# JOINT NUMERICAL RANGES AND COMMUTATIVITY OF MATRICES

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ABSTRACT. The connection between the commutativity of a family of $n \times n$ matrices and the generalized joint numerical ranges is studied. For instance, it is shown that $\mathcal{F}$ is a family of mutually commuting normal matrices if and only if the joint numerical range $W_k(A_1, \ldots, A_m)$ is a polyhedral set for some $k$ satisfying $|n/2 - k| \le 1$, where $\{A_1, \ldots, A_m\}$ is a basis for the linear span of the family; equivalently, $W_k(X, Y)$ is polyhedral for any two $X, Y \in \mathcal{F}$. More generally, characterization is given for the $c$-numerical range $W_c(A_1, \ldots, A_m)$ to be polyhedral for any $n \times n$ matrices $A_1, \ldots, A_m$. Other results connecting the geometrical properties of the joint numerical ranges and the algebraic properties of the matrices are obtained. Implications of the results to representation theory, and quantum information science are discussed.

## 1. INTRODUCTION

Denote by $M_n$ the set of $n \times n$ complex matrices. Let $c \in \mathbb{R}^n$ be a real vector with entries $c_1 \ge \cdots \ge c_n$. The joint $c$-numerical range of $\mathbf{A} = (A_1, \ldots, A_m) \in M_n^m$ is defined by

$$W_c(\mathbf{A}) = \left\{ \Big(\sum_{j=1}^{n} c_j x_j^* A_1 x_j, \ldots, \sum_{j=1}^{n} c_j x_j^* A_m x_j\Big) : \{x_1, \ldots, x_n\} \text{ is an orthonormal set} \right\}.$$

If $c_1 = c_n$, then $W_c(\mathbf{A}) = \{c_1(\mathrm{tr}A_1, \ldots, \mathrm{tr}A_m)\}$. **We will always assume that** $c_1 > c_n$ to avoid this trivial case. When $c_1 = \cdots = c_k = 1$ and $c_{k+1} = \cdots = c_n = 0$, $W_c(\mathbf{A})$ reduces to the joint $k$-numerical range of $\mathbf{A}$, denoted by $W_k(\mathbf{A})$. In particular, if $k = 1$, we get the classical joint numerical range $W(\mathbf{A})$. The joint $c$-numerical range is useful in studying the behavior of the family of matrices $\{A_1, \ldots, A_m\}$. One may see [2, 3, 9, 13] for some background. Even for a single matrix $A \in M_n$, there is interesting interplay between the geometrical properties of $W_c(A)$ and the algebraic and analytic properties of $A \in M_n$; see [11, 14, 16, 21]. If a matrix $A$ is given, it is not hard to determine the properties of $W_c(A)$. In applications and theoretical study, it is useful to deduce the properties of the (hidden) matrix based on the geometrical properties of its $c$-numerical range. Here, we list a few results, which are pertinent to our study.

(1.1) ([21]) $W_c(A)$ is always convex.

(1.2) ([11, Corollary 4.4]) $W_c(A)$ is a singleton if and only if $A = \mu I$ is a scalar matrix.

(1.3) ([11, Proposition 4.3]) $W_c(A)$ is a line segment if and only if $A = \alpha I + \beta H$ for a Hermitian matrix $H \in M_n$ and $\alpha, \beta \in \mathbb{C}$.

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(1.4) ([16, Theorem 1]) If $A$ is normal, then $W_c(A)$ is polyhedral, i.e., the convex hull of a finite set in $\mathbb{C}$.

(1.5) ([11, Theorem 4.9], [14, Theorem 2.2], [16, Theorem 3]) The following conditions are equivalent.

(a) $A$ is normal.

(b) There is a positive integer $k$ with $|n/2-k| \le 1$ such that $W_k(A)$ is polyhedral.

(c) There is $c = (c_1, \ldots, c_n)^t \in \mathbb{R}^n$ with $c_1 \ge \cdots \ge c_n$ satisfying $c_k > c_{k+1}$ for some $k$ with $|n/2-k| \le 1$ such that $W_c(A)$ is polyhedral.

(d) For any $c \in \mathbb{R}^n$, $W_c(A)$ is polyhedral.

Moreover, we have the following characterization of $A \in M_n$ such that $W_k(A)$ or $W_c(A)$ is polyhedral for general $k$ and $c$. For $c \in \mathbb{R}^n$ with entries arranged in descending order $c_1 \ge \cdots \ge c_n$, let

$$\gamma(c) = \max(\{j \le n/2 : c_j > c_{j+1}\} \cup \{n-j \le n/2 : c_j > c_{j+1}\}). \tag{1}$$

(1.6) ([11, Theorem 4.9], [14, Theorem 2.2 and 2.3]) Let $k \in \{1, \ldots, \lfloor n/2 \rfloor\}$. The following conditions are equivalent.

(a) $W_k(A)$ is polyhedral.

(b) $A$ is unitarily similar to $D \oplus Q$ such that $D \in M_\ell$ is a diagonal matrix with $\ell \ge k$, and $W_k(A) = W_k(D)$.

(c) There is $c \in \mathbb{R}^n$ with $\gamma(c) = k$ such that $W_c(A)$ is polyhedral.

(d) For any $c \in \mathbb{R}^n$ with $\gamma(c) \le k$, $W_c(A)$ is polyhedral.

It is known that (1.1) may fail, i.e., $W_c(A_1, \ldots, A_m)$ may not be convex, if $m > 1$; see [2, 13]. In this paper, we will extend Properties (1.2) - (1.6) to the joint $c$-numerical range. Some other results concerning the geometrical properties of $W_c(A_1, \ldots, A_m)$ and the algebraic properties of $A_1, \ldots, A_m$ will also be obtained. Again, if the matrices $A_1, \ldots, A_m \in M_n$ are given, then one can deduce the properties of $W_c(A_1, \ldots, A_m)$. Our study illustrates that useful information about the family of matrices $\{A_1, \ldots, A_m\}$ may be obtained from the geometrical properties of $W_c(A_1, \ldots, A_m)$. In particular, we show that the joint $c$-numerical range is useful for studying the commutativity of a (finite or infinite) family of matrices. For instance, we show in Section 3 that a family $\mathcal{F} \subseteq M_n$ consists of mutually commuting normal matrices if and only if the joint $k$-numerical range $W_k(A_1, \ldots, A_m)$ is polyhedral for some $k$ satisfying $|n/2-k| \le 1$, where $\{A_1, \ldots, A_m\}$ is a basis for span $(\mathcal{F})$, the linear span of $\mathcal{F}$; equivalently, $W_k(X, Y)$ is polyhedral for any two $X, Y \in \mathcal{F}$. The same conclusion holds if we replace $W_k(\cdot)$ by $W_c(\cdot)$ for any $c$ with $|n/2-\gamma(c)| \le 1$, where $\gamma(c)$ is defined as in (1). Furthermore, we characterize $\mathbf{A} = (A_1, \ldots, A_m)$ such that $W_c(\mathbf{A})$ is a singleton, or a line segment in $\mathbb{C}^m$, i.e., the convex hull of two points.

Our paper is organized as follows. In Section 2, we present some preliminary results. In Section 3, we characterize a (finite or infinite) subset of (mutually) commuting normal matrices in terms of the geometrical properties of the $c$-numerical ranges. Some implications of the result to representation theory and quantum information science are discussed. Other results connecting the geometric properties of $W_c(A_1, \ldots, A_m)$ and algebraic properties of $A_1, \ldots, A_m$ are obtained in Section 4. In Section 5, we characterize $\mathbf{A} = (A_1, \ldots, A_m) \in M_n^m$ such that $W_c(\mathbf{A})$ is polyhedral for a general real vector $c = (c_1, \ldots, c_n)$.

## 2. Preliminaries

Suppose $\mathbf{A} = (A_1, \ldots, A_m) \in M_n^m$, and $c = (c_1, \ldots, c_n)^t \in \mathbb{R}^n$. Let $C$ be the diagonal matrix $\operatorname{diag}(c_1, \ldots, c_n)$. Then it is easy to check that

$$W_c(\mathbf{A}) = W_C(\mathbf{A}) = \{(\mathrm{tr} CU^*A_1U, \ldots, \mathrm{tr} CU^*A_mU) : U \in M_n \text{ is unitary}\}.$$

The set $W_C(\mathbf{A})$ is referred to as the joint $C$-numerical range of $\mathbf{A}$. We will use the formulation $W_C(\mathbf{A})$ in our discussion. The following result is easy to verify, and can be viewed as an extension of the results corresponding to $W_k(A)$ and $W_C(A)$ in [11, 12, 16]. In particular, following the proof of [16, Theorem 1], we can extend Property (1.4) to condition (c) below.

**Proposition 2.1.** *Let $C = \operatorname{diag}(c_1, \ldots, c_n)$ be a real diagonal matrix, and $\mathbf{A} = (A_1, \ldots, A_m) \in M_n^m$.*

(a) *For any unitary $U, V \in M_n$, if $D = U^*CU$ and $B_j = V^*A_jV$ for $j = 1, \ldots, m$, then*

$$W_C(A_1, \ldots, A_m) = W_D(B_1, \ldots, B_m).$$

(b) *For any real vector $(a_1, \ldots, a_m)$,*

$$W_C(A_1 - a_1I, \ldots, A_m - a_mI) = W_C(A_1, \ldots, A_m) - (\mathrm{tr} C)(a_1, \ldots, a_m),$$

*and*

$$W_{(aC+bI)}(A_1, \ldots, A_m) = aW_C(A_1, \ldots, A_m) + b(\mathrm{tr} A_1, \ldots, \mathrm{tr} A_m),$$

(c) *If $A_1, \ldots, A_m$ are diagonal matrices, then*

$$W_C(\mathbf{A}) = \mathbf{conv}\,\{(\mathrm{tr}(CP^tA_1P), \ldots, \mathrm{tr}(CP^tA_mP)) : P \text{ is a permutation matrix}\}$$

*is polyhedral.*

(d) *Suppose $A_j = H_{2j-1} + iH_{2j}$ for two Hermitian matrices $H_{2j-1}, H_{2j}$ for $j = 1, \ldots, m$. Then $W_C(\mathbf{A})$ can be identified with $W_C(H_1, \ldots, H_{2m}) \subseteq \mathbb{R}^{2m}$.*

(e) *Suppose $\{A_1, \ldots, A_k\}$ is a basis for the linear span of $\{A_1, \ldots, A_m\}$, and $A_i = \sum_{j=1}^k r_{ij}A_j$ for some $r_{ij} \in \mathbb{C}$ with $1 \le j \le k < i \le m$. Then $(\mu_1, \ldots, \mu_m) \in W_C(A_1, \ldots, A_m)$ if and only if $(\mu_1, \ldots, \mu_k) \in W_C(A_1, \ldots, A_k)$ and $\mu_i = \sum_{j=1}^k r_{ij}\mu_j$ for $i = k+1, \ldots, m$.*

(f) *Suppose $\{A_1, \ldots, A_m\}$ is linearly independent and $R = (r_{ij}) \in M_m$ is invertible such that $B_i = \sum_{j=1}^m r_{ij}A_j$ for $i = 1, \ldots, m$. Then $(\mu_1, \ldots, \mu_m) \in W_C(B_1, \ldots, B_m)$ if and only if $(\mu_1, \ldots, \mu_m)^t = R(\nu_1, \ldots, \nu_m)^t$ with $(\nu_1, \ldots, \nu_m) \in W_C(A_1, \ldots, A_m)$.*

When $C = I_k \oplus 0_{n-k}$, we see that $I_k \oplus 0_{n-k} = I_n - (0_k \oplus I_{n-k})$. By condition (b), we have

$$W_k(\mathbf{A}) = (\mathrm{tr} A_1, \ldots, \mathrm{tr} A_m) - W_{n-k}(\mathbf{A}).$$

So, we can always focus on $W_k(A_1, \ldots, A_m)$ for $k \le n/2$.

Also, by conditions (e) – (f) above, one can focus on the study of $W_C(A_1,\dots,A_m)$ such that $\{A_1,\dots,A_m\}$ is a linear independent set of trace zero matrices by the following reduction. Note that $(\mu_1,\dots,\mu_m) \in W_C(A_1,\dots,A_m)$ if and only if $(\mathrm{tr}C,\mu_1,\dots,\mu_m) \in W_C(I_n,A_1,\dots,A_m)$. Then we may permute the components of $(A_1,\dots,A_m)$ and assume that $\{I_n,A_1,\dots,A_k\}$ is a basis for span $\{I_n,A_1,\dots,A_m\}$. Then we can find a basis $\{I_n,B_1,\dots,B_k\}$ such that $B_1,\dots,B_k \in M_n$ are trace zero matrices. Then $(\mu_1,\dots,\mu_k) \in W_C(B_1,\dots,B_k)$ if and only if $(\mathrm{tr}C,\mu_1,\dots,\mu_k) \in W_C(I_n,B_1,\dots,B_k)$. Equivalently, $(\mathrm{tr}C,\mu_1,\dots,\mu_k)^t = R(\mathrm{tr}C,\nu_1,\dots,\nu_k)^t$ with $(\mathrm{tr}C,\nu_1,\dots,\nu_k) \in W_C(I_n,A_1,\dots,A_k)$, where $R=(r_{ij}) \in M_{k+1}$ satisfies $B_i = \sum_{j=0}^k r_{ij}A_j$ for $i=0,\dots,k$, with $A_0=B_0=I_n$. Consequently, there is an injective affine map converting $W_C(B_1,\dots,B_k)$ to $W_C(A_1,\dots,A_m)$. So, the convexity and polyhedral properties of the two sets will be preserved. Of course, if we apply (d) and assume that $A_1,\dots,A_m$ are Hermitian, then $B_1,\dots,B_k$ can be chosen to be linearly independent Hermitian matrices with trace zero. Nevertheless, we will state most of our results in terms of general complex matrices so that one does not need to impose the additional assumption when the result is applied.

It is easy to check that if $\{A_1,\dots,A_m\}$ is a family of mutually commuting normal matrices, then for all real diagonal matrix $C$, $W_C(A_1,\dots,A_m)$ is polyhedral and therefore is convex. In the next section, we will show that the converse is also valid. In fact, one only needs to check that $W_C(A_1,\dots,A_m)$ is polyhedral for some special $C$, it will follow that $\{A_1,\dots,A_m\}$ is a commuting family of normal matrices.

## 3. Commuting normal matrices

If $\mathcal{F}$ is a family of (mutually) commuting normal matrices, then $W(A_1,\dots,A_m)$ is polyhedral for any subset $\{A_1,\dots,A_m\}$ of $\mathcal{F}$. But the converse may not hold as shown in the following example; for example see [16].

**Example 3.1.** *Let* $w = e^{i2\pi/3}$ *and* $A = A_1 + iA_2 = \mathrm{diag}\,(1,w,w^2) \oplus \begin{pmatrix} 0 & 0.1 \\ 0 & 0 \end{pmatrix}$. *Then* $W(A_1,A_2) \equiv W(A) = \mathbf{conv}\,\{1,w,w^2\}$ *is a triangle, but* $A_1,A_2$ *do not commute, equivalently,* $A$ *is not normal.*

Even if we assume that the family of matrices have nice property, say, it consists of unitary matrices, we still cannot get nice conclusion.

**Example 3.2.** *Let* $A = A_1 + iA_2 = \mathrm{diag}\,(1+i,1-i,-1+i,-1-i) \oplus \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$. *Then* $A_1,A_2$ *are unitary, and* $W(A) = \mathbf{conv}\,\{1+i,1-i,-1+i,-1-i\}$. *But* $A_1,A_2$ *do not commute.*

It turns out that one can detect the commutativity of a family of matrices using the $C$-numerical range or $k$-numerical range for some special $C$ and $k$. The following is an extension of property (1.5).

**Theorem 3.3.** *Let* $A_1,\dots,A_m \in M_n$. *The following conditions are equivalent.*

(a) *$\{A_1, \ldots, A_m\}$ consists of mutually commuting normal matrices.*
(b) *There is a positive integer $k$ with $|n/2 - k| \leq 1$ such that $W_k(A_1, \ldots, A_m)$ is polyhedral.*
(c) *There is a Hermitian $C \in M_n$ with eigenvalues $c_1 \geq \cdots \geq c_n$ satisfying $c_k > c_{k+1}$ for some $k$ with $|n/2 - k| \leq 1$ such that $W_C(A_1, \ldots, A_m)$ is polyhedral.*
(d) *For any Hermitian $C$, $W_C(A_1, \ldots, A_m)$ is polyhedral.*

*Proof.* Suppose (a) holds. Then there is a unitary $U \in M_n$ such that $U^*A_jU$ is a diagonal matrix for $j = 1, \ldots, m$. By Proposition 2.1 (a) and (c), we see that $W_C(A_1, \ldots, A_m)$ is polyhedral for any Hermitian $C \in M_n$. Thus (d) holds.

If (d) holds, then clearly (c) and (b) hold.

Suppose (c) holds. We can let $A_j = H_{2j-1} + iH_{2j}$ such that $H_{2j-1}, H_{2j}$ are Hermitian for $j = 1, \ldots, m$. Then $W_C(A_1, \ldots, A_m) \subseteq \mathbb{C}^m$ can be identified with $W_C(H_1, \ldots, H_{2m}) \subseteq \mathbb{R}^{2m}$, which is polyhedral. Thus, $W_C(H_r, H_s)$ is polyhedral for any $r, s$. Thus, by Property (1.5), $H_r + iH_s$ is normal, i.e., $H_rH_s = H_sH_r$ for any $1 \leq r, s \leq 2m$. Hence, $\{H_1, \ldots, H_{2m}\}$ is a commuting family of Hermitian matrices so that $\{A_1, \ldots, A_m\}$ is a commuting family of normal matrices. Hence (a) holds. If (b) holds, then (c) holds. Thus, (a) holds. □

Note that Theorem 3.3 can also be deduced from Theorem 5.1, whose proof is more involved. In the proof of Theorem 3.3, we use the fact that one only needs to check any two matrices in $\{H_1, \ldots, H_{2m}\}$ commute to conclude that $\{A_1, \ldots, A_m\}$ is a commuting family of normal matrices. In fact, it is difficult to visualize $W_C(A_1, \ldots, A_m) \subseteq \mathbb{C}^m$ or $W_C(H_1, \ldots, H_{2m}) \subseteq \mathbb{R}^{2m}$ if $m > 1$. It is more practical to check $W_C(H_r, H_s) \subseteq \mathbb{R}^2$ for all $1 \leq r < s \leq 2m$. Of course, one may let $\{G_1, \ldots, G_r\}$ be a maximal linearly independent subset of $\{H_1, \ldots, H_{2m}\}$ and examine $W_C(G_u, G_v) \subseteq \mathbb{R}^2$ for $1 \leq u < v \leq r$ to deduce the desired conclusion.

Even for an infinite family $\mathcal{F} \subseteq M_n$, if we take the Hermitian and skew-Hermitian parts of the matrices in $\mathcal{F}$ and show that any two of them commute, then $\mathcal{F}$ will be a family of commuting normal matrices. Also, if we take a basis $\mathcal{B} = \{B_1, \ldots, B_m\}$ for the linear span of $\mathcal{S}$ and show that $\mathcal{B}$ is a family of commuting normal matrices, then so is the family $\mathcal{S}$. By these observations, we can extend Theorem 3.3 to the following.

**Theorem 3.4.** *Suppose $\mathcal{F} \subset M_n$ is a non-empty set of matrices, and $\mathcal{F}^* = \{A^* : A \in \mathcal{F}\}$. Let $\mathcal{B} = \{B_1, \ldots, B_r\}$ be a basis for* span$(\mathcal{F})$, span$(\mathcal{F}^*)$, *or* span$(\mathcal{F} \cup \mathcal{F}^*)$. *In the last case, we may assume that $B_1, \ldots, B_r$ are Hermitian matrices. The following conditions are equivalent.*

(a) *One of / all the sets $\mathcal{F}, \mathcal{F} \cup \mathcal{F}^*$ or $\mathcal{B}$ consists of mutually commuting normal matrices.*
(b) *For any Hermitian $C$ and $\{A_1, \ldots, A_m\} \subseteq$ span$(\mathcal{F} \cup \mathcal{F}^*)$, $W_C(A_1, \ldots, A_m)$ is polyhedral.*
(c) *There is a Hermitian $C \in M_n$ with eigenvalues $c_1 \geq \cdots \geq c_n$ and $c_k > c_{k+1}$ for some $k$ satisfying $n/2 - 1 \leq k \leq n/2 + 1$ such that $W_C(X, Y)$ is polyhedral for any $X, Y \in \mathcal{S}$, where $\mathcal{S}$ can be any one of the sets $\mathcal{F}, \mathcal{F}^*, \mathcal{F} \cup \mathcal{F}^*, \mathcal{B}$.*
(d) *There is a positive integer $k$ with $n/2 - 1 \leq k \leq n/2 + 1$ such that $W_k(X, Y)$ is polyhedral for any $X, Y \in \mathcal{S}$, where $\mathcal{S}$ can be any one of the sets $\mathcal{F}, \mathcal{F}^*, \mathcal{F} \cup \mathcal{F}^*, \mathcal{B}$.*

(e) *There is a Hermitian $C \in M_n$ with eigenvalues $c_1 \geq \cdots \geq c_n$ and $c_k > c_{k+1}$ for some $k$ satisfying $n/2 - 1 \leq k \leq n/2 + 1$ such that $W_C(B_1, \ldots, B_r)$ is polyhedral.*

We include many equivalent conditions in the statement of Theorem 3.4 so that it can be applied to different situations. For instance, Theorem 3.4 can be used to check whether $\mathcal{F} = \Phi(\mathcal{G})$ consists of commutative matrices if $\Phi$ is a finite dimensional unitary representation of a group $\mathcal{G}$. Therefore, it can be used to check whether a finite group $\mathcal{G}$ is Abelian if $\Phi$ is the left regular representation of $\mathcal{G}$.

More generally, for every bounded group $\mathcal{G}$ of matrices in $M_n$, there is an invertible matrix $S \in M_n$ such that $S^{-1}\mathcal{G}S = \{S^{-1}AS : A \in \mathcal{G}\}$ is a group of unitary matrices; see [1] and also [5]. Then the above results can be used to check whether the group $S^{-1}\mathcal{G}S$ consists of commutative unitary matrices. Of course, $\mathcal{G}$ is Abelian if and only if $S^{-1}\mathcal{G}S$ is Abelian.

In quantum information science, if $A_1, \ldots, A_m \in M_n$ are Hermitian matrices corresponding to $m$ observables on a quantum system with quantum states represented as density matrices in $M_n$, i.e., positive semidefinite matrices of trace one, then

$$\mathbf{conv}\, W(A_1, \ldots, A_m) = \{(\mathrm{tr} A_1 P, \ldots, \mathrm{tr} A_m P) : P \text{ is a density matrix}\}$$

is the set of joint measurements of different quantum states $P$. As mentioned before, even if $\mathbf{conv}\, W(A_1, \ldots, A_m)$ is polyhdral, we may not be able to conclude that $\{A_1, \ldots, A_m\}$ is a commuting family. By Theorem 3.3, suppose we consider the subset $\mathcal{S}_k$ of states consisting of $\frac{1}{k}A$, where $A$ is a convex combination of rank $k$-orthogonal projections. Then [6, Lemma 1.4]

$$\mathcal{S}_k = \{A \in M_n : \mathrm{tr} A = 1,\ 0 \leq A \leq I/k\},$$

where $X \geq Y$ means $X - Y$ is positive semidefinite for $X, Y \in M_n$, and

$$\mathbf{conv}\, W_k(A_1, \ldots, A_m) = \{k(\mathrm{tr} A_1 P, \ldots, \mathrm{tr} A_m P) : P \in \mathcal{S}_k\}.$$

Hence, $\{A_1, \ldots, A_m\}$ is a commutative family of Hermitian matrices if and only if the joint measurements of the states in $\mathcal{S}_k$ form a polyhedral set for some $k$ satisfying $|n/2 - k| \leq 1$.

Recall that an operator system $\mathcal{S}$ in $M_n$ is a subspace containing $I_n$ and satisfies $A^* \in \mathcal{S}$ whenever $A \in \mathcal{S}$. Operator systems are useful structure in the study of operator algebras and functional analysis; see [18]. Recently, it is shown that operator systems are useful in studying the properties of quantum channels; see [10]. Every operator system $\mathcal{S} \subseteq M_n$ has a basis $\{I, B_1, \ldots, B_m\}$ consisting of Hermitian matrices. So, one can use Theorem 3.3 to check whether an operator system is commutative. This turns out to be equivalent to the condition that the associated quantum channel is a Schur channel; see [7].

A referee pointed out another connection of our result to quantum information science research, namely, for a given density matrix $\rho$ and Hermitian matrices $A_1, \ldots, A_m$, one may define the Wigner distribution function $\mathcal{W}_\rho : \mathbb{R}^m \to \mathbb{R}$, and it was known that for a full rank density matrix $\rho$, $\mathcal{W}_\rho$ is positive if and only if $\{A_1, \ldots, A_m\}$ is a commuting family; see [20, Property 2]. Evidently, our result is related to this study.

We can use the $C$-numerical range to see that a family of matrices are commuting normal matrices with special structure. The following result extends Properties (1.2) and (1.3).

**Theorem 3.5.** *Let $C \in M_n$ be a non-scalar Hermitian matrix. Let $\mathbf{A} = (A_1, \ldots, A_m) \in M_n^m$.*

(a) *$W_C(\mathbf{A})$ is a singleton if and only if $A_j = a_j I$ is a scalar matrix for each $j$.*

(b) *$W_C(\mathbf{A})$ is a line segment in $\mathbb{C}^m$ if and only if there is a Hermitian matrix $H$ such that $A_j \in \operatorname{span}\{I, H\}$ for each $j$.*

*Proof.* (a) If $W_C(\mathbf{A})$ is a singleton, then so is $W_C(A_j)$ for each $j$. By (1.2), $A_j$ is a scalar matrix. The converse is clear.

(b) Let $A_j = H_{2j-1} + iH_{2j}$ for $j = 1, \ldots, m$. Then $W_C(H_u + iH_v)$ is a line segment for any $1 \le u < v \le 2m$. If all the line segments are degenerate (with length zero), then $H_u + iH_v$ is a scalar matrix by (1.2) for all $u, v$. Else, we may assume that $W_C(H_1 + iH_2)$ is a non-degenerate line segment and $H_1 = (\mathrm{tr} H_1)I/n + H$ for a nonzero Hermitian matrix $H$ with trace 0 by (1.3). Now, $W_C(H_1 + iH_v)$ is a line segment for each $v > 1$. By (1.3) again, we see that for each $v > 1$, $H_v = (\mathrm{tr} H_v)I/n + b_v H$ for some $b_v \in \mathbb{R}$.

The converse is clear. $\square$

## 4. Other properties

We establish some other properties connecting the geometric properties of $W_C(A_1, \ldots, A_m)$ and the algebraic properties of $A_1, \ldots, A_m$. These results have their own interest, and will be useful in studying the polyhedral property of $W_C(A_1, \ldots, A_m)$ in the next section. By the comments in Section 2, we will focus on Hermitian matrices $C, A_1, \ldots, A_m \in M_n$.

First we give a description of the convex hull of $W_C(A_1, \ldots, A_m)$. The result is an extension of [14, Theorem 2.1]. Denote by $\lambda_1(A) \ge \cdots \ge \lambda_n(A)$ the eigenvalues of a Hermitian matrix $A \in M_n$.

**Theorem 4.1.** *Let $C, A_1, \ldots, A_m \in M_n$ be Hermitian such that $C = \operatorname{diag}(c_1, \ldots, c_n)$ with $c_1 \ge \cdots \ge c_n$. Then for $\mathbf{A} = (A_1, \ldots, A_m)$,*

$$\mathbf{conv}\, W_C(\mathbf{A}) = \cap\{\mathcal{P}_v(\mathbf{A}) : v \in \mathbb{R}^m, v^t v = 1\},$$

*where for $v = (v_1, \ldots, v_m)^t \in \mathbb{R}^m$,*

$$\mathcal{P}_v(\mathbf{A}) = \left\{ (a_1, \ldots, a_m) : \sum_{j=1}^m v_j a_j \le \sum_{j=1}^n c_j \lambda_j (v_1 A_1 + \cdots + v_m A_m) \right\}.$$

*Proof.* To prove "$\subseteq$", let $v = (v_1, \ldots, v_m)^t$ be a unit vector in $\mathbb{R}^m$, and let $U \in M_n$ be unitary such that

$$(a_1, \ldots, a_m) = (\mathrm{tr} CU^* A_1 U, \ldots, \mathrm{tr} CU^* A_m U) \in W_C(\mathbf{A}).$$

Then by [14, Theorem 2.1] and also [11],

$$\sum_{j=1}^{m} v_j a_j = \sum_{j=1}^{m} v_j(\mathrm{tr} CU^*A_jU) = \mathrm{tr}[CU^*(\sum_{j=1}^{m} v_jA_j)U] \le \sum_{j=1}^{n} c_j\lambda_j(v_1A_1 + \cdots + v_mA_m).$$

Hence, $W_C(A_1, \ldots, A_m)$ is a subset of the convex set $\cap\{\mathcal{P}_v : v \in \mathbb{R}^m, v^tv = 1\}$, and so is **conv** $W_C(A_1, \ldots, A_m)$.

For the reverse inclusion, suppose $(b_1, \ldots, b_n) \notin$ **conv** $W_C(A_1, \ldots, A_m)$. Then there is a linear functional $f : \mathbb{R}^m \to \mathbb{R}$ of the form

$$(x_1, \ldots, x_m) \to v_1x_1 + \cdots + v_mx_m$$

for a unit vector $(v_1, \ldots, v_m)^t \in \mathbb{R}^m$ such that $f(b_1, \ldots, b_m) > f(a_1, \ldots, a_m)$ for all $(a_1, \ldots, a_m) \in$ **conv** $W_C(A_1, \ldots, A_m)$, and hence $f(b_1, \ldots, b_m) > f(\mathrm{tr}CU^*A_1U, \ldots, \mathrm{tr}CU^*A_mU)$ for any unitary $U \in M_n$. Hence, if $V \in M_n$ is unitary such that $V^*(v_1A_1+\cdots+v_mA_m)V = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \ge \cdots \ge \lambda_n$, then

$$\sum_{j=1}^{m} v_jb_j > \sum_{j=1}^{m} v_j(\mathrm{tr}CV^*A_jV) = \mathrm{tr}[CV^*(\sum_{j=1}^{m} v_jA_j)V] = \sum_{j=1}^{n} c_j\lambda_j.$$

Thus, $(b_1, \ldots, b_m) \notin \mathcal{P}_v(\mathbf{A})$ with $v = (v_1, \ldots, v_m)^t$. □

Let $\mathcal{S} \subset \mathbb{R}^m$. A point $p \in \mathcal{S}$ is a conical point if there is an invertible affine transform $f : \mathbb{R}^m \to \mathbb{R}^m$ such that $f(p) = 0$ and $f(\mathcal{S}) \subseteq \{(x_1, \ldots, x_m) : x_j \le 0 \text{ for all } j = 1, \ldots, m\}$.

In the following, we also use $\mathbb{R}^m$ to denote the set of row vectors. It is known that if $p = (p_1, \ldots, p_m)$ is a conical point of $W(A_1, \ldots, A_m)$, where $A_1, \ldots, A_m \in M_n$ are Hermitian, then there is a unit vector $v \in \mathbb{C}^n$ such that $A_jv = p_jv$; see [3]. In other words, the matrices $A_1, \ldots, A_m$ have a common eigenvector $v$. We will extend this result to the $C$-numerical range.

**Theorem 4.2.** *Let $A_1, \ldots, A_m \in M_n$ be Hermitian matrices. Suppose $C = \mathrm{diag}\,(c_1, \ldots, c_n) = \xi_1I_{n_1} \oplus \cdots \oplus \xi_rI_{n_r}$ such that $\xi_1 > \cdots > \xi_r$ and $n_1 + \cdots + n_r = n$. If $U \in M_n$ is unitary such that $(\mathrm{tr}CU^*A_1U, \ldots, \mathrm{tr}CU^*A_mU)$ is a conical point of $W_C(A_1, \ldots, A_m)$, then each $U^*A_jU = A_{j1} \oplus \cdots \oplus A_{j_r} \in M_{n_1} \oplus \cdots \oplus M_{n_r}$ has the same direct sum structure as $C$.*

*Proof.* Let $\mathbf{A} = (A_1, \ldots, A_m)$. We may assume that all $A_i$ are positive definite and $U = I_n$. By an affine transform, we may assume that $W_C(\mathbf{A})$ lies in the set $\{(a_1, \ldots, a_m) : a_1, \ldots, a_m \in (-\infty, 0]\}$ and $\mathrm{tr}CA_j = 0$ for all $1 \le j \le m$. Then for each $A_j = (a_{uv}^{(j)})$, we see that $W_C(A_j) \subseteq (-\infty, 0]$ and

$$0 = \mathrm{tr}CA_j = \sum_{u=1}^{n} c_ua_{uu}^{(j)} = \sum_{u=1}^{n} c_u\lambda_u(A_j).$$

Note that sum of the first $v$ diagonal entries of $A_j$ is always smaller than or equal to the sum of the $v$ largest eigenvalues of $A_j$. So,

$$\begin{aligned}\sum_{u=1}^{n} c_u a_{uu}^{(j)} &= (\xi_1-\xi_2)\sum_{u=1}^{n_1} a_{uu}^{(j)} + (\xi_2-\xi_3)\sum_{u=1}^{n_1+n_2} a_{uu}^{(j)} + \ldots.. + \xi_r(\mathrm{tr}A_j)\\ &\le (\xi_1-\xi_2)\sum_{u=1}^{n_1}\lambda_u(A_j) + \ldots. + \xi_r(\mathrm{tr}A_j) = \sum_{u=1}^{n} c_u\lambda_u(A_j).\end{aligned}$$

As a result, the equality holds implies that $\sum_{u=1}^{\ell} a_{uu}^{(j)} = \sum_{u=1}^{\ell}\lambda_u(A_j)$ for $\ell = n_1, n_1+n_2, \ldots, n_1+\cdots+n_{r-1}$. It follows [15, Lemma 5.10] that $A_j = A_{j1}\oplus\cdots\oplus A_{jr}$. □

By Theorem 4.2, we have the following result on general matrices $A_1,\ldots,A_m \in M_n$.

**Corollary 4.3.** *Suppose $C \in M_n$ is Hermitian with $n$ distinct eigenvalues. Let $A_1,\ldots,A_m \in M_n$. If $W_C(A_1,\ldots,A_m)$ has a conical point, then $\{A_1,\ldots,A_m\}$ is a commuting family of normal matrices.*

The next result shows that if $A_1,\ldots,A_m \in M_{n_1}\oplus\cdots\oplus M_{n_r}$ has common direct sum structure, then we can find a containment regions for $W_k(A_1,\ldots,A_m)$ using the joint $\ell$-numerical ranges of the smaller matrices in the component of the direct sum. The result will be useful in the study of polyhedral property of $W_C(A_1,\ldots,A_m)$.

**Theorem 4.4.** *Suppose $A_1,\ldots,A_m \in M_n$ are Hermitian such that $A_j = A_{j1}\oplus\cdots\oplus A_{jr} \in M_{n_1}\oplus\cdots\oplus M_{n_r}$. Then*

$$W_k(A_1,\ldots,A_m) \subseteq \mathbf{conv}\,\mathcal{W} = \mathbf{conv}\,W_k(A_1,\ldots,A_m),$$

*where*

$$\mathcal{W} = \cup\{W_{k_1}(A_{11},\ldots,A_{m1}) + \cdots + W_{k_r}(A_{1r},\ldots,A_{mr}) : k_1,\ldots,k_r \ge 0,\ \sum_{j=1}^{r} k_j = k\},$$

*with the convention that $W_0(B_1,\ldots,B_m) = \{(0,\ldots,0)\}$ for any $B_1,\ldots,B_m \in M_q$.*

*Proof.* First, we prove $W_k(A_1,\ldots,A_m) \subseteq \mathbf{conv}\,\mathcal{W}$. Suppose $r = 2$. Let $(\mathrm{tr}A_1P,\ldots,\mathrm{tr}A_mP) \in W_k(A_1,\ldots,A_m)$, where $P$ is a rank $k$ orthogonal projection. Suppose $P = \begin{pmatrix} P_{11} & P_{12}\\ P_{12}^* & P_{22}\end{pmatrix}$ with $P_{11} \in M_{n_1}$.

We **claim** that $P_{11}\oplus P_{22}$ is a convex combination of rank $k$ orthogonal projections of the form $Q_1\oplus Q_2$ with $Q_1^2 = Q_1$ and $Q_2^2 = Q_2$, i.e., $Q_1, Q_2$ are orthogonal projections. Then $(\mathrm{tr}A_jP)_{j=1}^m = (\mathrm{tr}A_j(P_{11}\oplus P_{22}))_{j=1}^m$ will be a convex combination of the form $(\mathrm{tr}A_{j1}Q_1 + \mathrm{tr}A_{j2}Q_2)_{j=1}^m$. So, $W_k(A_1,\ldots,A_m)$ is a subset of the convex hull of

$$\cup\{W_{k_1}(A_{11},\ldots,A_{m1}) + W_{k_2}(A_{12},\ldots,A_{m2}) : k_1,k_2 \ge 0,\quad k_1+k_2 = k\}.$$

To prove our **claim**, since $P$ is a rank $k$ orthogonal projection, there exist $U \in M_{k,n}$ with orthonormal rows such that $P = U^*U$. Let $U = [U_1U_2]$ with $U_1 \in M_{k,n_1}$ and $U_2 \in M_{k,n_2}$.

Then $P_{11} = U_1^*U_1$ and $P_{22} = U_2^*U_2$. Also, $I_k = UU^* = U_1U_1^* + U_2U_2^*$. Suppose $P_{11}$ has eigenvalues $d_1 \geq \cdots \geq d_{n_1}$. Let $d_i = 1$ for $i \leq p$ and $d_i = 0$ for $q < i$, where $p = \max(0, k - n_2)$ and $q = \min(k, n_1)$. Note that $U_iU_i^*$ and $U_i^*U_i$ have the same nonzero eigenvalues, including multiplicities, and $U_2U_2^* = I_k - U_1U_1^*$. Therefore, $P_{22}$ has eigenvalues $1-d_k \geq \cdots \geq 1-d_{k+1-n_2}$. So we can choose unitary matrices $V_1 \in M_{n_1}$ and $V_2 \in M_{n_2}$ such that $R_{11} = V_1^*P_{11}V_1 = \operatorname{diag}(d_1, \ldots, d_{n_1})$ and $R_{22} = V_2^*P_{22}V_2 = \operatorname{diag}(1 - d_k, \ldots, 1 - d_{k+1-n_2})$. For $p \leq \ell \leq q$, define

$$T_\ell = (I_\ell \oplus 0_{n_1-\ell}) \oplus (I_{k-\ell} \oplus 0_{n_2-k+\ell}).$$

Since $d_\ell = 1$ for $\ell \leq p$ and $d_\ell = 0$ for $\ell > q$, we have

$$R_{11} \oplus R_{22} = \sum_{\ell=p}^{q}(d_\ell - d_{\ell+1})T_\ell \quad \text{and} \quad \sum_{\ell=p}^{q}(d_\ell - d_{\ell+1}) = 1.$$

Hence, if $\hat{T}_\ell = VT_\ell V^*$ for $p \leq \ell \leq q$, where $V = V_1 \oplus V_2$,

$$P_{11} \oplus P_{22} = \sum_{\ell=p}^{q}(d_\ell - d_{\ell+1})\hat{T}_\ell.$$

The general case follows from induction on $r$ .

It is clear that $W_{k_1}(A_{11}, \ldots, A_{m1}) + \cdots + W_{k_r}(A_{1r}, \ldots, A_{mr}) \subseteq W_k(A_1, \ldots, A_m)$ whenever $k_1, \ldots, k_r \geq 0$ satisfy $\sum_{j=1}^r k_j = k$. Thus, $\mathbf{conv}\, \mathcal{W} \subseteq \mathbf{conv}\, W_k(A_1, \ldots, A_m)$. By the result in the preceding paragraph, we have the reverse inclusion. $\square$

A referee pointed out that our **claim** is related to the study of the two projections; see [8] and also [4]. The results in that area might be useful in deducing our claim.

## 5. Polyhedral property

The following theorem characterizes $(A_1, \ldots, A_m) \in M_n^m$ such that $W_C(A_1, \ldots, A_m)$ is polyhedral. The result extends Property (1.6). We will focus on Hermitian matrices $A_1, \ldots, A_m \in M_n$ by the comment in Section 2.

Supppose $C \in M_n$ is Hermitian with eigenvalues $c_1 \geq \cdots \geq c_n$. Let

$$\gamma(C) = \max(\{j \leq n/2 : c_j > c_{j+1}\} \cup \{n - j \leq n/2 : c_j > c_{j+1}\}). \tag{2}$$

**Theorem 5.1.** *Let $A_1, \ldots, A_m \in M_n$ be Hermitian matrices, and let $k \in \{1, 2, \ldots, \lfloor n/2 \rfloor\}$. The following are equivalent.*

- (a) *There is a Hermitian matrix $C \in M_n$ with $\gamma(C) = k$ such that $\mathbf{conv}\, W_C(A_1, \ldots, A_m)$ or $W_C(A_1, \ldots, A_m)$ is polyhedral.*
- (b) *There exist $\ell \geq 2k$ and a unitary $U \in M_n$ such that for each $j = 1, \ldots, m$, $U^*A_jU = D_j \oplus Q_j$, where $D_j \in M_\ell$ is a diagonal matrix, and $W_k(A_1, \ldots, A_m) = W_k(D_1, \ldots, D_m)$.*
- (c) *There exist $\ell \geq 2k$ and a unitary matrix $U \in M_n$ such that for each $j = 1, \ldots, m$, $U^*A_jU = D_j \oplus Q_j$, where $D_j \in M_\ell$ is a diagonal matrix, and for any Hermitian $C \in M_n$ with eigenvalues $c_1 \geq \cdots \geq c_n$ and $\gamma(C) = k$, we have $W_{(C-c_{k+1}I)}(A_1, \ldots, A_m) =$*

$W_{\hat{C}}(D_1, \dots, D_m)$, *where* $\hat{C} = \mathrm{diag}\,(c_1 - c_{k+1}, \dots, c_k - c_{k+1}, c_{k+n-\ell+1} - c_{k+1}, \dots, c_n - c_{k+1}) \in M_\ell$.

(d) $W_C(A_1, \dots, A_m)$ *is polyhedral for any Hermitian* $C$ *with* $\gamma(C) \le k$.

*Proof.* (a) $\Rightarrow$ (b). Suppose $C \in M_n$ is Hermitian with $\gamma(C) = k$, and $\mathbf{conv}\, W_C(A_1, \dots, A_m)$ is polyhedral. Let $p = (p_1, \dots, p_m)$ be a conical point of $\mathbf{conv}\, W_C(A_1, \dots, A_m)$. We may assume that $C = \mathrm{diag}\,(c_1, \dots, c_n)$ with $c_1 \ge \cdots \ge c_n$ and $c_k > c_{k+1}$. We may further assume that $C = \xi_1 I_{n_1} \oplus \cdots \oplus \xi_r I_{n_r}$ with $\xi_1 > \cdots > \xi_r$ and $n_1 + \cdots + n_r = n$. Applying an affine transform, we may assume that $(p_1, \dots, p_m) = (0, \dots, 0)$ and

$$W_C(A_1, \dots, A_m) \subseteq \{(x_1, \dots, x_m) : x_1, \dots, x_m \in (-\infty, 0]\}.$$

By Theorem 4.2, there is a unitary $U \in M_n$ such that $U^*A_jU = A_{j_1} \oplus \cdots \oplus A_{j_r} \in M_{n_1} \oplus \cdots \oplus M_{n_r}$.

Let $q$ be such that $n_1 + \cdots + n_q = k$. From the proof of Theorem 4.2, if $B_j = A_{j1} \oplus \cdots \oplus A_{jq}$, then $b_j = \mathrm{tr} B_j = \sum_{u=1}^k \lambda_u(A_j)$. Hence,

$$(b_1, \dots, b_m) \in W_k(A_1, \dots, A_m) \subseteq \{(x_1, \dots, x_m) : x_1, \dots, x_m \in (-\infty, b_j]\}.$$

So, $(b_1, \dots, b_m)$ lies in the intersection of the $m$ support planes: $\mathcal{P}_j = \{(x_1, \dots, x_m) : x_j \le b_j\}$ for $j = 1, \dots, m$, and is a conical point of $W_k(A_1, \dots, A_m)$.

Now, for any $1 \le u, v \le m$, $W_C(A_u + iA_v) \subseteq \{x + iy : x, y \in (-\infty, 0]\}$ is polyhedral with a vertex 0. By the results in [14], we see that $W_k(A_u + iA_v)$ is polyhedral, and $(b_u + ib_v)$ is a vertex and hence $B_uB_v = B_vB_u$. Since this is true for all $u, v$, we see that $\{B_1, \dots, B_m\}$ is a commuting family and hence we may assume that $B_1, \dots, B_m$ are in diagonal form.

Now, let $\ell \in \{k, \dots, n\}$ be the maximum integer for the existence of a unitary $V \in M_n$ such that $V^*A_jV = D_j \oplus Q_j$, where $D_j \in M_\ell$ is a diagonal matrix and $Q_j \in M_{n-\ell}$ for $j = 1, \dots, m$. Without loss of generality, we may assume that $A_j = D_j \oplus Q_j$.

If every conical point of $W_k(A_1, \dots, A_k)$ lies in $W_k(D_1, \dots, D_m)$, then $W_k(D_1, \dots, D_m) = W_k(A_1, \dots, A_m)$. Suppose there is a conical point $(a_1, \dots, a_m)$ of $W_k(A_1, \dots, A_m)$ not lying in $W_k(D_1, \dots, D_m)$. We may apply an affine transform to the matrices $A_1, \dots, A_m$, and assume that $(a_1, \dots, a_m) = (0, \dots, 0)$ and $W_k(A_1, \dots, A_m) \subseteq \{(x_1, \dots, x_m) : x_1, \dots, x_m \in (-\infty, 0]\}$. So, $0 = \sum_{u=1}^k \lambda_u(A_j)$ for $j = 1, \dots, m$.

By Theorem 4.4, $(a_1, \dots, a_m) = (\mathrm{tr} A_1 P, \dots, \mathrm{tr} A_m P)$ for some rank $k$ orthogonal projection $P$ so that $(a_1, \dots, a_m)$ is a convex combination of elements of the form $(\mathrm{tr} A_1 R, \dots, \mathrm{tr} A_m R)$, where $R = R_1 \oplus R_2 \in M_\ell \oplus M_{n-\ell}$. Since $(a_1, \dots, a_m)$ is an extreme point, $P$ must be equal to one of the $R = R_1 \oplus R_2$. Clearly, $R_2 \ne 0$. Else, $(\mathrm{tr} A_1 R, \dots, \mathrm{tr} A_m R) \in W_k(D_1, \dots, D_m)$. Now, there is a unitary $V = V_1 \oplus V_2 \in M_\ell \oplus M_{n-\ell}$ such that $V^*(R_1 \oplus R_2)V = I_q \oplus 0_{n-k} \oplus I_{k-q}$. Then

$$\mathrm{tr}(V^*A_jVV^*(R_1 \oplus R_2)V) = \mathrm{tr}(A_j(R_1 \oplus R_2)) = a_j, \qquad j = 1, \dots, m.$$

Hence, for each $j$, the first $q$ diagonal entries and the last $k - q$ diagonal entries of $V^*A_jV$ summing up to $0 = a_j = \sum_{u=1}^k \lambda_u(A_j)$; as a result,

$$V^*A_jV = V_1^*D_jV_1 \oplus V_2^*Q_jV_2 = (T_j \oplus S_j) \oplus (\hat{Q}_j \oplus \hat{D}_j),$$

where $T_j \in M_q$ and $\hat{D}_j \in M_{k-q}$. If $1 \le u < v \le m$, then $W_k(A_u + iA_v) \subseteq \{x + iy : x, y \le 0\}$ is polyhedral and $0 = \mathrm{tr}(T_u + iT_v) + \mathrm{tr}(\hat{D}_u + i\hat{D}_v)$ is a conical point. By [14, Lemma 2.6], $T_u + iT_v$ and $\hat{D}_u + i\hat{D}_v$ are normal matrices, i.e., $T_uT_v = T_vT_u$ and $\hat{D}_u\hat{D}_v = \hat{D}_v\hat{D}_u$. Since this is true for all $1 \le u < v \le m$, up to unitarily similarity, we may assume that $T_1, \ldots, T_m$ are diagonal matrices, and so are $\hat{D}_1, \ldots, \hat{D}_m$. So, there is $\hat{V} \in M_{n-\ell}$ such that $\hat{V}^*Q_j\hat{V} = \hat{D}_j \oplus \hat{Q}_j$ for each $j$. Consequently,

$$(I_\ell \oplus \hat{V})^* A_j (I_\ell \oplus \hat{V}) = D_j \oplus \hat{D}_j \oplus \hat{Q}_j, \qquad j = 1, \ldots, m,$$

contradicting the choice of $\ell$.

Now, we show that $\ell \ge 2k$, where $\ell \in \{k, \ldots, n\}$. Suppose the contrary that $\ell < 2k \le n$. Note that for every $j$, $W_k(D_j) = W_k(A_j)$. Then we have

$$\lambda_i(D_j) = \lambda_i(A_j) \ \text{ and } \ \lambda_{n-i+1}(A_j) = \lambda_{\ell-i+1}(D_j), \text{ for all } 1 \le i \le k.$$

It follows that

$$\lambda_{\ell-k+1}(D_j) \le \lambda_{n-\ell}(Q_j) \le \lambda_1(Q_j) \le \lambda_k(D_j) \le \lambda_{\ell-k+1}(D_j)$$

because $\ell - k + 1 \le k$. So we have $\lambda_{\ell-k+1}(D_j) = \lambda_k(D_j)$ and $Q_j = \lambda_k(A_j)I_{n-\ell}$. Hence, we have $V^*A_jV = D_j$ for each $j$ and $\ell = n \ge 2k$, a contradiction.

(b) $\Rightarrow$ (c). Suppose (b) holds. Without loss of generality, assume that $A_j = D_j \oplus Q_j$ for $j = 1, \ldots, m$ and

$$W_k(A_1, \ldots, A_m) = W_k(D_1, \ldots, D_m).$$

If $v = (v_1, \ldots, v_m)^t \in \mathbb{R}^m$ is a unit vector, $A_v = v_1A_1 + \cdots + v_mA_m$ and $D_v = v_1D_1 + \cdots + v_mD_m$, then

$$\begin{aligned}
& [\sum_{j=1}^k \lambda_{n-j+1}(A_v), \sum_{j=1}^k \lambda_j(A_v)] = W_k(v_1A_1 + \cdots + v_mA_m) \\
= \ & \{\sum_{j=1}^m v_ja_j : (a_1, \ldots, a_m) \in W_k(A_1, \ldots, A_m)\} = \{\sum_{j=1}^m v_ja_j : (a_1, \ldots, a_m) \in W_k(D_1, \ldots, D_m)\} \\
= \ & W_k(v_1D_1 + \cdots + v_mD_m) = \left[\sum_{j=1}^k \lambda_{\ell-j+1}(D_v), \sum_{j=1}^k \lambda_j(D_v)\right].
\end{aligned}$$

So,

$$\lambda_i(D_v) = \lambda_i(A_v) \ \text{ and } \ \lambda_{n-i+1}(A_v) = \lambda_{\ell-i+1}(D_v), \text{ for all } 1 \le i \le k.$$

Now, if $C \in M_n$ is Hermitian with eigenvalues $c_1 \ge \cdots \ge c_n$ and $\gamma(C) = k$, then $C - c_{k+1}I$ has at most $k$ positive eigenvalues and at most $k$ negative eigenvalues. Moreover, all the nonzero eigenvalues of $C - c_{k+1}I$ will also be those of $\hat{C}$. As a result, for any unit vector $v = (v_1, \ldots, v_m)^t \in \mathbb{R}^m$, if $A_v = v_1A_1 + \cdots + v_mA_m$ and $D_v = v_1D_1 + \cdots + v_mD_m$, then $W_{C-c_{k+1}I}(A_v) = W_{\hat{C}}(D_v)$. So,

$$\mathbf{conv}\, W_{C-c_{k+1}I}(A_1, \ldots, A_m) \subseteq \mathbf{conv}\, W_{\hat{C}}(D_1, \ldots, D_m) = W_{\hat{C}}(D_1, \ldots, D_m).$$

Clearly, if we assume that $A_j = D_j \oplus Q_j$ for $j = 1, \ldots, m$, and $D = \hat{C} \oplus 0_{n-\ell}$ which is unitarily similar to $C - c_{k+1}I$, then for any unitary $V \in M_\ell$, we can let $\hat{V} = V \oplus I_{n-\ell}$ so that

$$(\mathrm{tr}\hat{C}V^*D_1V, \ldots, \mathrm{tr}\hat{C}V^*D_mV) = (\mathrm{tr}D\hat{V}^*A_1\hat{V}, \ldots, \mathrm{tr}D\hat{V}^*A_m\hat{V}) \in W_{C-c_{k+1}I}(A_1, \ldots, A_m).$$

Hence, we have

$$\mathbf{conv}\, W_{C-c_{k+1}I}(A_1, \ldots, A_m) \subseteq W_{\hat{C}}(D_1, \ldots, D_m) \subseteq W_{C-c_{k+1}I}(A_1, \ldots, A_m).$$

Thus, condition (c) holds.

Suppose (c) holds. Then for any Hermitian $C$ with $\gamma(C) \leq k$, $W_{C-c_{k+1}I}(A_1, \ldots, A_m)$ is polyhedral and so is $W_C(A_1, \ldots, A_m)$. Thus, (d) holds.

The implication (d) $\Rightarrow$ (a) is clear. □

A referee pointed out condition (b) is an improvement of [14, Theorem 2.3] that deals with two Hermitian matrices $A_1, A_2$, and asserts that for $k \leq n/2$, $W_k(A_1, A_2)$ is polyhedral if and only if there is a unitary matrix $U$ such that $U^*A_1U = D_1 \oplus B_1$ and $U^*A_2U = D_2 \oplus B_2$, where $D_j \in M_\ell$ is a diagonal matrix with $\ell \geq k+1$. One cannot deduce from this result that there is $k$ with $|n/2 - k| \leq 1$ such that $W_k(A_1, A_2)$ is polyhedral if and only if $A_1, A_2$ commute. Because of the improvement of the value $\ell$ in Theorem 5.1 (b), one can now readily deduce Theorem 3.3 from Theorem 5.1.

By Theorem 5.1, we see that if $\mathbf{conv}\, W_C(A_1, \ldots, A_m)$ is polyhedral, then $W_{\hat{C}}(A_1, \ldots, A_m)$ is polyhedral for any Hermitian $\hat{C} \in M_n$ with $\gamma(\hat{C}) \leq \gamma(C)$. In particular, we can choose $C = \hat{C}$ so that $W_C(A_1, \ldots, A_m)$ is polyhedral. Similarly, if $\mathbf{conv}\, W_k(A_1, \ldots, A_m)$ is polyhedral for some $k \leq n/2$, then $W_r(A_1, \ldots, A_m)$ is polyhedral for any $r \leq k$.

Note that checking $\mathcal{F} \subseteq M_n$ is a set of commuting normal matrices can be reduced to checking whether $XY = YX$ for any two matrices $X, Y \in \mathcal{F}$. That is why we can focus on the polyhedral property of $W_C(X, Y)$ is normal for any two matrices in $X, Y \in \mathcal{F}$ for a suitable $C$ in Theorem 3.3. We cannot use the same strategy for Theorem 5.1 because $W_C(X, Y)$ is polyhedral for all $X, Y \in \mathcal{B}$, where $\mathcal{B}$ is a basis of the span of $\mathcal{F}$.

**Example 5.2.** *Let* $A_1 = \mathrm{diag}\,(1, 1, -1-1, 1, -1)$, $A_2 = \mathrm{diag}\,(1, -1, 1, -1) \oplus \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $A_3 = [1] \oplus \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \oplus \mathrm{diag}\,(1, -1, -1)$, *then* $W(X, Y) = \mathbf{conv}\,\{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$ *for all* $X, Y \in \{A_1, A_2, A_3\}$, *but* $W(A_1, A_2, A_3)$ *is not polyhedral as it has only two conical points* $(1, 1, 1)$ *and* $(-1, -1, 1)$ *associated with the two common reducing eigenvectors* $e_1$ *and* $e_4$ *of the matrices* $A_1, A_2, A_3$.

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