}^{(23)} = 0.$$

Since $\Phi(U_{ij})^2 = E_{ii} \otimes I_{k_i} + E_{jj} \otimes I_{k_j}$ for $(i, j) = (1, 3)$ and $(2, 3)$, we have

$$B_{13}^{(13)}B_{31}^{(13)} = I_{k_1} \quad \text{and} \quad B_{23}^{(23)}B_{32}^{(23)} = I_{k_2}.$$

Therefore,

$$\begin{aligned} B_{11}^{(12)} &= B_{11}^{(12)}I_{k_1} = B_{11}^{(12)}B_{13}^{(13)}B_{31}^{(13)} = 0_{k_1} \quad \text{and} \\ B_{22}^{(12)} &= B_{22}^{(12)}I_{k_2} = B_{22}^{(12)}B_{23}^{(23)}B_{32}^{(23)} = 0_{k_2}. \end{aligned}$$

Again, since $\Phi(U_{12})^2 = E_{11} \otimes I_{k_1} + E_{22} \otimes I_{k_2}$, the form (2.8) can be rewritten as

$$\Phi(U_{12}) = \Phi(E_{12} + E_{21}) = E_{12} \otimes B_{12}^{(12)} + E_{21} \otimes B_{21}^{(12)},$$

for some $B_{12}^{(12)} \in \mathbf{M}_{k_1 k_2}(\mathbb{Z}_3)$ and $B_{21}^{(12)} \in \mathbf{M}_{k_2 k_1}(\mathbb{Z}_3)$ with

$$B_{12}^{(12)}B_{21}^{(12)} = I_{k_1} \quad \text{and} \quad B_{21}^{(12)}B_{12}^{(12)} = I_{k_2}.$$

Hence, $k_1 = k_2$ and $B_{21}^{(12)} = {B_{12}^{(12)}}^{-1}$.

We can apply the arguments on e_1, e_2, e_3, e_4 to any other quartet of e_i, e_j, e_p, e_q whenever $1 \leq i < j < p < q \leq n$ to conclude that $k_1 = k_2 = \cdots = k_n := k$, $B_{ii}^{(ij)} = B_{jj}^{(ij)} = 0_k$, and $B_{ji}^{(ij)} = {B_{ij}^{(ij)}}^{-1}$ whenever $1 \leq i < j \leq n$. Consequently, we can write $B_{ij} = B_{ij}^{(ij)}$ and $B_{ji} = B_{ji}^{(ij)}$, and (2.8) becomes

$$\Phi(E_{ij} + E_{ji}) = E_{ij} \otimes B_{ij} + E_{ji} \otimes B_{ji} \quad \text{with } B_{ji} = B_{ij}^{-1} \in \mathbf{M}_k(\mathbb{F})$$

whenever $1 \leq i < j \leq n$.

We may replace Φ with the map

$$X \mapsto (I_k \oplus B_{12} \oplus B_{13} \oplus \cdots \oplus B_{1n})\Phi(X)(I_k \oplus B_{12}^{-1} \oplus B_{13}^{-1} \oplus \cdots \oplus B_{1n}^{-1}),$$

so that

$$B_{1j} = B_{j1} = I_k \quad \text{for } j = 2, \dots, n.$$

Comparing the $(2,3)$-th blocks in (2.9), we have

$$B_{23} = B_{21}B_{13} = I_k, \quad \text{and then} \quad B_{32} = B_{23}^{-1} = I_k.$$

As a result,

$$\Phi(E_{ij} + E_{ji}) = E_{ij} \otimes I_k + E_{ji} \otimes I_k \quad \text{for all } i, j = 1, 2, 3. \tag{2.10}$$

Replacing $1, 2, 3$ with any other indices $1, i, j$, we see that (2.10) holds for any distinct $i, j = 1, \dots, n$. Since $\{E_{ii}, E_{ij} + E_{ji} : i, j = 1, \dots, n, i \neq j\}$ is a basis for $\mathbf{S}_n(\mathbb{Z}_3)$, the result follows. □

Proposition 2.4. *The conclusion in Theorem 1.2 holds when $n \geq 3$ and $|\mathbb{F}| = 4$.*

Proof. It suffices to check the necessity. Up to isomorphism, we can assume that $\mathbb{F} = \mathbb{F}_4 = \{a + bx : a, b \in \mathbb{Z}_2\}$ with $1 + x + x^2 = 0$. Note that $\mathbb{F}_4$ contains two distinct nonzero elements $a = x$ and $a = 1 + x$ such that $a^2 + 1 \neq 0$. We can proceed as in Step 1 of the proof of Proposition 2.2. More precisely, we can conclude that, after a similarity transformation if necessary,

$$\Phi(E_{jj}) = 0_{k_1} \oplus \cdots \oplus 0_{k_{j-1}} \oplus I_{k_j} \oplus 0_{k_{j+1}} \oplus \cdots \oplus 0_{k_n} \oplus 0_{r-s} \quad \text{for } j = 1, \dots, n,$$

where $s = r - k_1 - k_2 - \cdots - k_n$, and

$$\Phi(E_{12} + E_{21}) = B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \oplus 0_{k_3} \oplus \cdots \oplus 0_{k_n} \oplus B_{n+1,n+1}$$

with $B_{jj} \in \mathbf{M}_{k_j}(\mathbb{F}_4)$. Putting $a = x$ in (2.5), we have

$$B^2 = \begin{pmatrix} I_{k_1} + B_{11} & 0 \\ 0 & I_{k_2} + B_{22} \end{pmatrix} \oplus 0_{s'}. \tag{2.11}$$

Consider the rank one symmetric idempotents:

$$\begin{aligned}
X_1 &= \big((1+x)e_1 + xe_2\big)\big((1+x)e_1 + xe_2\big)^{\mathrm{t}}, \\
X_2 &= \big(xe_1 + (1+x)e_2\big)\big(xe_1 + (1+x)e_2\big)^{\mathrm{t}}, \\
A_1 &= (1+x)\big(xe_1 + (1+x)e_2 + xe_3\big)\big(xe_1 + (1+x)e_2 + xe_3\big)^{\mathrm{t}}, \\
A_2 &= (1+x)\big((1+x)e_1 + xe_2 + xe_3\big)\big((1+x)e_1 + xe_2 + xe_3\big)^{\mathrm{t}}.
\end{aligned}$$

Since $\Phi(X_1)$ is an idempotent, we have

$$\begin{aligned}0_r &= \Phi(X_1) - \Phi(X_1)^2 \\ &= \begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'} + B + B^2 + \begin{pmatrix} 0 & B_{12} \\ B_{21} & 0 \end{pmatrix}.\end{aligned}$$

It follows that $B_{n+1,n+1} = 0$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \oplus 0_{s'}$.

In a similar way, with $U_{ij} = E_{ij} + E_{ji}$, we obtain that

$$\Phi(U_{ij}) = \Phi(E_{ij} + E_{ji}) = E_{ii} \otimes B_{ii}^{(ij)} + E_{ij} \otimes B_{ij}^{(ij)} + E_{ji} \otimes B_{ji}^{(ij)} + E_{jj} \otimes B_{jj}^{(ij)} \quad (2.12)$$

for some $B_{ii}^{(ij)} \in \mathbf{M}_{k_i}(\mathbb{F}_4)$, $B_{ij}^{(ij)} \in \mathbf{M}_{k_i k_j}(\mathbb{F}_4)$, $B_{ji}^{(ij)} \in \mathbf{M}_{k_j k_i}(\mathbb{F}_4)$, $B_{jj}^{(ij)} \in \mathbf{M}_{k_j}(\mathbb{F}_4)$ with

$$\Phi(U_{ij})^2 = E_{ii} \otimes (I_{k_i} + B_{ii}^{(ij)}) + E_{jj} \otimes (I_{k_j} + B_{jj}^{(ij)}).$$

Since $X_1 A_1 = X_2 A_2 = X_1 X_2 = X_1 E_{33} = X_2 E_{33} = 0_n$, we have

$$\begin{aligned}0_r &= \Phi(X_1)\Phi(A_1) = (1+x)\Phi(X_1)\left[\Phi(X_2) + (1+x)\Phi(U_{13}) + \Phi(U_{23}) + (1+x)\Phi(E_{33})\right] \\ &= (1+x)\Phi(X_1)\left[(1+x)\Phi(U_{13}) + \Phi(U_{23})\right] \\ &= (1+x)\left[\begin{pmatrix} xI_{k_1} & 0 \\ 0 & (1+x)I_{k_2} \end{pmatrix} \oplus 0_{s'} + \Phi(U_{12})\right]\left[(1+x)\Phi(U_{13}) + \Phi(U_{23})\right] \quad (2.13)\end{aligned}$$

and

$$\begin{aligned}0_r &= \Phi(X_2)\Phi(A_2) = (1+x)\Phi(X_2)\left[\Phi(X_1) + \Phi(U_{13}) + (1+x)\Phi(U_{23}) + (1+x)\Phi(E_{33})\right] \\ &= (1+x)\Phi(X_2)\left[\Phi(U_{13}) + (1+x)\Phi(U_{23})\right] \\ &= (1+x)\left[\begin{pmatrix} (1+x)I_{k_1} & 0 \\ 0 & xI_{k_2} \end{pmatrix} \oplus 0_{s'} + \Phi(U_{12})\right]\left[\Phi(U_{13}) + (1+x)\Phi(U_{23})\right], \quad (2.14)\end{aligned}$$

where $s' = r - k_1 - k_2$. It follows from (2.13) and (2.14) that

$$\begin{aligned}&\left[\begin{pmatrix} I_{k_1} & 0 \\ 0 & xI_{k_2} \end{pmatrix} \oplus 0_{s'}\right]\Phi(U_{13}) + \left[\begin{pmatrix} xI_{k_1} & 0 \\ 0 & (1+x)I_{k_2} \end{pmatrix} \oplus 0_{s'}\right]\Phi(U_{23}) \\ &\qquad + (1+x)\Phi(U_{12})\Phi(U_{13}) + \Phi(U_{12})\Phi(U_{23}) = 0_r \quad (2.15)\end{aligned}$$

and

$$\begin{aligned}&\left[\begin{pmatrix} (1+x)I_{k_1} & 0 \\ 0 & xI_{k_2} \end{pmatrix} \oplus 0_{s'}\right]\Phi(U_{13}) + \left[\begin{pmatrix} xI_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'}\right]\Phi(U_{23}) \\ &\qquad + \Phi(U_{12})\Phi(U_{13}) + (1+x)\Phi(U_{12})\Phi(U_{23}) = 0_r. \quad (2.16)\end{aligned}$$

Multiplying (2.16) by x and then adding it to (2.15), together with (2.12) we have

$$\begin{aligned}\Phi(U_{12})\Phi(U_{13}) &= \left[\begin{pmatrix} 0 & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'}\right]\Phi(U_{13}) + \left[\begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'}\right]\Phi(U_{23}) \\ &= \left[\begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'}\right]\Phi(U_{23}).\end{aligned}$$

Multiplying (2.15) by x and then adding it to (2.16), together with (2.12) we have

$$\begin{aligned}\Phi(U_{12})\Phi(U_{23}) &= \left[\begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'}\right]\Phi(U_{13}) + \left[\begin{pmatrix} I_{k_1} & 0 \\ 0 & 0 \end{pmatrix} \oplus 0_{s'}\right]\Phi(U_{23}) \\ &= \left[\begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'}\right]\Phi(U_{13}).\end{aligned}$$

At this stage, we arrive at the same conclusions as those established in Step 1 of the proof of Proposition 2.3. Employing an argument analogous to that of its Step 2, we get the desired conclusion. □

Proposition 2.5. *The conclusion in Theorem 1.2 holds when $n \geq 3$ and $\mathbb{F} = \mathbb{Z}_5$.*

Proof. It suffices to check the necessity. Note that $\mathbb{Z}_5$ contains a nonzero element $a = 1$ such that $a^2 + 1 \neq 0$. We can proceed as in Step 1 of the proof of Proposition 2.2. More precisely, we can conclude that, after a similarity transformation if necessary,

$$\Phi(E_{jj}) = 0_{k_1} \oplus \cdots \oplus 0_{k_{j-1}} \oplus I_{k_j} \oplus 0_{k_{j+1}} \oplus \cdots \oplus 0_{k_n} \oplus 0_{r-s} \quad \text{for } j = 1, \ldots, n,$$

where $s = r - k_1 - k_2 - \cdots - k_n$, and with $U_{ij} = E_{ij} + E_{ji}$, that

$$\Phi(U_{ij}) = \Phi(E_{ij} + E_{ji}) = E_{ii} \otimes B_{ii}^{(ij)} + E_{ij} \otimes B_{ij}^{(ij)} + E_{ji} \otimes B_{ji}^{(ij)} + E_{jj} \otimes B_{jj}^{(ij)}, \tag{2.17}$$

for some $B_{ii}^{(ij)} \in \mathbf{M}_{k_i}(\mathbb{Z}_5)$, $B_{ij}^{(ij)} \in \mathbf{M}_{k_i k_j}(\mathbb{Z}_5)$, $B_{ji}^{(ij)} \in \mathbf{M}_{k_j k_i}(\mathbb{Z}_5)$, $B_{jj}^{(ij)} \in \mathbf{M}_{k_j}(\mathbb{Z}_5)$ with

$$\Phi(U_{ij})^2 = E_{ii} \otimes I_{k_i} + E_{jj} \otimes I_{k_j}.$$

Consider the rank one symmetric idempotents:

$$\begin{aligned}X_1 &= -2(e_1 + e_2)(e_1 + e_2)^{\mathrm{t}}, \qquad X_2 = -2(e_1 - e_2)(e_1 - e_2)^{\mathrm{t}}, \\ A_1 &= 2(e_1 - e_2 + e_3)(e_1 - e_2 + e_3)^{\mathrm{t}}, \quad A_2 = (e_1 + e_2 + 2e_3)(e_1 + e_2 + 2e_3)^{\mathrm{t}}.\end{aligned}$$

Since $X_1A_1 = X_2A_2 = X_1X_2 = X_1E_{33} = X_2E_{33} = 0_n$, we have

$$\begin{aligned}0_r = \Phi(X_1)\Phi(A_1) &= \Phi(X_1)\left[\Phi(-X_2) + 2\Phi(U_{13}) - 2\Phi(U_{23}) + 2\Phi(E_{33})\right] \\ &= \Phi(X_1)\left[2\Phi(U_{13}) - 2\Phi(U_{23})\right], \\ &= \left[\begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'} + \Phi(U_{12})\right]\left[\Phi(U_{13}) - \Phi(U_{23})\right]\end{aligned}$$

and

$$\begin{aligned} 0_r &= \Phi(X_2)\Phi(A_2) = \Phi(X_2)\left[2\Phi(X_1) + 2\Phi(U_{13}) + 2\Phi(U_{23}) - \Phi(E_{33})\right] \\ &= \Phi(X_2)\left[2\Phi(U_{13}) + 2\Phi(U_{23})\right] \\ &= \left[\begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'} - \Phi(U_{12})\right]\left[\Phi(U_{13}) + \Phi(U_{23})\right], \end{aligned}$$

where $s' = r - k_1 - k_2$. This implies

$$\Phi(U_{12})\Phi(U_{13}) = \left[\begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'}\right]\Phi(U_{23})$$

and

$$\Phi(U_{12})\Phi(U_{23}) = \left[\begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'}\right]\Phi(U_{13}).$$

At this stage, we arrive at the same conclusions as those established in Step 1 of the proof of Proposition 2.3. Employing an argument analogous to that of its Step 2, we get the desired conclusion. □

Lemma 2.6. *Every nonzero idempotent in* $\mathbf{M}_n(\mathbb{F})$ *is a sum of disjoint rank one idempotents. When* char $\mathbb{F} \neq 2$, *every nonzero symmetric idempotent in* $\mathbf{M}_n(\mathbb{F})$ *is a sum of disjoint rank one symmetric idempotents.*

Proof. Let E be an idempotent in $\mathbf{M}_n(\mathbb{F})$ of rank $m \geq 1$. Since $E(I_n - E) = (I_n - E)E = 0$, we can decompose $\mathbb{F}^n$ as a direct sum $\mathbb{F}^n = E\mathbb{F}^n + (I_n - E)\mathbb{F}^n$ of the range space and the kernel space of E. Choose a basis $\{u_1, \dots, u_m\}$ for $E\mathbb{F}^n$ and a basis $\{u_{m+1}, \dots, u_n\}$ for $(I_n - E)\mathbb{F}^n$, and let S be the invertible matrix with column vectors $u_1, \dots, u_n$. If S^{-1} has row vectors $v_1^{\mathrm{t}}, \dots, v_n^{\mathrm{t}}$, then $v_j^{\mathrm{t}} u_i = \delta_{ij}$ for all $i, j = 1, \dots, n$. It follows that

$$E = S(I_m \oplus 0_{n-m})S^{-1} = u_1 v_1^{\mathrm{t}} + \cdots + u_m v_m^{\mathrm{t}}$$

is a sum of disjoint rank one idempotents.

Next, assume that $E = E^{\mathrm{t}} = E^2$ is a symmetric idempotent in $\mathbf{M}_n(\mathbb{F})$ of rank $m \geq 1$ and $\mathbb{F}$ has characteristic not 2. We claim that there is a column vector $u = Eu$ from $E\mathbb{F}^n$ such that $u^{\mathrm{t}}u \neq 0$. Suppose on the contrary that all vectors u in $E\mathbb{F}^n$ satisfy $u^{\mathrm{t}}u = 0$. If v is an other column vector in $E\mathbb{F}^n$, then so is $u+v$ and thus $0 = (u+v)^{\mathrm{t}}(u+v) = 2u^{\mathrm{t}}v$. If $\mathbb{F}$ does not have characteristic 2, then $u^{\mathrm{t}}v = 0$. Let S be invertible in $\mathbf{M}_n(\mathbb{F})$ such that $E = S(I_m \oplus 0_{n-m})S^{-1}$. Note that the first m column vectors of S are basic vectors of $E\mathbb{F}^n$, and thus the invertible matrix $S^{\mathrm{t}}S$ has a zero $m \times m$ upper left block. Consequently,

$$E = E^2 = E^{\mathrm{t}}E = (S^{-1})^{\mathrm{t}}(I_m \oplus 0_{n-m})S^{\mathrm{t}}S(I_m \oplus 0_{n-m})S^{-1} = 0,$$

a contradiction.

Let $u = Eu$ such that $u^{\mathrm{t}}u = \lambda \neq 0$. Since $u^{\mathrm{t}}E = (E^{\mathrm{t}}u)^{\mathrm{t}} = (Eu)^{\mathrm{t}} = u^{\mathrm{t}}$, we see that $F = E - uu^{\mathrm{t}}/\lambda$ is a symmetric idempotent disjoint from the rank one symmetric idempotent uu^{t}/λ. In particular, F has rank $m-1$. Then the assertion follows from induction. □

When char $\mathbb{F} = 2$, we see in (1.3) a rank two symmetric idempotent which cannot be written as a sum of disjoint rank one symmetric idempotents.

Corollary 2.7. *Assume that* char $\mathbb{F} \neq 2$ *and* $\mathbf{S}_n(\mathbb{F}) \neq \mathbf{S}_3(\mathbb{Z}_3), \mathbf{S}_2(\mathbb{F})$ *with* $|\mathbb{F}| \leq 5$. *Let* $\Phi\colon \mathbf{S}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ *be a linear map. The following conditions are equivalent.*

(1) Φ *has the form (1.2):*

$$\Phi(A) = S \begin{pmatrix} A \otimes I_k & 0 \\ 0 & 0_{r-nk} \end{pmatrix} S^{-1} \quad \text{for all } A \in \mathbf{S}_n(\mathbb{F}).$$

(2) Φ *sends disjoint rank one symmetric idempotents to disjoint idempotents.*

(3) Φ *sends symmetric idempotents to idempotents.*

Proof. The case when $n = 1$ is trivial, and we thus assume $n \geq 2$ below.

The equivalence between (1) and (2) follows from Theorem 1.2.

Suppose that Φ sends disjoint rank one symmetric idempotents to disjoint idempotents. We claim that Φ sends symmetric idempotents to idempotents. Let E be a symmetric idempotent of rank at least two. It follows from Lemma 2.6 that $E = \sum_j E_j$ is a (finite) sum of disjoint rank one symmetric idempotents. The assumption ensures that $\Phi(E) = \sum_j \Phi(E_j)$ is a sum of disjoint idempotents, and thus $\Phi(E)$ is an idempotent. In the case when E is a rank one symmetric idempotent, $I_n - E$ is also a (nonzero) symmetric idempotent in $\mathbf{S}_n(\mathbb{F})$ disjoint from E. By Lemma 2.6, the nonzero symmetric idempotent $I_n - E = F_1 + \cdots + F_{n-1}$ is a sum of disjoint rank one symmetric idempotents. It is clear that E and F_1 are disjoint. By assumption, $\Phi(E)$ and $\Phi(F_1)$ are disjoint idempotents. Hence, in any case, Φ sends symmetric idempotents to idempotents, as claimed.

Finally, suppose Φ sends symmetric idempotents to idempotents. We verify that Φ sends disjoint symmetric idempotents to disjoint idempotents. Let E, F be disjoint idempotents in $\mathbf{S}_n(\mathbb{F})$. Then $E + F$ is also a symmetric idempotent, and thus $\Phi(E)$, $\Phi(F)$ and $\Phi(E+F) = \Phi(E) + \Phi(F)$ are all idempotents in $\mathbf{M}_r(\mathbb{F})$. This implies $\Phi(E)\Phi(F) = -\Phi(F)\Phi(E)$. Since $\mathbb{F}$ does not have characteristic two,

$$\Phi(E)\Phi(F) = \Phi(E)^2\Phi(F) = -\Phi(E)\Phi(F)\Phi(E) = \Phi(F)\Phi(E)^2 = \Phi(F)\Phi(E) = 0.$$

This completes the proof. □

Remark 2.8. The necessity part in Theorem 1.2 may fail when $\mathbf{S}_n(\mathbb{F}) = \mathbf{S}_n(\mathbb{Z}_2)$, or $\mathbf{S}_3(\mathbb{Z}_3)$, or $\mathbf{S}_2(\mathbb{F})$ with $|\mathbb{F}| = 3, 4, 5$. In these cases, there are too few disjoint symmetric idempotent pairs for us to establish the conclusion. We shall give explicit examples below.

(1) When $\mathbb{F} = \mathbb{Z}_2$, observe that all rank one symmetric idempotents in $\mathbf{S}_n(\mathbb{Z}_2)$ are given by

$$A_u = uu^{\mathrm{t}} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix}$$

for a vector $u = \begin{pmatrix} u_1 & \dots & u_n \end{pmatrix}^{\mathrm{t}} \in \mathbb{Z}_2^n$ with $u_1 + \cdots + u_n = 1$ in $\mathbb{Z}_2$. Let $\mathcal{I}_1^n(\mathbb{Z}_2)$ be the set of all rank one symmetric idempotents in $\mathbf{S}_n(\mathbb{Z}_2)$.

Consider the rank 2 symmetric matrix

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 0_{n-2}.$$

We claim that Q cannot be written as a sum of rank one symmetric idempotents; namely, Q is linearly independent from $\mathcal{I}_1^n(\mathbb{Z}_2)$. Suppose otherwise $Q \in \operatorname{span}(\mathcal{I}_1^n(\mathbb{Z}_2))$, and there are vectors $v_i = \begin{pmatrix} v_{i1} & \dots & v_{in} \end{pmatrix}^{\mathrm{t}} \in \mathbb{Z}_2^n$ with $v_{i1} + \cdots + v_{in} = 1$ in $\mathbb{Z}_2$ for $i = 1, \dots, k$ such that $Q = A_{v_1} + \cdots + A_{v_k}$. Note that

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = Q \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \sum_{i=1}^{k} A_{v_i} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \sum_{i=1}^{k} \begin{pmatrix} v_{i1} \\ v_{i2} \\ v_{i3} \\ \vdots \\ v_{in} \end{pmatrix} \begin{pmatrix} v_{i1} & v_{i2} & v_{i3} & \cdots & v_{in} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$
$$= \sum_{i=1}^{k} \begin{pmatrix} v_{i1} \\ v_{i2} \\ v_{i3} \\ \vdots \\ v_{in} \end{pmatrix}.$$

This implies in particular $1 = \sum_{i=1}^{k} v_{i1}$, while the $(1,1)$th entry of $Q = A_{v_1} + \cdots + A_{v_k}$ is $0 = \sum_{i=1}^{k} v_{i1}$, a contradiction.

(a) When n is even, we consider a linear map $\Phi \colon \mathbf{S}_n(\mathbb{Z}_2) \to \mathbf{S}_n(\mathbb{Z}_2)$ such that $\Phi(A) = A$ if $A \in \operatorname{span}(\mathcal{I}_1^n(\mathbb{Z}_2))$ and $\Phi(Q) = RQR^{-1}$, where

$$R = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & 1 \\ 1 & \cdots & 1 & 1 & 0 \end{pmatrix} = R^{-1}.$$

It is clear that Φ sends disjoint rank one symmetric idempotents to disjoint rank one symmetric idempotents. By a direct calculation,

$$RQR^{-1} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 0 & \cdots & 0 \end{pmatrix} \neq Q.$$

Suppose there is an invertible $S \in \mathbf{M}_n(\mathbb{Z}_2)$ such that (1.2) holds. Then $SA = AS$ for every symmetric rank one idempotent A in $\mathbf{S}_2(\mathbb{Z}_2)$ implies $S = I_n$. However, $SQS^{-1} = Q \neq \Phi(Q) = RQR^{-1}$. This contradiction says that Φ does not have the form (1.2).

(b) When n is odd, we consider a linear map $\Phi\colon \mathbf{S}_n(\mathbb{Z}_2) \to \mathbf{S}_n(\mathbb{Z}_2)$ such that $\Phi(A) = A$ if $A \in \operatorname{span}(\mathcal{I}_1^n(\mathbb{Z}_2))$ and $\Phi(Q) = PQP^{-1}$, where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \cdots & 1 & 0 & 1 \\ 0 & 1 & \cdots & 1 & 1 & 0 \end{pmatrix} = P^{-1}.$$

By a direct calculation,

$$PQP^{-1} = \begin{pmatrix} 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \neq Q.$$

As in Case (a), we see that Φ does not have the form (1.2), either.

(2) When $\mathbb{F} = \mathbb{Z}_3 = \{0, \pm 1\}$, observe that all rank one symmetric idempotents in $\mathbf{S}_n(\mathbb{Z}_3)$ are given by

$$\pm \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \quad \text{such that } u_1^2 + \cdots + u_n^2 = \pm 1 \text{ in } \mathbb{Z}_3.$$

Let $\mathcal{I}_1^n(\mathbb{Z}_3)$ be the set of all rank one symmetric idempotents in $\mathbf{S}_n(\mathbb{Z}_3)$. Note that $E_{ii} \in \mathcal{I}_1^n(\mathbb{Z}_3)$ and

$$E_{ij} + E_{ji} = (e_i + e_j)(e_i + e_j)^{\mathrm{t}} - E_{ii} - E_{jj} \in \operatorname{span} \mathcal{I}_1^n(\mathbb{Z}_3).$$

Hence span $\mathcal{I}_1^n(\mathbb{Z}_3) = \mathbf{S}_n(\mathbb{Z}_3)$.

(i) All the symmetric idempotents in $\mathbf{S}_2(\mathbb{Z}_3)$ are

$$0_2,\ I_2,\ E_{11},\ E_{22}, \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The unital linear map $\Phi \colon \mathbf{S}_2(\mathbb{Z}_3) \to \mathbf{M}_2(\mathbb{Z}_3)$ defined by

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \mapsto \begin{pmatrix} a+b & 0 \\ 0 & b+d \end{pmatrix}$$

sends disjoint rank one symmetric idempotents to disjoint symmetric idempotents. Indeed, Φ preserves zero products. However, it does not have the form (1.2), as the nonzero matrix E_{22} is sent to zero by Φ.

(ii) All rank one symmetric idempotents in $\mathbf{S}_3(\mathbb{Z}_3)$ are $E_{11},\ E_{22},\ E_{33}$, and

$$A_1 = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix},$$
$$A_4 = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A_5 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix},\ A_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

All the disjoint pairs among them are:

$$\{E_{11}, E_{22}\},\quad \{E_{11}, E_{33}\},\quad \{E_{22}, E_{33}\},\quad \{E_{11}, A_3\},\quad \{E_{11}, A_6\},\quad \{E_{22}, A_2\},\quad \{E_{22}, A_5\},$$
$$\{E_{33}, A_1\},\quad \{E_{33}, A_4\},\quad \{A_1, A_4\},\quad \{A_2, A_5\},\quad \text{and}\quad \{A_3, A_6\}.$$

The unital linear map $\Phi \colon \mathbf{S}_3(\mathbb{Z}_3) \to \mathbf{S}_3(\mathbb{Z}_3)$ defined by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \mapsto \begin{pmatrix} a_{11}+a_{12}+a_{13} & & \\ & -a_{12}+a_{22}+a_{23} & \\ & & a_{13}-a_{23}+a_{33} \end{pmatrix}$$

sends disjoint rank one symmetric idempotents to disjoint symmetric idempotents, while it does not have the form (1.2) because $\Phi(A_5) = 0$.

(3) When $\mathbb{F}_4 = \{a + bx : a, b \in \mathbb{Z}_2\}$ is the field of 4 elements with $1 + x + x^2 = 0$, all the pairs of disjoint rank one symmetric idempotents are E_{11}, E_{22} and X_1, X_2, where

$$X_1 = \big((1+x)e_1 + xe_2\big)\big((1+x)e_1 + xe_2\big)^{\mathrm{t}} = \begin{pmatrix} x & 1 \\ 1 & 1+x \end{pmatrix}$$

and

$$X_2 = \big(xe_1 + (1+x)e_2\big)\big(xe_1 + (1+x)e_2\big)^{\mathrm{t}} = \begin{pmatrix} 1+x & 1 \\ 1 & x \end{pmatrix}.$$

The unital linear map $\Phi\colon \mathbf{S}_2(\mathbb{F}_4) \to \mathbf{S}_2(\mathbb{F}_4)$ defined by

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \mapsto \begin{pmatrix} a + b(1+x) & 0 \\ 0 & d + b(1+x) \end{pmatrix}$$

sends disjoint rank one symmetric idempotents to disjoint symmetric idempotents. In fact, $\Phi(X_1) = \Phi(E_{11}) = E_{11}$, $\Phi(X_2) = \Phi(E_{22}) = E_{22}$. However, it does not have the form (1.2) as it sends $\begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}$ to zero.

(4) When $\mathbb{F} = \mathbb{Z}_5$, all the disjoint rank one symmetric idempotent pairs in $\mathbf{S}_2(\mathbb{Z}_5)$ are

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}, \quad \text{and} \quad \left\{\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}\right\}.$$

The unital linear map $\Phi\colon \mathbf{S}_2(\mathbb{Z}_5) \to \mathbf{S}_2(\mathbb{Z}_5)$ defined by

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \mapsto \begin{pmatrix} a+b & 0 \\ 0 & b+d \end{pmatrix}$$

sends disjoint rank one symmetric idempotents to disjoint symmetric idempotents. But Φ does not have the form (1.2) as it sends $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ to zero.

Another counter example is the linear map $\Psi\colon \mathbf{S}_2(\mathbb{Z}_5) \to \mathbf{M}_2(\mathbb{Z}_5)$ defined by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}.$$

The unital linear map Ψ sends disjoint rank one symmetric idempotents to disjoint rank one idempotents, namely,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \quad \text{and}$$
$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}.$$

However, it does not have the form (1.2), i.e., $A \mapsto SAS^{-1}$. If it does, then $S\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}S^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ will imply that $S = \begin{pmatrix} a & -d \\ 0 & d \end{pmatrix}$ for some $a, d \in \mathbb{Z}_5$. But then $S\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}S^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ implies $a = d = 0$, a contradiction.

3. Linear maps preserving matrices annihilated by a polynomial

In this section, we characterize those linear maps Φ preserving matrices annihilated by a polynomial $f(x)$, i.e.,

$$f(\Phi(A)) = 0 \quad \text{whenever} \quad f(A) = 0. \tag{3.1}$$

Without loss of generality, we can always assume that the leading coefficient of $f(x)$ is 1. It was shown in [4] that when $f(x) = (x - a_1)(x - a_2) \cdots (x - a_m)$ is a complex polynomial with $m \geq 2$ distinct simple zeroes, a unital linear map Φ between unital complex algebras satisfying (3.1) exactly when Φ sends idempotents to idempotents. In [12] we studied such linear preservers between matrices over an arbitrary field $\mathbb{F}$. In the following, Theorem 3.4 extends [12, Theorem 3.1] by relaxing the condition to that $f(x)$ has at least one simple zero. Moreover, it now suffices to check the annihilating condition (3.1) for only symmetric matrices instead of all matrices, except for a few cases of small domains.

Let us begin with two observations.

Lemma 3.1. *Let $f(x) \in \mathbb{F}[x]$ be a polynomial with $f(0) = 0$. Let $E_1, E_2 \in \mathbf{M}_n(\mathbb{F})$ be disjoint idempotents. Then for any zeroes a_1, a_2 of $f(x)$, we have*

$$f(a_1E_1 + a_2E_2) = f(a_1)E_1 + f(a_2)E_2 = 0.$$

.

Proof. Let $f(x) = \alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_1 x \in \mathbb{F}[x]$ such that $f(a_1) = f(a_2) = 0$. Since E_1, E_2 are disjoint idempotents, we have

$$\begin{aligned}(a_1E_1 + a_2E_2)^k &= a_1^k E_1^k + a_1^{k-1}a_2(E_1^{k-1}E_2 + E_1^{k-2}E_2E_1 + \cdots + E_2E_1^{k-1}) + \cdots + a_2^k E_2^k \\ &= a_1^k E_1 + a_2^k E_2\end{aligned}$$

for $k = 1, 2, \ldots$. It follows that

$$\begin{aligned}f(a_1E_1+a_2E_2) &= \alpha_n(a_1^nE_1+a_2^nE_2)+\alpha_{n-1}(a_1^{n-1}E_1+a_2^{n-1}E_2)+\cdots+\alpha_1(a_1E_1+a_2E_2) \\ &= f(a_1)E_1 + f(a_2)E_2 = 0,\end{aligned}$$

as asserted. □

Lemma 3.2. *Let $f(x) \in \mathbb{F}[x]$ be a polynomial splitting in $\mathbb{F}$. Let $A \in \mathbf{M}_n(\mathbb{F})$ such that $f(A) = 0$. Then all eigenvalues of A are zeroes of $f(x)$.*

Proof. Note that $f(A) = 0$ if and only if the minimal polynomial of the matrix A divides $f(x)$. It amounts to saying that A has a Jordan block form representation, in which the diagonal entries in each Jordan block are annihilated by $f(x)$. In particular, all eigenvalues of A are zeroes of $f(x)$. □

Recall that a subset Z of $\mathbb{F}$ is called an *additive coset* if $a-b+c \in Z$ whenever $a,b,c \in Z$. An additive coset Z is an additive group if $0 \in Z$. Note that the set $Z(f)$ of zeroes of a polynomial $f(x) \in \mathbb{Z}_2[x]$ of degree at least two and with a simple zero is simply the additive group $\mathbb{Z}_2$.

Lemma 3.3. *Let $f(x) \in \mathbb{F}[x]$ be a polynomial splitting in $\mathbb{F}$ with at least two distinct zeroes such that the zero set $Z(f)$ is not an additive coset. Let $\mathbf{V}_n(\mathbb{F}) = \mathbf{S}_n(\mathbb{F})$ or $\mathbf{M}_n(\mathbb{F})$, and $E \in \mathbf{V}_n(\mathbb{F})$ be an idempotent. Suppose that $\Phi\colon \mathbf{V}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ is a unital linear map preserving matrices annihilated by $f(x)$. Then all eigenvalues of $\Phi(E)$ are either 0 or 1.*

Proof. Consider the polynomial $g(x) = f(ax+b)$ for any $a,b \in \mathbb{F}$, $a \neq 0$. Since $\Phi(I_n) = I_r$, for any $A \in \mathbf{V}_n(\mathbb{F})$ we have $g(\Phi(A)) = f(a\Phi(A) + bI_r)) = f(\Phi(aA + bI_n))$. It follows that Φ preserves matrices annihilated by f if and only if it preserves matrices annihilated by the polynomial $g(x) = f(ax+b)$. Replacing $f(x)$ with $g(x)$ for some suitable a, b, we can assume that

$$Z(f) = \{0, 1, a_3, \ldots, a_m\} \subseteq \mathbb{F},$$

is not an additive group. In particular, $f(x)$ has zero constant term.

Since $f(E) = f(1)E = 0$ by Lemma 3.1, $f(\Phi(E)) = 0$, and thus $\Phi(E)$ has a Jordan block form with eigenvalues listed in the diagonal entries, which are also the zeroes of $f(x)$ by Lemma 3.2. For any $a_j \in Z(f)$, we have similarly $f(a_jE) = f(a_j)E = 0$, and thus $f(a_j\Phi(E)) = 0$. It follows again from Lemma 3.2 that the eigenvalues of $a_j\Phi(E)$ are all from $Z(f)$ for $j = 1, \ldots, m$. This tells us that if λ is an eigenvalue of $\Phi(E)$, then $\lambda Z(f) \subseteq Z(f)$, and thus

$$\lambda Z(f) = Z(f) \quad \text{if } \lambda \neq 0,$$

because $Z(f)$ is a finite set in $\mathbb{F}$. In particular, $\lambda \in Z(f)$ since $0, 1 \in Z(f)$.

We claim that all eigenvalues of $\Phi(E)$ are either 0 or 1. Suppose $\Phi(E)$ has any eigenvalue $\lambda \neq 0, 1$. Then $\lambda Z(f) = Z(f)$ implies $Z(f)$ contains λ^l for all $l \in \mathbb{Z}$. With $I_n - E$ playing the role of E, since $\Phi(I_n - E) = I_n - \Phi(E)$ has $1 - \lambda$ as an eigenvalue, we see that $(1-\lambda)^l \in Z(f)$ for any $l \in \mathbb{Z}$. For any $a_i, a_j \in Z(f)$, by Lemma 3.1, we have $f(a_i(I_n - E) + a_jE) = f(a_i)(I_n - E) + f(a_j)E = 0$. It follows from Lemma 3.2 and $f(\Phi(a_i(I_n - E) + a_jE)) = f(a_i(I_n - \Phi(E)) + a_j\Phi(E)) = 0$ that

(†) all eigenvalues of $a_i(I_n - \Phi(E)) + a_j\Phi(E)$ are in $Z(f)$ for any $a_i, a_j \in Z(f)$.

Note that up to similarity $a_i(I_n - \Phi(E)) + a_j\Phi(E)$ is an upper triangular matrix with a diagonal entry $a_i(1-\lambda) + a_j\lambda$. Since $Z(f) = \lambda Z(f) = (1-\lambda)Z(f)$, we obtain that $a_i(1-\lambda)^{-1}$, $a_j\lambda^{-1} \in Z(f)$. Therefore,

$$a_i + a_j = [a_i(1-\lambda)^{-1}](1-\lambda) + [a_j\lambda^{-1}]\lambda \in Z(f)$$

for all $a_i, a_j \in Z(f)$, which means that $Z(f)$ is either an infinite set or forms an additive group. This contradiction ensures that all the eigenvalues of $\Phi(E)$ are 0 or 1. □

Theorem 3.4. *Let $f(x) = (x-a_1)(x-a_2)(x-a_3)\cdots(x-a_m)$ be a polynomial of degree $m \geq 2$ with at least one simple zero such that the zero set $Z(f) = \{a_1, a_2, \ldots, a_m\} \subseteq \mathbb{F}$ is not an additive coset. Let $n \geq 1$, and $\mathbf{V}_n(\mathbb{F}) = \mathbf{S}_n(\mathbb{F})$ or $\mathbf{M}_n(\mathbb{F})$. Suppose $\Phi\colon \mathbf{V}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ is a unital linear map with $\mathbf{V}_n(\mathbb{F}) \neq \mathbf{S}_3(\mathbb{Z}_3)$, or $\mathbf{S}_2(\mathbb{F})$ with $|\mathbb{F}| \leq 5$. The following conditions are equivalent.*

(a) *$f(\Phi(A)) = 0_r$ whenever $f(A) = 0_n$.*

(b) *Φ sends disjoint rank one idempotents in $\mathbf{V}_n(\mathbb{F})$ to disjoint idempotents in $\mathbf{M}_n(\mathbb{F})$.*

(c) *There are nonnegative integers p, q satisfying $r/n = p + q$ and an invertible matrix $S \in \mathbf{M}_r(\mathbb{F})$ such that Φ has the form*

$$A \mapsto S\begin{pmatrix} A \otimes I_p & \\ & A^{\mathrm{t}} \otimes I_q \end{pmatrix} S^{-1}; \tag{3.2}$$

when $\mathbf{V}_n(\mathbb{F}) = \mathbf{S}_n(\mathbb{F})$, the form reduces to

$$A \mapsto S(A \otimes I_{r/n})S^{-1}. \tag{3.3}$$

Proof. When $n = 1$, *any* unital linear map $\Phi : \mathbf{M}_1(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ preserves matrices annihilated by any but fixed polynomial $f(x) \in \mathbb{F}[x]$. In fact, Since $I_r = \Phi(1) \in \mathbf{M}_r(\mathbb{F})$, we have $f(\Phi(a)) = f(aI_r) = f(a)I_r = 0$ whenever $f(a) = 0$ for any $a \in \mathbf{M}_1(\mathbb{F}) \cong \mathbb{F}$. In this case, all stated conditions are satisfied automatically.

From now on, suppose $n \geq 2$. By the assumption, $\mathbb{F} \neq \mathbb{Z}_2$ for else $Z(f) = \mathbb{Z}_2$ is an additive group.

(1) We first assume that $\Phi\colon \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ is a unital linear map. If (c) holds with (3.2), then clearly (a) holds. The equivalence between (b) and (c) follows from Theorem 1.1 (the zero part does not appear since Φ is unital). Assume (a) holds. To verify (b), we follow the proof of [12, Theorem 3.1] with a slight modification due to the weakening of the assumption from that all zeroes are simple to that at least one of them is simple.

As in the proof of Lemma 3.3, by replacing $f(x)$ with $g(x) = f(ax+b)$ for some suitably chosen a, b, we can assume that $f(x)$ has zeroes $a_1 = 0$ and $a_2 = 1$. Furthermore, we can also arrange that 0 or 1 is a simple zero of $f(x)$ according to our wish. Let E be an idempotent in $\mathbf{M}_n(\mathbb{F})$. By Lemma 3.3, all eigenvalues of $\Phi(E)$ are either 0 or 1. Since

$f(\Phi(E)) = 0$, the minimal polynomial of $\Phi(E)$ divides f. If 0 (resp. 1) is a simple zero of $f(x)$, then so is the case for the minimal polynomial of $\Phi(E)$. Consequently, the minimal polynomial of $\Phi(E)$ can only be x, $x-1$ or $x(x-1)$. It follows that $\Phi(E)$ is diagonalizable with eigenvalues from 0 and 1, and thus an idempotent (see, e.g., [2, Theorems 7.12 and 7.16, and Exercise 7.13]). Therefore, Φ sends idempotents to idempotents.

Suppose E, F are disjoint idempotents in $\mathbf{M}_n(\mathbb{F})$. Since $E + F$ is an idempotent, we have $\Phi(E + F)^2 = \Phi(E + F)$, and thus $\Phi(E)\Phi(F) + \Phi(F)\Phi(E) = 0$. It follows

$$\Phi(E)\Phi(F) = -\Phi(E)\Phi(F)\Phi(E) = \Phi(F)\Phi(E).$$

When $\mathbb{F}$ does not have characteristic 2, we see that $\Phi(E), \Phi(F)$ are disjoint. When $\mathbb{F}$ has characteristic 2, after simultaneously diagonalizing the commuting idempotents, we can assume both $\Phi(E), \Phi(F)$ are diagonal matrices with diagonal entries 0 or 1. If $\Phi(E)\Phi(F) \neq 0$, then we can further assume that the $(1, 1)$-th entries of both $\Phi(E), \Phi(F)$ are 1. For any $a_i, a_j \in Z(f)$, since $f(a_iE + a_jF) = 0$ by Lemma 3.1, we have $f(a_i\Phi(E) + a_j\Phi(F)) = 0$. In particular, the $(1, 1)$-th entry $a_i + a_j$ of the diagonal matrix $a_i\Phi(E) + a_j\Phi(F)$ is a zero of $f(x)$ by Lemma 3.2. It follows that $a_i + a_j \in Z(f)$. Hence, the zero set $Z(f)$ is closed under additions (and differences), and thus forms an additive group, a contradiction. Therefore, Φ sends disjoint idempotents to disjoint idempotents in any case. This establishes (b).

(2) For the case when $\Phi\colon \mathbf{S}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ is a unital linear map, a careful examination of the proof of part (1) ensures the implication (a) $\implies$ (b), as it suffices to use only symmetric idempotents E in the argument. The equivalence between (b) and (c) (with (3.3) held) is due to Theorem 1.2 under the assumption that $\mathbf{S}_n(\mathbb{F}) \neq \mathbf{S}_3(\mathbb{Z}_3), \mathbf{S}_2(\mathbb{F}), |\mathbb{F}| \leq 5$ and that $\mathbb{F} \neq \mathbb{Z}_2$, while the implication (c) $\implies$ (a) is plain. □

Remark 3.5. (a) The unital linear map Φ in Remark 2.8 for $\mathbf{S}_3(\mathbb{Z}_3)$, and $\mathbf{S}_2(\mathbb{F})$ with $|\mathbb{F}| = 3, 4, 5$, respectively, sends disjoint rank one symmetric idempotents to disjoint symmetric idempotents. In particular, Φ preserves matrices annihilated by the polynomial $f(x) = x(x-1)$. The zero set $Z(f)$ of $f(x)$ consists of two distinct simple zeroes $0, 1$, and does not form an additive group. But these linear maps Φ do not have the form (3.3).

(b) The assumption that $f(x)$ has a simple zero in Theorem 3.4 is used to ensure that all the Jordan blocks of $\Phi(E)$ have size 1×1 for any idempotent E in the domain of Φ. Suppose the minimum order of the zeroes of the polynomial $f(x)$ is $s \geq 2$. For any idempotent E, the argument in the proof of Theorem 3.4 shows that $\Phi(E)$ has eigenvalues from 0 and 1, and all its Jordan blocks have size at most $s \times s$. This might be the best we can expect. For example, let $f(x) = x^m$ and $n \neq 0$ in $\mathbb{F}$. Consider the unital linear map $\Phi\colon \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ defined by

$$\Phi(A) = \frac{\operatorname{trace} A}{n} I_r + \operatorname{trace}(AB)C$$

with any $B \in \mathbf{M}_n(\mathbb{F})$ and $C \in \mathbf{M}_r(\mathbb{F})$ such that B has zero trace and $C^m = 0$. Since nilpotent matrices have zero trace, we see that Φ preserves matrices annihilated by $f(x)$. But Φ does not have the form (3.2).

In the following, we study how to remove the unital assumption on Φ.

Definition 3.6. Let $f(x) = (x - a_1)(x - a_2)\cdots(x - a_m) \in \mathbb{F}[x]$ be a polynomial splitting in $\mathbb{F}$. We call $\alpha \in \mathbb{F}$ a *zero multiplier* of $f(x)$ if $\alpha Z(f) \subseteq Z(f)$. Let

$$M(f) = \{\alpha \in \mathbb{F} : \alpha Z(f) \subseteq Z(f)\}$$

denote the set of zero multipliers of $f(x)$.

Lemma 3.7. *Let $f(x) = (x - a_1)(x - a_2)\cdots(x - a_m) \in \mathbb{F}[x]$ be a polynomial splitting in $\mathbb{F}$.*

- $0 \in M(f)$ *exactly when* $0 \in Z(f)$.
- $\alpha Z(f) = Z(f)$ *whenever* $\alpha \in M(f) \setminus \{0\}$.
- $M(f) \setminus \{0\} = \{1, \gamma, \dots, \gamma^{h-1}\}$ *is a finite multiplicative cyclic group, where γ is an hth primitive root of unity.*
- *If $f(0) \neq 0$, then h divides m; if $f(0) = 0$, then h divides $m - 1$.*
- $h^{-1} \in \mathbb{F}$.

Proof. The first assertion is clear. Since $Z(f)$ is a finite subset,

$$\alpha Z(f) = Z(f) \quad \text{whenever } \alpha \in M(f) \setminus \{0\}.$$

It then follows that $M(f) \setminus \{0\}$ is a finite multiplicative subgroup of $\mathbb{F} \setminus \{0\}$. In particular, $M(f) \setminus \{0\}$ is a cyclic group (see, e.g., [7, Theorem IV.1.9]).

Assume that $M(f) \setminus \{0\}$ has order h and generated by γ. Then for every nonzero $b \in Z(f)$, we see that the set $\{b, \gamma b, \dots, \gamma^{h-1} b\} \subseteq Z(f) \setminus \{0\}$. There are $b_1, \dots, b_r$ such that

$$Z(f) \setminus \{0\} = \bigcup_{j=1}^{r} \{b_j, \gamma b_j, \dots, \gamma^{h-1} b_j\}$$

as a disjoint union. Consequently, if $f(0) \neq 0$, then h divides m; if $f(0) = 0$, then h divides $m - 1$.

Finally, we verify that $h^{-1} \in \mathbb{F}$. This is true unless $\mathbb{F}$ has characteristic $k \neq 0$ and $h = km$ for some positive integer m. In this case, every element y in $M(f)$ satisfies the equation

$$0 = y^h - 1 = y^{mk} - 1 = (y^m - 1)^k,$$

which implies $M(f)$ contains at most m elements. This contradiction shows that $h^{-1} \in \mathbb{F}$. □

Lemma 3.8. *Let $f(x) = (x - a_1)(x - a_2) \cdots (x - a_m) \in \mathbb{F}[x]$ be a polynomial splitting in $\mathbb{F}$ having a nonzero simple zero. Let $A \in \mathbf{M}_n(\mathbb{F})$ such that $f(aA) = 0$ for all $a \in Z(f)$.*

(a) *All eigenvalues of A are zero multipliers of $f(x)$.*
(b) *All nonzero eigenvalues of A are not defective.*
(c) *If 0 is a simple zero of $f(x)$, then A is diagonalizable.*

Proof. (a) For any $a_j \in Z(f)$, since $f(a_jA) = 0$, Lemma 3.2 implies that $a_j\lambda \in Z(f)$ for every eigenvalue λ of A. In other words, $\lambda Z(f) \subseteq Z(f)$. That is, λ is a zero multiplier of $f(x)$.

(b) Assume that $a_i \neq 0$ is a simple zero of $f(x)$. If λ is a nonzero eigenvalue of A, then $\lambda \in M(f)$ by (a), and thus $\lambda^{-1} \in M(f)$ by Lemma 3.7. In particular, $\lambda^{-1}a_i \in Z(f)$. It follows from $f((\lambda^{-1}a_i)A) = 0$ that all the Jordan blocks of $(\lambda^{-1}a_i)A$ for its eigenvalue $(\lambda^{-1}a_i)\lambda = a_i$ are 1×1 blocks. Therefore, the algebraic multiplicity of any nonzero eigenvalue λ of A equals to its geometric multiplicity.

(c) If 0 is a simple zero of $f(x)$, as well as an eigenvalue of A, then $f(a_iA) = 0$ implies that all the Jordan blocks of A for its eigenvalue 0 are 1×1 blocks. Therefore, the zero eigenvalue of A, if any, is not defective either. Thus A is diagonalizable. □

Lemma 3.9. *Let $f(x) = (x - a_1)(x - a_2) \cdots (x - a_m) \in \mathbb{F}[x]$ be a polynomial splitting in $\mathbb{F}$. Let $\Phi\colon \mathbf{S}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ be a linear map preserving matrices annihilated by $f(x)$.*

(a) *All eigenvalues of $\Phi(I_n)$ are zero multipliers of $f(x)$.*
(b) *Suppose, in addition, that $f(x)$ has a nonzero simple zero. Then all nonzero eigenvalues of $\Phi(I_n)$ are not defective. Moreover,*
   - *if 0 is a simple zero of $f(x)$, then $\Phi(I_n)$ is diagonalizable;*
   - *if $f(0) \neq 0$, then $\Phi(I_n)$ is invertible;*
   - *if $\Phi(I_n)$ is invertible, then $\Phi(I_n)$ is diagonalizable;*
   - *if $\Phi(I_n)$ is diagonalizable and $M(f) = \{1\}$, then $\Phi(I_n) = I_r$.*

Proof. (a) For any $a_j \in Z(f)$, we have $f(a_jI_n) = f(a_j)I_n = 0$ by Lemma 3.1, and thus $f(a_j\Phi(I_n)) = 0$. Hence (a) follows from Lemma 3.8(a).

(b) Since $f(a_j\Phi(I_n)) = 0$ for every $a_j \in Z(f)$, it follows from Lemma 3.8(b) that all nonzero eigenvalues of $\Phi(I_n)$ are not defective.

If 0 is a simple zero of $f(x)$, then $\Phi(I_n)$ is diagonalizable by Lemma 3.8(c).

If $f(0) \neq 0$, and thus $0 \notin M(f)$, then $\Phi(I_n)$ has only nonzero eigenvalues and thus it is invertible. If $\Phi(I_n)$ is invertible, then $\Phi(I_n)$ is diagonalizable since all its eigenvalues are nonzero and thus not defective. Finally, if $\Phi(I_n)$ is diagonalizable and $M(f) = \{1\}$, then $\Phi(I_n) = I_r$ since all eigenvalues of $\Phi(I_n)$ are the zero multipliers 1 of $f(x)$ by (a). □

Corollary 3.10. *Let $\mathbb{F}$ be a field of characteristic zero. Let $f(x) \in \mathbb{F}[x]$ be a polynomial splitting in $\mathbb{F}$ of degree at least two with a nonzero simple zero. Suppose that $f(0) \neq 0$ and $Z(f) \neq \gamma Z(f)$ for any primitive root $\gamma \neq 1$ of the unity. Let $n \geq 2$, $\mathbf{V}_n(\mathbb{F}) = \mathbf{M}_n(\mathbb{F})$ or $\mathbf{S}_n(\mathbb{F})$, and $\Phi\colon \mathbf{V}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ be a linear map preserving matrices annihilated by $f(x)$. Then Φ has the form*

$$A \mapsto S \begin{pmatrix} A \otimes I_p & \\ & A^{\mathrm{t}} \otimes I_q \end{pmatrix} S^{-1}$$

for an invertible matrix $S \in \mathbf{M}_r(\mathbb{F})$ and nonnegative integers p, q with $r = n(p+q)$.

Proof. We first claim that the zero set $Z(f) = \{a_1, a_2, \ldots\}$ does not form an additive coset, where $a_1 \neq a_2$. Supposing otherwise, we would see that $a_1 - a_2 + a_1 = 2a_1 - a_2 \in Z(f)$, and inductively, $(t+1)a_1 - ta_2 = a_1 - a_2 + [ta_1 - (t-1)a_2] \in Z(f)$ for $t = 1, 2, \ldots$. Since $\mathbb{F}$ has characteristic zero, they are all distinct. This gives a contradiction to the finiteness of $Z(f)$. On the other hand, it is easy to see that 1 is the unique zero multiplier of $f(x)$ since all other roots of unity cannot be (see Lemma 3.7). By Lemma 3.9, we see that $\Phi(I_n) = I_r$. The assertion follows from Theorem 3.4. □

The proofs of the following Lemma 3.11 and Theorem 3.12 are similar to those of [12, Theorems 3.5 and 3.6]. We give the details for clarity.

Lemma 3.11. *Let $f(x) = (x - a_1)(x - a_2) \cdots (x - a_m) \in \mathbb{F}[x]$ be a polynomial splitting in $\mathbb{F}$ with two simple zeroes including 0. Let $\mathbf{V}_n(\mathbb{F}) = \mathbf{M}_n(\mathbb{F})$ or $\mathbf{S}_n(\mathbb{F})$. Suppose $\Phi\colon \mathbf{V}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ is a linear map preserving matrices annihilated by $f(x)$. Then $\Phi(I_n)$ is diagonalizable and commutes with all elements in the range of Φ.*

Proof. Let $a_1 \neq 0$ be a simple zero of $f(x)$. Let $E \in \mathbf{V}_n(\mathbb{F})$ be an idempotent. Since $f(0) = 0$, for any $a_j \in Z(f)$, we have $f(a_jE) = f(a_j)E + f(0)(I_n - E) = 0$ by Lemma 3.1, and thus $f(a_j\Phi(E)) = 0$. Let λ be an eigenvalue of $\Phi(E)$. Because 0 is a simple zero of $f(x)$, it follows from Lemma 3.8 that λ is a zero multiplier of $f(x)$ and is not defective. In other words, $\Phi(E)$ is diagonalizable with eigenvalues from $M(f)$. Similarly, $\Phi(I_n - E)$ is also diagonalizable with eigenvalues from $M(f)$.

Let h be the order of the finite cyclic subgroup $M(f)\backslash\{0\} = \{\lambda, \ldots, \lambda^h = 1\}$ of $\mathbb{F}\backslash\{0\}$. For any $a \in Z(f)$ and $t = 1, \ldots, h$, we have $a\lambda^t \in Z(f)$. Consequently, by Lemma 3.1,

$$f(a(E + \lambda^t(I_n - E))) = f(a)E + f(a\lambda^t)(I_n - E) = 0.$$

Hence,

$$f(a(\Phi(E) + \lambda^t\Phi(I_n - E))) = 0, \quad \text{for all } a \in Z(f) \text{ and for } t = 1, \ldots, h.$$

It follows from Lemma 3.2 that all eigenvalues of $a(\Phi(E)+\lambda^t\Phi(I_n-E))$ are zeroes of $f(x)$. Furthermore, an argument similar to the above for $\Phi(E)$ shows that $\Phi(E)+\lambda^t\Phi(I_n-E)$ is diagonalizable with eigenvalues from $M(f)$.

Since $M(f)$ is a finite multiplicative cyclic group of order h, and all eigenvalues of the diagonalizable matrices $\Phi(E), \Phi(I_n - E)$ and $\Phi(E) + \lambda^t\Phi(I_n - E)$ are from $M(f)$, we have

$$\Phi(E)^{h+1} = \Phi(E), \quad \Phi(I_n - E)^{h+1} = \Phi(I_n - E),$$

and

$$(\Phi(E) + \lambda^t\Phi(I_n - E))^{h+1} = \Phi(E) + \lambda^t\Phi(I_n - E), \quad \text{for } t = 1, \dots, h.$$

Denote by $E' = \Phi(E)$ and $F' = \Phi(I_n - E)$. Then

$$E'^{h+1} = E', \quad F'^{h+1} = F', \quad \text{and} \quad (E' + \lambda^t F')^{h+1} = E' + \lambda^t F', \quad \text{for } t = 1, \dots, h. \tag{3.4}$$

For $j = 1, \dots, h$, let

$$q_j = \text{ the sum of all noncommutative products consisting of } h + 1 - j \ E'\text{'s and } j \ F'\text{'s}.$$

Expanding the third equation in (3.4), we have

$$E'^{h+1} + \sum_{j=1}^{h} \lambda^{jt} q_j + \lambda^{t(h+1)} F'^{h+1} = E' + \lambda^t F'.$$

Since $E'^{h+1} = E'$, $F'^{h+1} = F'$ and $\lambda^h = 1$, we have

$$\sum_{j=1}^{h} \lambda^{jt} q_j = 0, \quad \text{for all } t = 1, \dots, h.$$

Since $\sum_{t=1}^{h} \lambda^{jt} = \lambda^j(1 - \lambda^{jh})(1 - \lambda^j)^{-1} = 0$ for $j = 1, \dots, h - 1$, we have

$$\sum_{t=1}^{h}\sum_{j=1}^{h} \lambda^{jt} q_j = \sum_{j=1}^{h} q_j \sum_{t=1}^{h} \lambda^{jt} = hq_h = 0.$$

Since h^{-1} exists in $\mathbb{F}$ by Lemma 3.7, we have $q_h = 0$. It follows from

$$\begin{aligned} q_h F' &= (E'F'^h + F'E'F'^{h-1} + \cdots + F'^h E')F' = 0 \\ = F'q_h &= F'(E'F'^h + F'E'F'^{h-1} + \cdots + F'^h E') \end{aligned}$$

that

$$E'F' = -F'E'F'^h - \cdots - F'^h E'F' = F'E'.$$

Consequently, $\Phi(E)\Phi(I_n) = E'(E' + F') = (E' + F')E' = \Phi(E)\Phi(I_n)$ for every idempotent E in the domain of Φ. By Lemma 2.1, the assertion is established. □

Using Lemmas 3.9 and 3.11, we obtain the following extension of Theorem 3.4 relaxing the unital assumption.

Theorem 3.12. *Let $f(x) \in \mathbb{F}[x]$ be a polynomial splitting in a field $\mathbb{F}$ of degree at least two and with a nonzero simple zero such that the zero set $Z(f)$ is not an additive coset. Let $n \geq 2$, $\mathbf{V}_n(\mathbb{F}) = \mathbf{M}_n(\mathbb{F})$ or $\mathbf{S}_n(\mathbb{F})$, but $\mathbf{V}_n(\mathbb{F}) \neq \mathbf{S}_3(\mathbb{Z}_3)$ or $\mathbf{S}_2(\mathbb{F})$ with $|\mathbb{F}| \leq 5$. Suppose $\Phi\colon \mathbf{V}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ is a linear map preserving matrices annihilated by $f(x)$. Assume that either*

- *$f(0) \neq 0$, and $\Phi(I_n)$ commutes with all elements in the range of Φ, or*
- *$f(0) = 0$, and 0 is a simple zero of $f(x)$.*

Then there are nonnegative integers p, q with $s = r - np - nq \geq 0$, an invertible matrix $S \in \mathbf{M}_r(\mathbb{F})$, and invertible diagonal matrices $D_1 \in \mathbf{M}_p(\mathbb{F})$, $D_2 \in \mathbf{M}_q(\mathbb{F})$ with diagonal entries from $M(f)$ such that Φ has the form

$$A \mapsto S \begin{pmatrix} A \otimes D_1 & & \\ & A^{\mathrm{t}} \otimes D_2 & \\ & & 0_s \end{pmatrix} S^{-1}, \tag{3.5}$$

and $s = 0$ if $f(0) \neq 0$. Here, we may assume that the part $A^t \otimes D_2$ is vacuous if $\mathbf{V}_n(\mathbb{F}) = \mathbf{S}_n(\mathbb{F})$.

Proof. By the assumption, $\mathbb{F} \neq \mathbb{Z}_2$ for else $Z(f) = \mathbb{Z}_2$ is an additive group. By Lemma 3.9, $\Phi(I_n)$ is diagonalizable with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_l$, which are zero multipliers of $f(x)$. After a similarity transformation, we can assume that

$$\Phi(I_n) = \begin{pmatrix} \lambda_1 I_{r_1} & & & \\ & \lambda_2 I_{r_2} & & \\ & & \ddots & \\ & & & \lambda_l I_{r_l} \end{pmatrix}$$

When 0 is a simple zero of $f(x)$, Lemma 3.11 ensures that $\Phi(I_n)$ commutes with all elements in the range of Φ. Since $\Phi(I_n)$ is central in the range of Φ in any case, there are linear maps Φ_j from $\mathbf{M}_n(\mathbb{F})$ or $\mathbf{S}_n(\mathbb{F})$ into $\mathbf{M}_{r_j}(\mathbb{F})$ preserving matrices annihilated by $f(x)$ such that Φ has the form

$$A \mapsto \begin{pmatrix} \Phi_1(A) & & & \\ & \Phi_2(A) & & \\ & & \ddots & \\ & & & \Phi_l(A) \end{pmatrix}.$$

Suppose some $\lambda_j = 0$. We claim that $\Phi_j(E) = 0$ for every idempotent E, and thus $\Phi_j = 0$ by Lemma 2.1. Note that 0 is a simple zero of $f(x)$ in this case. Replacing $f(x)$ with $f(ax)$ for some suitable a, we can also assume that 1 is an other zero of $f(x)$. It follows from $\Phi_j(I_n) = 0$ that $\Phi_j(E) = -\Phi_j(I_n - E)$. Let α be an eigenvalue of $\Phi_j(E)$. For any zeroes a, b of $f(x)$, by Lemma 3.1, we have $f(aE + b(I_n - E)) = 0$. Hence $f(a\Phi_j(E) + b\Phi_j(I_n - E)) = f((a - b)\Phi_j(E)) = 0$. By Lemma 3.2, $\alpha(a - b) \in Z(f)$. In particular, with $b = 0$ we see that $\alpha Z(f) = Z(f)$ if $\alpha \neq 0$. But in this case, $\alpha^{-1} \in M(f)$ by Lemma 3.7, and thus $Z(f) = \alpha^{-1}Z(f)$ is an additive group, a contradiction. Consequently, all eigenvalues of $\Phi_j(E)$ are zero. With $a = 1$ and $b = 0$, we see that $f(\Phi_j(E)) = 0$. Since 0 is a simple zero of $f(x)$, it forces $\Phi_j(E) = 0$ as claimed.

For those nonzero λ_j we consider the polynomial $f_j(x) = f(x/\lambda_j)$. Then

$$f_j(A) = 0 \quad \Longrightarrow \quad f(A/\lambda_j) = 0 \quad \Longrightarrow \quad f(\Phi_j(A/\lambda_j)) = f(\lambda_j^{-1}\Phi_j(A)) = 0$$

for all $A \in \mathbf{V}_n(\mathbb{F})$. Consequently, the unital linear map ${\lambda_j}^{-1}\Phi_j$ preserves matrices annihilated by $f_j(x)$. Since $Z(f_j) = \lambda_j Z(f)$ is not an additive coset and f_j also has a simple zero, we can then apply Theorem 3.4 to establish the desired assertions. □

Remark 3.13. When $n = 1$, we have already seen in Theorem 3.4 that any unital linear map $\Phi : \mathbf{M}_1(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ preserves matrices annihilated by any but fixed polynomial $f(x) \in \mathbb{F}[x]$. In general, let $J = \Phi(1)$ which might not be the $r \times r$ identity matrix I_r. Assume that Φ preserves matrices annihilated by $f(x)$. Let $f(a) = 0$ for some $a \neq 0$ in $\mathbb{F}$. Since $f(\Phi(a)) = f(aJ) = 0$, we can suppose that J is in its Jordan form with all diagonal (eigenvalue) entries λ from $\mathbb{F}$ such that $\lambda Z(f) \subseteq Z(f)$, and $\lambda Z(f) = Z(f)$ whenever $\lambda \neq 0$.

Let $a_1 \neq 0$ be a simple zero of $f(x)$. Suppose λ is a nonzero eigenvalue of J. Then $\lambda Z(f) = Z(f)$ implies that $a_1\lambda^{-1} \in Z(f)$. Since $f(a_1\lambda^{-1}J) = 0$, we see that the Jordan blocks of J corresponding to any nonzero eigenvalue λ have size 1×1.

We then divide into 2 cases. First, assume that $f(0) \neq 0$, or 0 is another simple zero of $f(x)$. In this case, the triangular blocks of a_1J with 0 diagonal, if any, have size at most 1×1. As a result, $\Phi(1) = J$ is a diagonalizable matrix with all eigenvalues λ satisfying $\lambda Z(f) \subseteq Z(f)$. It is plain that this condition is also sufficient for Φ preserves matrices annihilated by $f(x)$.

Assume that 0 is a zero of $f(x)$ of order $p > 1$. In this case, the triangular blocks of a_1J corresponding to the eigenvalue 0 can have size up to $p \times p$. Consequently, $J = \Phi(1)$ assumes the form

$$S \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_q & \\ & & & & J_0 \end{pmatrix} S^{-1},$$

where $S \in \mathbf{M}_r(\mathbb{F})$ is invertible, $\lambda_j Z(f) \subseteq Z(f)$ for $j = 1, \ldots, q$, and J_0 is a sum of Jordan blocks with zero diagonal entries and sizes adding up to $p \times p$ with $p+q = r$. It is plain that this condition is also sufficient for Φ preserves matrices annihilated by $f(x)$.

Finally, the case $f(x)$ has no nonzero simple zero will be more tedious, and we leave it to the interested readers.

Recall that a matrix A is called an *m-potent* matrix if $A^m = A$. Together with Theorems 1.1 and 1.2 (for the case $m = 2$), Corollary 3.14 below extends some known results; see, e.g., Theorems 6.6.1 and 6.6.3 and their footnotes in [21].

Corollary 3.14. *Suppose that $n \geq 2$, $m \geq 3$ and $\mathbb{F}$ contains all distinct $(m-1)$th roots of unity, which do not form an additive group with 0. Consider a linear map $\Phi\colon \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$, or $\Phi\colon \mathbf{S}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ in which $\mathbf{S}_n(\mathbb{F}) \neq \mathbf{S}_2(\mathbb{Z}_5)$ when $m = 3$. Then Φ preserves m-potent matrices if and only if Φ has the form*

$$A \mapsto S \begin{pmatrix} A \otimes D_1 & & \\ & A^{\mathrm{t}} \otimes D_2 & \\ & & 0_s \end{pmatrix} S^{-1},$$

such that all the diagonal entries of the diagonal matrices D_1, D_2 are $(m-1)$th roots of unity.

Proof. The linear map Φ preserves matrices annihilated by the polynomial $f(x) = x^m - x$ which splits in $\mathbb{F}$ and has simple zeroes 0 and 1. Evidently, $M(f) = Z(f)$. Note that the assumption prevents from the cases when $|\mathbb{F}| = 2, 3, 4$. However, $\mathbb{Z}_5$ contains square roots $1, 4$ of unity, and $0, 1, 4$ do not form an additive group. Thus the special case $\mathbf{S}_n(\mathbb{F}) = \mathbf{S}_2(\mathbb{Z}_5)$ and $m = 3$ needs to be explicitly excluded. Then, Theorem 3.12 applies. □

Example 3.15. (a) Apart from those symmetric idempotents given in Remark 2.8(4), the only other tripotents in $\mathbf{S}_2(\mathbb{Z}_5)$ are

$$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}.$$

Therefore, the unital linear map Φ in Remark 2.8(4) preserves symmetric tripotents in $\mathbf{S}_2(\mathbb{Z}_5)$, but it does not have the form stated in Corollary 3.14.

(b) Consider $f(x) = x^2 g(x)$ where $g(x)$ is any polynomial in $\mathbb{F}[x]$ for any field $\mathbb{F}$. Note that 0 is not a simple zero of $f(x)$. The linear map $\Phi\colon \mathbf{M}_2(\mathbb{F}) \to \mathbf{M}_2(\mathbb{F})$ defined by $\Phi(A) = a_{11}E_{12}$ for $A = (a_{ij}) \in \mathbf{M}_2(\mathbb{F})$ satisfies that $f(\Phi(A)) = 0$ for any $A \in \mathbf{M}_2(\mathbb{F})$.

The non-diagonalizable non-invertible matrix $\Phi(I_2) = E_{12}$ commutes with all $\Phi(A)$ in the range of Φ. But Φ does not have the form (3.5).

4. Preservers between symmetric matrices

With different preserving properties including those in previous sections, one may be able to show that a linear map $\Phi\colon \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ has the form

$$\Phi(A) = S \begin{pmatrix} A \otimes I_p & & \\ & A^{\mathrm{t}} \otimes I_q & \\ & & 0_{r-n(p+q)} \end{pmatrix} S^{-1} \quad \text{for all } A \in \mathbf{M}_n(\mathbb{F}). \tag{4.1}$$

In particular,

$$\Phi(A) = S \begin{pmatrix} A \otimes I_k & 0 \\ 0 & 0_{r-nk} \end{pmatrix} S^{-1} \quad \text{for all } A \in \mathbf{S}_n(\mathbb{F}). \tag{4.2}$$

If Φ preserves transpositions, that is, $\Phi(A^{\mathrm{t}}) = \Phi(A)^{\mathrm{t}}$, and thus sends symmetric matrices to symmetric matrices, one can say more about the map.

Proposition 4.1. *Let $\mathbb{F} \neq \mathbb{Z}_2$ and $n \geq 2$.*

(a) *Suppose $\Phi\colon \mathbf{S}_n(\mathbb{F}) \to \mathbf{S}_r(\mathbb{F})$ has the form (4.2). Let S_0 be the matrix obtained from the first kn columns of S. Then $S_0^{\mathrm{t}} S_0 = I_n \otimes B$ for an invertible symmetric matrix $B \in \mathbf{S}_k(\mathbb{F})$, and Φ also has the form*

$$A \mapsto S_0(A \otimes B^{-1})S_0^{\mathrm{t}}. \tag{4.3}$$

(b) *Suppose $\Phi\colon \mathbf{M}_n(\mathbb{F}) \to \mathbf{M}_r(\mathbb{F})$ has the form (4.1) and preserves transposition. Let S_1 and S_2 be the matrices obtained from the first np columns and the next nq columns of S. Then $S_1^{\mathrm{t}} S_1 = I_n \otimes B_1$ and $S_2^{\mathrm{t}} S_2 = I_n \otimes B_2$ for some invertible symmetric matrices $B_1 \in \mathbf{S}_p(\mathbb{F})$ and $B_2 \in \mathbf{S}_q(\mathbb{F})$ such that Φ also has the form*

$$A \mapsto S_1(A \otimes B_1^{-1})S_1^{\mathrm{t}} + S_2(A^{\mathrm{t}} \otimes B_2^{-1})S_2^{\mathrm{t}}. \tag{4.4}$$

Moreover, $S_1^{\mathrm{t}} S_2 = 0_{np\times nq}$ and thus the two summand maps have zero product.

Proof. (a) Assume that Φ satisfies (4.2). If $A \in \mathbf{S}_n(\mathbb{F})$, then $\Phi(A) = \Phi(A)^{\mathrm{t}}$, or

$$S[(A \otimes I_k) \oplus 0_{r-nk}]S^{-1} = (S^{-1})^{\mathrm{t}}[(A \otimes I_k) \oplus 0_{r-nk}]S^{\mathrm{t}}.$$

Thus

$$S^{\mathrm{t}} S[(A \otimes I_k) \oplus 0_{r-nk}] = [(A \otimes I_k) \oplus 0_{r-nk}]S^{\mathrm{t}} S \tag{4.5}$$

for all symmetric $A \in \mathbf{S}_n(\mathbb{F})$. Putting $A = I_n$ in (4.5), we see that $S^{\mathrm{t}}S = W \oplus Z$ for some invertible symmetric matrices $W \in \mathbf{S}_{kn}(\mathbb{F})$ and $Z \in \mathbf{S}_{r-nk}(\mathbb{F})$. Putting $A = E_{ii}$ and $A = E_{ij} + E_{ji}$ in (4.5), $1 \le i \le j \le n$, we see further that

$$S^{\mathrm{t}}S = (I_n \otimes B) \oplus Z, \tag{4.6}$$

for some invertible symmetric matrices $B \in \mathbf{S}_k(\mathbb{F})$ and $Z \in \mathbf{S}_{r-nk}(\mathbb{F})$.

For any $A \in \mathbf{M}_n(\mathbb{F})$, It follows from (4.6) that

$$\begin{aligned}\Phi(A) &= \Phi(A)\Phi(I_n) = \Phi(A)\Phi(I_n)^{\mathrm{t}} \\ &= S[(A \otimes I_k) \oplus 0_{r-nk}]S^{-1}(S^{-1})^{\mathrm{t}}[(I_n \otimes I_k) \oplus 0_{r-nk}]S^{\mathrm{t}} \\ &= S[(A \otimes I_k) \oplus 0_{r-nk}][(I_n \otimes B^{-1}) \oplus Z^{-1}][(I_n \otimes I_k) \oplus 0_{r-nk}]S^{\mathrm{t}} \\ &= S[(A \otimes B^{-1}) \oplus 0_{r-nk}]S^{\mathrm{t}} \\ &= S_0[(A \otimes B^{-1})]S_0^{\mathrm{t}},\end{aligned}$$

where S_0 is the matrix consisting of the first nk columns of S.

(b) Suppose Φ has the form (4.1) and preserves transpositions. Observe

$$S\begin{pmatrix} A^{\mathrm{t}} \otimes I_p & & \\ & A \otimes I_q & \\ & & 0_{r-n(p+q)}\end{pmatrix}S^{-1} = (S^{-1})^{\mathrm{t}}\begin{pmatrix} A^{\mathrm{t}} \otimes I_p & & \\ & A \otimes I_q & \\ & & 0_{r-n(p+q)}\end{pmatrix}S^{\mathrm{t}}$$

for all $A \in \mathbf{M}_n(\mathbb{F})$. It follows

$$S^{\mathrm{t}}S[(A \otimes I_p) \oplus (A^{\mathrm{t}} \otimes I_q) \oplus 0_{r-n(p+q)}] = [(A \otimes I_p) \oplus (A^{\mathrm{t}} \otimes I_q) \oplus 0_{r-n(p+q)}]S^{\mathrm{t}}S \tag{4.7}$$

for all $A \in \mathbf{M}_n(\mathbb{F})$. Putting $A = I_n$, $A = E_{ii}$, and $A = E_{ij}$ for $i \neq j$, respectively in (4.7), we can derive

$$S^{\mathrm{t}}S = ((I_n \otimes B_1) \oplus (I_n \otimes B_2)) \oplus Z \tag{4.8}$$

for some invertible symmetric matrices $B_1 \in \mathbf{S}_p(\mathbb{F})$, $B_2 \in \mathbf{S}_q(\mathbb{F})$ and $Z \in \mathbf{S}_{r-n(p+q)}$. Direct computations show that

$$S_1^{\mathrm{t}}S_1 = I_n \otimes B_1, \quad S_2^{\mathrm{t}}S_2 = I_n \otimes B_2, \quad S_1^{\mathrm{t}}S_2 = 0_{np\times nq}, \quad \text{and} \quad S_2^{\mathrm{t}}S_1 = 0_{nq\times np},$$

where S_1 and S_2 are the matrices consisting of the first np columns and the next nq columns of S, respectively.

For any $A \in \mathbf{M}_n(\mathbb{F})$, it follows from (4.8) that

$$\Phi(A) = \Phi(A)\Phi(I_n) = \Phi(A)\Phi(I_n)^{\mathrm{t}}$$

$$
\begin{aligned}
&= S[(A\otimes I_p)\oplus(A^{\mathrm t}\otimes I_q)\oplus 0_{r-n(p+q)}](S^{\mathrm t}S)^{-1}[(I_n\otimes I_p)\oplus(I_n\otimes I_q)\oplus 0_{r-n(p+q)}]S^{\mathrm t}\\
&= S[(A\otimes B_1^{-1})\oplus(A^{\mathrm t}\otimes B_2^{-1})\oplus 0_{r-n(p+q)}]S^{\mathrm t}\\
&= S[(A\otimes B_1^{-1})\oplus 0_{nq}\oplus 0_{r-n(p+q)}]S^{\mathrm t}+S[0_{np}\oplus(A^{\mathrm t}\otimes B_2^{-1})\oplus 0_{r-n(p+q)}]S^{\mathrm t}\\
&= S_1(A\otimes B_1^{-1})S_1^{\mathrm t}+S_2(A^{\mathrm t}\otimes B_2^{-1})S_2^{\mathrm t}. \qquad \square
\end{aligned}
$$

Remark 4.2. (a) The assertion in Proposition 4.1(b) might not hold if we assume only that Φ sends symmetric matrices to symmetric matrices. Indeed, Φ might not preserve transposition, while it should if Φ has the form (4.4). For example, consider the linear map from $\mathbf{M}_n(\mathbb{F})$ into $\mathbf{M}_{2n}(\mathbb{F})$ defined by

$$
\begin{aligned}
A \mapsto &\begin{pmatrix} I_n & I_n \\ I_n/2 & 0_n \end{pmatrix}\begin{pmatrix} A & \\ & A^{\mathrm t} \end{pmatrix}\begin{pmatrix} I_n & I_n \\ I_n/2 & 0_n \end{pmatrix}^{-1}\\
&= \begin{pmatrix} I_n & I_n \\ I_n/2 & 0_n \end{pmatrix}\begin{pmatrix} A & \\ & A^{\mathrm t} \end{pmatrix}\begin{pmatrix} 0_n & 2I_n \\ I_n & -2I_n \end{pmatrix} = \begin{pmatrix} A^{\mathrm t} & 2(A-A^{\mathrm t}) \\ 0_n & A \end{pmatrix}.
\end{aligned}
$$

It has the form (4.1) and sends symmetric matrices to symmetric matrices. But it does not preserve transposition.

(b) When $p = 0$ or $q = 0$, if Φ has the form (4.1) and sends symmetric matrices to symmetric matrices, then Φ preserves transpositions, and thus it has the form (4.4). This happens in particular when $n = r$.

Let $q = 0$. In view of (4.6), we have

$$
S^{\mathrm t} = [(I_n \otimes B) \oplus Z]S^{-1},
$$

and thus

$$
\begin{aligned}
\Phi(A)^{\mathrm t} &= (S^{-1})^{\mathrm t}[(A^{\mathrm t}\otimes I_p)\oplus 0_{r-np}]S^{\mathrm t}\\
&= S[(I_n\otimes B^{-1})\oplus Z^{-1}][(A^{\mathrm t}\otimes I_p)\oplus 0_{r-np}][(I_n\otimes B)\oplus Z]S^{-1}\\
&= S[(A^{\mathrm t}\otimes I_p)\oplus 0_{r-np}]S^{-1} = \Phi(A^{\mathrm t}).
\end{aligned}
$$

Therefore, Φ preserves transposition and we can apply Proposition 4.1(b). The case when $p = 0$ is similar.

Maps in the form (4.3) or (4.4) admit even simpler descriptions if the underlying field $\mathbb{F}$ has some special properties. For instance, using Proposition 4.8 below, if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and Φ has the form (4.3) or (4.4), then we may assume that S is orthogonal, i.e., $S^{\mathrm t}S = I_r$. In general, it is also the case when $\mathbb{F}$ is *Pythagorean*, that is, for any $v_1, \dots, v_m \in \mathbb{F}$, there is $v \in \mathbb{F}$ such that $v^2 = \sum_{j=1}^m v_j^2$.

The following result is a consequence of a well known result about diagonalization of symmetric bilinear forms; see, e.g., [2, Theorem 6.35].

Lemma 4.3. *Suppose that the characteristic of* $\mathbb{F}$ *is not* 2*. For every symmetric matrix* $B \in \mathbf{S}_n(\mathbb{F})$*, there is an invertible matrix* R *and a diagonal matrix* D *in* $\mathbf{M}_n(\mathbb{F})$ *such that* $B = R^{\mathrm{t}} D R$*.*

The conclusion of Lemma 4.3 does not hold when $\mathbb{F}$ has characteristic 2.

Example 4.4. Consider the symmetric matrix $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbf{S}_2(\mathbb{F})$, where $\mathbb{F}$ has characteristic 2. Suppose that $B = R^{\mathrm{t}} D R$, or $(R^{-1})^{\mathrm{t}} B R^{-1} = D$, for some invertible matrix $R^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and diagonal matrix D in $\mathbf{M}_2(\mathbb{F})$. Direct computation gives that

$$D = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 2\alpha\gamma & \beta\gamma + \alpha\delta \\ \beta\gamma + \alpha\delta & 2\beta\delta \end{pmatrix} = \begin{pmatrix} 0 & \beta\gamma + \alpha\delta \\ \beta\gamma + \alpha\delta & 0 \end{pmatrix}.$$

This forces the diagonal matrix $D = 0$, and thus $B = 0$, a contradiction. Hence, it is impossible to diagonal B as in Lemma 4.3.

Proposition 4.5. *Suppose the characteristic of* $\mathbb{F}$ *is not* 2*. Assume that* Φ *is a linear map of the form* (4.2) *(resp.* (4.1) *and preserving transpositions). Then there are matrices* $\hat{S}_0, \hat{S}_1, \hat{S}_2$ *of appropriate sizes such that* Φ *also has the form*

$$A \mapsto \hat{S}_0 (A \otimes D^{-1}) \hat{S}_0^{\mathrm{t}} \quad \text{for all } A \in \mathbf{S}_n(\mathbb{F}) \tag{4.9}$$

$$(\text{resp. } A \mapsto \hat{S}_1 (A \otimes D_1^{-1}) \hat{S}_1^{\mathrm{t}} + \hat{S}_2 (A^{\mathrm{t}} \otimes D_2^{-1}) \hat{S}_2^{\mathrm{t}} \quad \text{for all } A \in \mathbf{M}_n(\mathbb{F})) \tag{4.10}$$

for some invertible diagonal matrices D_0, D_1, D_2*.*

Proof. Use the notations in Proposition 4.1. By Lemma 4.3, the symmetric invertible matrices B, B_1, B_2 in (4.3) and (4.4) admit the decomposition

$$B = R^{\mathrm{t}} D R, \quad B_1 = R_1^{\mathrm{t}} D_1 R_1, \quad B_2 = R_2^{\mathrm{t}} D_2 R_2$$

for some invertible matrices $R \in \mathbf{M}_k(\mathbb{F}), R_1 \in \mathbf{M}_p(\mathbb{F}), R_2 \in \mathbf{M}_q(\mathbb{F})$, and invertible diagonal matrices $D \in \mathbf{M}_k(\mathbb{F}), D_1 \in \mathbf{M}_p(\mathbb{F}), D_2 \in \mathbf{M}_q(\mathbb{F})$. Set $\hat{S}_0 = S_0(I_n \otimes R^{-1})$ and $\hat{S}_j = S_j(I_n \otimes R_j^{-1})$ for $j = 1, 2$. The form (4.9) or (4.10) can be verified readily. □

The following lemma follows from Witt's theorem (see, e.g., [7, Theorem XV.10.2]).

Lemma 4.6. *Suppose* $\mathbb{F}$ *is a Pythagorean field with characteristic not* 2*, and* $1 \le m \le r$*. Then every orthonormal family* $\{u_1, \ldots, u_m\}$ *of vectors in* $\mathbb{F}^r$ *can be extended to an orthonormal basis* $\{u_1, \ldots, u_m, \ldots, u_r\}$ *of* $\mathbb{F}^r$*; namely,*

$$u_i^{\mathrm{t}} u_j = \delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

When $\mathbb{F}$ has characteristic 2, not every orthonormal family of $\mathbb{F}^r$ can be enlarged to an orthonormal basis.

Example 4.7. Let $\mathbb{F}$ have characteristic 2. Consider the unit vector $u_1 = (1 \quad 1 \quad 1)^{\mathrm{t}}$ in $\mathbb{F}^3$. Suppose there are two vectors $u_2 = (a \quad b \quad c)^{\mathrm{t}}$ and $u_3 = (d \quad e \quad f)^{\mathrm{t}}$ such that $\{u_1, u_2, u_3\}$ is an orthonormal basis of $\mathbb{F}^3$. Then the mutual orthogonality implies that

$$a + b + c = d + e + f = ad + be + cf = 0.$$

It forces the determinant of the invertible matrix formed by u_1, u_2, u_3 becomes

$$\begin{aligned}\det\begin{pmatrix}1 & a & d\\ 1 & b & e\\ 1 & c & f\end{pmatrix} &= bf + ce + af + cd + ae + bd\\ &= a(e+f) + b(d+f) + c(d+e) = ad + be + cf = 0.\end{aligned}$$

This contradiction shows that the conclusion of Lemma 4.6 does not hold in this case.

Proposition 4.8. *Suppose $\mathbb{F}$ is a Pythagorean field with characteristic not 2. Assume that Φ is a linear map of the form (4.2) (resp. (4.1) and preserving transpositions). Then the matrices S can be chosen to be orthogonal, i.e., $S^{\mathrm{t}} = S^{-1}$.*

Proof. We use the notation in Propositions 4.1 and 4.5. Assume Φ has the form (4.2), Note that $I_n \otimes B = S_0^{\mathrm{t}} S_0$. Let S_{00} be the matrix obtained from the first k columns of S_0. Then ${S_{00}}^{\mathrm{t}} S_{00} = B$ and the diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_k) = (R^{\mathrm{t}})^{-1} B R^{-1} = (R^{\mathrm{t}})^{-1} {S_{00}}^{\mathrm{t}} S_{00} R^{-1}$. Hence, $d_j = v_j^{\mathrm{t}} v_j$ where v_j is the jth column of $S_{00} R^{-1}$ for $j = 1, \ldots, k$. Since $\mathbb{F}$ is Pythagorean, there are $q_1, \ldots, q_k$ such that $D = \operatorname{diag}(q_1^2, \ldots, q_k^2)$. Replacing R by $\operatorname{diag}(q_1, \ldots, q_k) R$, we have that $B = R^{\mathrm{t}} R$. In this way, we can assume $D = I_k$ in the form (4.9). Similarly, we can also choose $D_1 = I_p$ and $D_2 = I_q$ in the form (4.10).

Now, (4.9) becomes

$$\Phi(A) = \hat{S}_0 (A \otimes I_k) \hat{S}_0^{\mathrm{t}} \quad \text{for all } A \in \mathbf{S}_n(\mathbb{F}).$$

In view of (4.2),

$$\Phi(I_n)\Phi(A) = \Phi(A)\Phi(I_n) = \Phi(A),$$

in other words,

$$\hat{S}_0 \hat{S}_0^{\mathrm{t}} \hat{S}_0 (A \otimes I_k) \hat{S}_0^{\mathrm{t}} = \hat{S}_0 (A \otimes I_k) \hat{S}_0^{\mathrm{t}} \hat{S}_0 \hat{S}_0^{\mathrm{t}} = \hat{S}_0 (A \otimes I_k) \hat{S}_0^{\mathrm{t}} \quad \text{for all } A \in \mathbf{S}_n(\mathbb{F}). \tag{4.11}$$

Since the $r \times nk$ matrix $\hat{S}_0 = S_0(I_n \otimes R^{-1})$ and S_0 consists of the first nk columns of the invertible $r \times r$ matrices S, the column rank of $\hat{S}_0$ is nk. Hence, there is an $nk \times r$

matrix L_0 such that $L_0\hat{S}_0 = I_{nk}$. Multiplying (4.11) from the left by L_0 and from the right by L_0^{t}, we have

$$\hat{S}_0^{\mathrm{t}}\hat{S}_0(A \otimes I_k) = (A \otimes I_k)\hat{S}_0^{\mathrm{t}}\hat{S}_0 = A \otimes I_k \quad \text{for all } A \in \mathbf{S}_n(\mathbb{F}). \tag{4.12}$$

Putting $A = I_n$, E_{ii} and $E_{ij} + E_{ji}$ for $1 \le i \le j \le n$ in (4.12), we see that $\hat{S}_0^{\mathrm{t}}\hat{S}_0 = I_n \otimes I_k = I_{nk}$.

Let $u_1, \ldots, u_{nk}$ be the columns of the $r \times nk$ matrix $\hat{S}_0$. Since $\hat{S}_0^{\mathrm{t}}\hat{S}_0 = I_{nk}$, we have $u_i^{\mathrm{t}} u_j = \delta_{ij}$ for all $i, j = 1, \ldots, nk$, and thus $\{u_1, \ldots, u_{nk}\}$ is an orthonormal family of $\mathbb{F}^r$. By Lemma 4.6, we can enlarge it to an orthonormal basis $\{u_1, \ldots, u_{nk}, \ldots, u_r\}$ of $\mathbb{F}^r$. We may replace S in the form (4.2) by the matrix with columns $u_1, \ldots, u_r$. It then follows that Φ has the form (4.2) with $S^{\mathrm{t}}S = I_r$.

The assertion that the matrix S in (4.1) can be chosen to be orthogonal can be verified with similar arguments, by noting that S_1, S_2 consist of orthogonal columns from S by Proposition 4.1(b). □

The following examples show that if the underlying field is not Pythagorean, then we may not replace D, D_1, D_2 by I_k in the conclusion of Proposition 4.1.

Example 4.9. Let $\Phi\colon \mathbf{S}_2(\mathbb{Z}_5) \to \mathbf{S}_2(\mathbb{Z}_5)$ be defined by

$$A \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} A \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}^{-1} = 3 \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} A \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}.$$

Let $S = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$. The algebra isomorphism $\Phi(A) = SAS^{-1} = 3SAS^{\mathrm{t}}$ sends symmetric matrices to symmetric matrices and preserves transposition.

Suppose there is an orthogonal matrix $S_1 \in \mathbf{S}_2(\mathbb{Z}_5)$ such that $\Phi(A) = S_1 A S_1^{\mathrm{t}}$ for all $A \in \mathbf{S}_2(\mathbb{Z}_5)$. Then $S_1^{\mathrm{t}} S A = A S_1^{\mathrm{t}} S$ for all $A \in \mathbf{S}_2(\mathbb{Z}_5)$. This implies $S_1^{\mathrm{t}} S = \lambda I_2$, and thus $S = \lambda S_1$, for some nonzero $\lambda \in \mathbb{Z}_5$. Consequently,

$$I_2 = S_1^{\mathrm{t}} S_1 = \lambda^{-2} S^{\mathrm{t}} S = \lambda^{-2} \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}^2 = \lambda^{-2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

It implies $\lambda^2 = 2$. But $2 = 1^2 + 1^2$ has no square root in the non-Pythagorean field $\mathbb{Z}_5$. This contradiction shows that Φ cannot be implemented by an orthogonal matrix.

Declaration of competing interest

There is no competing interest.

Acknowledgement

We thank the referee for many helpful comments, which fixed some issues in the proofs and improved the presentation. We also thank Ching-Ting Tu for performing some computer experiments on matrices in $S_n(\mathbb{F})$ when $\mathbb{F} = \mathbb{Z}_2, \mathbb{Z}_3$ and $\mathbb{Z}_5$.

Li is an affiliate member of the Institute for Quantum Computing, University of Waterloo; his research was partially supported by the Simons Foundation Grant 851334. M.-C. Tsai, Y.-S. Wang and N.-C. Wong are supported by Taiwan NSTC grants 114-2115-M-027-001, 113-2115-M-005-008-MY2 and 114-2115-M-110-002-MY2, respectively.

Data availability

Data will be made available on request.

References

[1] M.A. Chebotar, W.-F. Ke, P.-H. Lee, N.-C. Wong, Mappings preserving zero products, Stud. Math. 155 (1) (2003) 77–94.
[2] S.H. Friedberg, A.J. Insel, L.E. Spence, Linear Algebra, fourth edition, Pearson, 2002.
[3] A. Guterman, C.-K. Li, P. Šemrl, Some general techniques on linear preserver problems, Linear Algebra Appl. 315 (2000) 61–81.
[4] J. Hou, S. Hou, Linear maps on operator algebras that preserve elements annihilated by a polynomial, Proc. Am. Math. Soc. 130 (2002) 2383–2395.
[5] R. Howard, Linear maps that preserve matrices annihilated by a polynomial, Linear Algebra Appl. 30 (1980) 167–176.
[6] B. Kuzma, Additive idempotent preservers, Linear Algebra Appl. 355 (2002) 103–117.
[7] S. Lang, Algebra, rev. 3rd ed., Grad. Texts in Math., vol. 211, Springer-Verlag, New York, 2002.
[8] C.-K. Li, S. Pierce, Linear operators preserving similarity classes and related results, Can. Math. Bull. 37 (1994) 374–383.
[9] C.-K. Li, S. Pierce, Linear preserver problems, Am. Math. Mon. 108 (2001) 591–605.
[10] C.-K. Li, M.-C. Tsai, Y.-S. Wang, N.-C. Wong, Nonsurjective maps between rectangular matrix spaces preserving disjointness, triple products, or norms, J. Oper. Theory 83 (2020) 27–53.
[11] C.-K. Li, M.-C. Tsai, Y.-S. Wang, N.-C. Wong, Additive Hermitian idempotent preservers between operator algebras, J. Math. Anal. Appl. 505 (2022) 125522.
[12] C.-K. Li, M.-C. Tsai, Y.-S. Wang, N.-C. Wong, Linear maps preserving matrices annihilated by a fixed polynomial, Linear Algebra Appl. 674 (2023) 46–67.
[13] C.-K. Li, M.-C. Tsai, Y.-S. Wang, N.-C. Wong, Linear maps preserving real symmetric or complex Hermitian matrices annihilated by a fixed polynomial, preprint.
[14] C.-K. Li, N.-K. Tsing, Linear preserver problems: a brief introduction and some special techniques, Linear Algebra Appl. 162–164 (1992) 217–235.
[15] J.-H. Liu, C.-Y. Chou, C.-J. Liao, N.-C. Wong, Linear disjointness preservers of operator algebras and related structures, Acta Sci. Math. 84 (2018) 277–307.
[16] L. Monlar, Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces, Springer-Verlag, Berlin, 2007.
[17] P. Šemrl, Linear mappings that preserve operators annihilated by a polynomial, J. Oper. Theory 36 (1996) 45–583.
[18] P. Šemrl, Maps on matrix spaces, Linear Algebra Appl. 413 (2006) 364–393.
[19] N.-C. Wong, Triple homomorphisms of operators algebras, Southeast Asian Bull. Math. 29 (2005) 401–407.
[20] N.-C. Wong, Zero product preservers of C^*-algebras, Contemp. Math. 435 (2007) 377–380.
[21] X. Zhang, X.-M. Tang, C.-G. Cao, Preserver Problems on Spaces of Matrices, Science Press, Beijing, 2006.