# Numerical Range, Dilation, and Maximal Operator Systems

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**Abstract**

An operator system is a unital self-adjoint subspace of bounded linear operators. It is maximal if every positive linear map from it to another operator system is completely positive. In this paper, characterizations of maximal operator systems in terms of the joint numerical range are presented. New families of maximal operator systems are identified. These results admit formulations in terms of numerical range inclusion and dilation of operators that unify and extend earlier results on the topic.

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## 1 Introduction

Let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators acting on a Hilbert space $\mathcal{H}$ with inner product $\langle \mathbf{x}, \mathbf{y}\rangle$. If $\mathcal{H}$ has dimension $n$, we identify $\mathcal{B}(\mathcal{H})$ with $M_n$ and $\mathcal{H} = \mathbb{C}^n$ with the usual inner product $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{y}^*\mathbf{x}$. The *numerical range of* $A \in \mathcal{B}(\mathcal{H})$ is defined and denoted by

$$W(A) = \{\langle A\mathbf{x}, \mathbf{x}\rangle : \mathbf{x} \in \mathcal{H}, \langle \mathbf{x}, \mathbf{x}\rangle = 1\}.$$

The numerical range is a useful tool for studying operators and matrices; e.g., see [11]. In particular, researchers have used numerical range inclusion relation of two operators to determine whether one is a dilation of the other. Recall that an operator $B \in \mathcal{B}(\mathcal{H})$ admits a dilation $A \in \mathcal{B}(\mathcal{K})$ if there is a partial isometry $X : \mathcal{H} \to \mathcal{K}$ such that $X^*X = I_{\mathcal{H}}$ and $X^*AX = B$. For simplicity, we will say that $B$ admits a dilation of the form $A \otimes I$ if there is a Hilbert space $\mathcal{L}$ such that $B$ admits a dilation of the form $A \otimes I_{\mathcal{L}}$.

It is easy to show that if $B$ admits a dilation of the form $A \otimes I$, then $W(B) \subseteq W(A)$. But the converse may not hold. Mirman [15] (see also [16]) showed that if $A \in M_3$ is normal, then the following condition holds:

*Every operator $B \in \mathcal{B}(\mathcal{H})$ satisfying $W(B) \subseteq W(A)$ admits a dilation of the form $A \otimes I$.* (1.1)

Ando [1] and Arveson [3] proved that condition (1.1) holds for $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. These results were further extended by Choi and Li [7, 8]. It was shown that condition (1.1) holds if $A \in M_2$, or $A \in M_3$ unitarily similar to $[a_0] \oplus A_1$ with $A_1 \in M_2$. However, let

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{or} \qquad A_2 = \operatorname{diag}(1, i, -1, -i).$$

If $B = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, then $W(B) = \{\mu \in \mathbb{C} : |\mu| \le 1/\sqrt{2}\} = W(A_1) \subseteq W(A_2) = \mathrm{conv}\,\{1, i, -1, i\}$. But $B$ does not admit a dilation of the form $A_1 \otimes I$ or $A_2 \otimes I$ as

$$\|B\| = \sqrt{2} > 1 = \|A_1\| = \|A_2\|.$$

So, (1.1) may not hold for a general matrix $A \in M_n$ for $n \ge 3$ or a general normal matrix $A \in M_m$ with $m \ge 4$. The classification of $A \in M_n$ satisfying (1.1) is an open problem for $n \ge 3$.

In Section 3, we show that if $A \in M_n$ and there are $a, b, c \in \mathbb{C}$ such that $aI + bA + cA^*$ is rank one, then (1.1) holds. As a consequence, if $A \in M_3$ is such that the boundary of $W(A)$ contains a non-trivial line segment, then condition (1.1) holds. Also, we characterize those matrices $A$ which are direct sums of matrices in $M_1$ and $M_2$ satisfying condition (1.1). As a result, if $A \in M_n$ is a normal matrix such that $W(A)$ has four or more vertices, then $A$ does not satisfy (1.1). These cover and extend the results by previous authors.

In addition to the study of a single operator $A$, we extend the study to $m$-tuple of matrices and identify $\mathbf{A} = (A_1, \dots, A_m) \in \mathcal{B}(\mathcal{H})^m$ such that:

*Any $m$-tuple of operators $\mathbf{B} = (B_1, \dots, B_m) \in \mathcal{B}(\mathcal{K})^m$ satisfying the joint numerical range inclusion $W(\mathbf{B}) \subseteq \mathrm{conv}\, W(\mathbf{A})$ will have a joint dilation of the form $(A_1 \otimes I, \dots, A_m \otimes I)$, i.e., there is a partial isometry $X$ such that $B_j = X^*(A_j \otimes I)X$ for $j = 1, \dots, m$.*

It turns out that the study can be reformulated in terms of maximal operator systems. Recall that an *operator system* $S$ of $\mathcal{B}(\mathcal{H})$ is a self-adjoint subspace of $\mathcal{B}(\mathcal{H})$ which contains $I_{\mathcal{H}}$. A linear map $\Phi : S \to \mathcal{B}(\mathcal{K})$ is *unital* if $\Phi(I_{\mathcal{H}}) = I_{\mathcal{K}}$, $\Phi$ is *positive* if $\Phi(A)$ is positive semi-definite for every positive semi-definite $A \in \mathcal{S}$, and $\Phi$ is *completely positive* if $I_k \otimes \Phi : M_k(\mathcal{S}) \to M_k(\mathcal{B}(\mathcal{K}))$ defined by $(S_{ij}) \mapsto (\Phi(S_{ij}))$ is positive for every $k \ge 1$; e.g., see [6] and [18] for some general background.

Suppose $A \in M_n$ and $B \in \mathcal{B}(\mathcal{H})$. Let $\mathcal{S} = \mathrm{span}\,\{I_n, A, A^*\}$. Define a unital linear map $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ by $\Phi(aI_n + bA + cA^*) = aI_{\mathcal{H}} + bB + cB^*$ for $a, b, c \in \mathbb{C}$. By [8, Lemma 4.1], $\Phi$ is positive if and only if $W(B) \subseteq W(A)$. On the other hand, $\Phi$ is completely positive if and only if $B = \Phi(A)$ has a dilation of the form $I \otimes A$; see Proposition 2.2 in the next section. Therefore, the results of Choi and Li can be restated as follows.

**Theorem 1.1** *Suppose $A = A_0$ or $[a] \oplus A_0$ with $A_0 \in M_2$. Then for every Hilbert space $\mathcal{H}$, a unital linear map $\Phi : \mathrm{span}\,\{I, A, A^*\} \to \mathcal{B}(\mathcal{H})$ is positive if and only if $\Phi$ is completely positive.*

Following the discussion in [19, Theorem 3.22], we say that an operator system $\mathcal{S}$ is *maximal* if every unital positive map $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ is completely positive for every Hilbert space $\mathcal{H}$. In such a case, we will say that $\mathcal{S}$ is an OMAX. In particular, it was shown in [19] that $\mathcal{S}$ is an OMAX if and only if for every positive integer $n$, a positive semi-definite operator operator $(b_{ij}) \in M_n(\mathcal{S})$ is the limit of a finite sum of operators of the form $T \otimes B$, where $T \in \mathcal{S}$ and $B \in M_n$ are positive semi-definite operators. Despite this nice characterization, it is not easy to check or construct OMAX.

Suppose an operator system $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ has a basis $\{I, A_1, \dots, A_m\}$. It turns out that one can use the joint numerical range of $(A_1, \dots, A_m) \in \mathcal{B}(\mathcal{H})^m$ defined by

$$W(A_1, \dots, A_m) = \{(\langle A_1 x, x\rangle, \dots, \langle A_m x, x\rangle) : x \in \mathcal{H}, \langle x, x\rangle = 1\} \subseteq \mathbb{C}^m$$

to determine whether $\mathcal{S}$ is maximal. Here, $\mathbb{C}^m$ denotes the set of row or column vectors with $m$ complex entries. Again, a key concept involved is dilation. We say that $(B_1, \dots, B_m) \in \mathcal{B}(\mathcal{K})^m$ has a joint dilation $(A_1, \dots, A_m) \in \mathcal{B}(\mathcal{H})^m$ if there is a partial isometry $V : \mathcal{K} \to \mathcal{H}$ such that $V^* A_j V = B_j$ for all $j = 1, \dots, m$. If there is a Hilbert space $\mathcal{L}$ such that $(B_1, \dots, B_m)$ has a joint dilation $(A_1 \otimes I_{\mathcal{L}}, \dots, A_m \otimes I_{\mathcal{L}}) \in \mathcal{B}(\mathcal{H} \otimes \mathcal{L})$, we will simply say that $(B_1, \dots, B_m)$ has a joint dilation $(A_1 \otimes I, \dots, A_m \otimes I)$.

We will obtain characterizations of maximal operator systems in terms of the inclusion relations of joint numerical ranges. Using these characterizations, we extend earlier results on the topic, and identify new families of maximal operator systems. For instance, we have the following result, which is a consequence Proposition 2.2 in the next section.

**Theorem 1.2** *Let $\mathcal{S} \subseteq M_n$ be an operator system with a basis $\{I, A_1, \dots, A_m\}$ consisting of Hermitian matrices. Then $\mathcal{S}$ is a maximal operator system if and only if every $(B_1, \dots, B_m) \in \mathcal{B}(\mathcal{H})^m$ with $W(B_1, \dots, B_m) \subseteq \operatorname{conv} W(A_1, \dots, A_m)$ has a joint dilation of the form $(A_1 \otimes I, \dots, A_m \otimes I)$.*

By Corollary 3.4, we have the following new example of OMAX.

**Example 1.3** *Let $A = \operatorname{diag}(1+i, 1-i) \oplus \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. Then $\operatorname{span}\{I, A, A^*\}$ is an OMAX. As a result, if $B$ is a bounded linear operator with $W(B) \subseteq W(A) = \operatorname{conv}(\{1+i, 1-i\} \cup \{\mu \in \mathbb{C} : |\mu| \le 1\})$, then $B$ admits a dilation of the form $A \otimes I$.*

Our paper is organized as follows. We first present some preliminary results concerning numerical range inclusion, dilation, and operator systems in Section 2, and consider maximal operator systems generated by one operator $A$ in Section 3. Maximal operator systems generated by two or more operators will be considered in Section 4.

## 2 Preliminary results

It is easy to show that $(B_1, \dots, B_m)$ admits a joint dilation of the form $(A_1 \otimes I, \dots, A_m \otimes I)$ if and only if $(B_1, \dots, B_m)$ admits a dilation of the form $(I \otimes A_1, \dots, I \otimes A_m)$. We will use these two equivalent conditions in our discussion.

We first summarize some basic results on the joint numerical range $W(A_1, \dots, A_m)$; e.g., see [13] and its references. Since $A_j = H_j + iG_j$ with $(H_j, G_j) = (H_j^*, G_j^*)$ for $j = 1, \dots, m$, $W(A_1, \dots, A_m) \subseteq \mathbb{C}^m$ can be identified with $W(H_1, G_1, \dots, H_m, G_m) \subseteq \mathbb{R}^{2m}$. We can focus on the joint numerical range of self-adjoint operators. Below are some basic properties of the joint numerical range; see [13] and its references.

**Proposition 2.1** *Let $T_1, \dots, T_m \in \mathcal{B}(\mathcal{H})$ be self-adjoint operators.*

(a) *The set $W(T_1, \dots, T_m)$ is bounded.*

(b) *The set $W(T_1, \dots, T_m)$ is closed if $\dim \mathcal{H} < \infty$. Otherwise, it may not be closed.*

(c) *When $\dim \mathcal{H} = 2$, $W(T_1, \dots, T_m)$ is convex if and only if $\dim \operatorname{span} \{I, T_1, \dots, T_m\} \le 3$.*

(d) *Suppose $\dim \mathcal{H} \ge 3$, and $\dim \operatorname{span} \{I, T_1, \dots, T_m\} \le 4$. Then $W(T_1, \dots, T_m)$ is convex.*

(e) *Suppose $\dim \mathcal{H} \ge 3$ and $\dim \operatorname{span} \{I, T_1, \dots, T_m\} \ge 4$. Then there is a rank 2 orthgonal projection $T_0$ such that $W(T_0, T_1, \dots, T_m)$ is not convex.*

Note that an operator system $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ always has a basis $\{I, A_1, \dots, A_m\}$ consisting of self-adjoint operators. The following is an extension of [8, Lemma 4.1]. The assertions are probably well known to researchers in the area (see [9, 12] for related results). We include a proof here for completeness.

**Proposition 2.2** *Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ be an operator system with a basis $\{I, A_1, \dots, A_m\}$. Suppose $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{K})$ is a unital linear map and $(B_1, \dots, B_m) = (\Phi(A_1), \dots, \Phi(A_m))$.*

(a) *The map $\Phi$ is positive if and only if*

$$W(B_1, \dots, B_m) \subseteq \mathbf{cl}\,(\operatorname{conv} W(A_1, \dots, A_m)),$$

*where $\mathbf{cl}\,(S)$ denotes the closure of $S \subset \mathbb{R}^m$.*

(b) *If $(B_1 \dots, B_m)$ admits a dilation of the form $(A_1 \otimes I, \dots, A_m \otimes I)$, then $\Phi$ is completely positive. If $\Phi$ is completely positive and $\dim \mathcal{H} < \infty$, then $(B_1, \dots, B_m)$ admits a dilation of the form*

$$(A_1 \otimes I, \dots, A_m \otimes I).$$

*Proof.* (a) Note that $(a_1, \dots, a_m) \in \mathbf{cl}\,(\operatorname{conv} W(\mathbf{A}))$ if and only if for any real vector $(u_0, u_1, \dots, u_m)$,

$$u_0 + u_1 a_1 + \cdots + u_m a_m \le \max \sigma(u_0 I + u_1 A_1 + \cdots + u_m A_m).$$

Here $\sigma(H)$ denotes the spectrum of $H \in \mathcal{B}(\mathcal{H})$. Thus, $W(B_1, \dots, B_m) \subseteq \mathbf{cl}\,(\operatorname{conv} W(A_1, \dots, A_m))$ if and only if $u_0 I + u_1 B_1 + \cdots + u_m B_m \ge 0$ whenever the real vector $(u_0, \dots, u_m)$ satisfies $u_0 I + u_1 A_1 + \cdots + u_m A_m \ge 0$. The assertion follows.

(b) Suppose $(B_1, \dots, B_m)$ admits a dilation of the form $(A_1 \otimes I_{\mathcal{L}}, \dots, A_m \otimes I_{\mathcal{L}})$ for some Hilbert space $\mathcal{L}$, then there exists $V : \mathcal{K} \to \mathcal{H} \otimes \mathcal{L}$, such that $V^*V = I_{\mathcal{K}}$ and $B_i = V^*(A_i \otimes I_{\mathcal{L}})V$ for $i = 1, \dots, m$. Therefore, $\Phi$ is completely positive.

Now, suppose $A_1, \dots, A_m \in M_n$ and $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{K})$ is completely positive. Then by Arveson's Theorem [2], $\Phi$ can be extended to $\Phi : M_n \to \mathcal{B}(\mathcal{K})$. By a result of Choi (see [6] and [18, Theorem

3.14]), if $\{E_{11}, E_{12}, \ldots, E_{nn}\}$ is the standard basis for $M_n$, then $C = (\Phi(E_{ij})) \in M_n(\mathcal{B}(\mathcal{K}))$ is a positive operator. Let $C^{1/2} = [C_1 \ldots C_n]$ so that $C_j : \mathcal{K} \to \mathbb{C}^n \otimes \mathcal{K}$. Because $\Phi$ is unital, if $V^* = [C_1^* \cdots C_n^*]$, then

$$I_{\mathcal{K}} = \sum_{j=1}^{n} \Phi(E_{jj}) = \sum_{j=1}^{n} C_j^* C_j = V^* V.$$

Suppose $I_{\mathcal{L}} = I_n \otimes I_{\mathcal{K}}$. Then for $\ell \in \{1, \ldots, m\}$,

$$V^*(A_\ell \otimes I_{\mathcal{L}})V = \sum_{i,j} (A_\ell)_{ij} (C_i^* C_j) = \sum_{i,j} (A_\ell)_{ij} \Phi(E_{ij}) = \Phi(A_\ell) = B_\ell.$$

Thus, $(B_1, \ldots, B_m)$ admits a joint dilation of the form $(A_1 \otimes I_{\mathcal{L}}, \ldots, A_m \otimes I_{\mathcal{L}})$. ■

**Remark 2.3** Note that the second statement in (b) may not hold if $\mathcal{H}$ is infinite dimensional. For example, if $A = \operatorname{diag}(1, 1/2, \ldots)$ and $B = \operatorname{diag}(0, 1)$, then $aI + bA \mapsto aI + bB$ is a unital completely positive map, but $B$ has no dilation of the form $A \otimes I$ because $0 \in W(B)$ and $0 \notin W(A \otimes I)$. This example shows a subtle difference between the condition that $(B_1, \ldots, B_m)$ has a dilation of the form $(A_1 \otimes I, \ldots, A_m \otimes I)$ and the condition that the unital positive map $\phi$ sending $A_j$ to $B_j$ for $j = 1, \ldots, m$ is completely positive.

Recall that $f : \mathbb{R}^m \to \mathbb{R}^m$ is an affine map if it has the form $\mathbf{x} \mapsto \mathbf{x}R + \mathbf{x}_0$ for a real matrix $R \in M_m$ and $\mathbf{x}_0 \in \mathbb{R}^m$, here $\mathbb{R}^m$ denotes the set of $1 \times m$ real vectors. The affine map is invertible if $R$ is invertible, and the inverse of $f$ has the form $y \mapsto \mathbf{y}R^{-1} - \mathbf{x}_0 R^{-1}$. One can extend the definition of affine map to an $m$-tuple of self-adjoint operators in $\mathcal{B}(\mathcal{H})$ by

$$(A_1, \ldots, A_m) \mapsto (A_1, \ldots, A_m)(r_{ij} I_{\mathcal{H}}) + (x_1 I_{\mathcal{H}}, \ldots, x_m I_{\mathcal{H}})$$

for a real matrix $R = (r_{ij}) \in M_m$ and $(x_1, \ldots, x_m) \in \mathbb{R}^m$. It turns out that real affine maps on $\mathbb{R}^m$ and $\mathcal{B}(\mathcal{H})^m$ behave nicely in connection to positive maps, completely positive maps, and the joint numerical range. We have the following result which can be easily verified.

**Proposition 2.4** *Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ be an operator system with a basis $\{I, A_1, \ldots, A_m\}$, and $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{K})$ a unital linear map such that $B_j = \Phi(A_j) \in \mathcal{B}(\mathcal{K})$ for $j = 1, \ldots, m$, where $A_1, \ldots, A_m$, $B_1, \ldots, B_m$ are self-adjoint. Suppose $f$ is an invertible affine map such that $f(A_1, \ldots, A_m) = (\tilde{A}_1, \ldots, \tilde{A}_m)$ and $f(B_1, \ldots, B_m) = (\tilde{B}_1, \ldots, \tilde{B}_m)$.*

(a) *Then $\Phi$ is positive (respectively, completely positive) if and only if the unital map $\tilde{\Phi}$ defined by $\tilde{\Phi}(\tilde{A}_j) = \tilde{B}_j$ for $j = 1, \ldots, m$, is positive (respectively, completely positive).*

(b) *The $m$-tuple of operators $(B_1, \ldots, B_m)$ has a joint dilation of the form $(I \otimes A_1, \ldots, I \otimes A_m)$ if and only if $(\tilde{B}_1, \ldots, \tilde{B}_m)$ has a joint dilation of the form $(I \otimes \tilde{A}_1, \ldots, I \otimes \tilde{A}_m)$.*

(c) *For any real (unit) vector* $(u_1, \dots, u_m)$

$$W(u_1 B_1 + \cdots + u_m B_m) \subseteq W(u_1 A_1 + \cdots + u_m A_m)$$

*if and only if for any real (unit) vector* $(v_1, \dots, v_m)$

$$W(v_1 \tilde{B}_1 + \cdots + v_m \tilde{B}_m) \subseteq W(v_1 \tilde{A}_1 + \cdots + v_m \tilde{A}_m).$$

## 3 Operator systems generated by a single operator

In this section, we consider operator system of the form $\mathrm{span}\,\{I, A, A^*\}$ for a single matrix $A$. Alternatively, we can write $A = A_1 + iA_2$ for two Hermitian matrices $A_1$ and $A_2$ and consider $\mathcal{S} = \mathrm{span}\,\{I, A_1, A_2\}$. Clearly, if $\mathcal{S}$ has dimension 1, i.e., $A_1$ and $A_2$ are scalar operators, then $\mathcal{S}$ is an OMAX. We will consider the cases when $\mathcal{S}$ has dimension 2 and 3 in the following.

**Proposition 3.1** *Suppose $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ has a basis $\{I, A\}$, with $A = A^*$. Then $\mathcal{S}$ is an OMAX. Furthermore, if $W(A)$ is closed, then a bounded linear operator $B \in \mathcal{B}(\mathcal{K})$ has a dilation of the form $A \otimes I$ whenever $W(B) \subseteq W(A)$.*

*Proof.* We may replace $A$ by $\mu I + A$ and assume that $0 \in W(A)$ and $A$ has an operator matrix with the $(1,1)$ entry equal to zero. Suppose $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{K})$ is a unital positive map and $B = \Phi(A)$. Then $W(B) \subseteq \mathbf{cl}\, W(A)$ by Proposition 2.2 (a).

We will show that $\Phi$ is completely positive. Suppose $k \geq 1$ and $C_0, C_1 \in M_k$ is such that $I_{\mathcal{H}} \otimes C_0 + A \otimes C_1$ is positive semidefinite. Since the $(1,1)$ entry of the operator matrix $A$ is assumed to be 0, we see that the corresponding $(1,1)$ block of $I_{\mathcal{H}} \otimes C_0 + A \otimes C_2$ equal to $C_0$ is positive semi-definite. We may focus on the case when $C_0$ is positive definite, and obtain the conclusion by continuity argument. Replacing $C_j$ by $U^* C_0^{-1/2} C_j C_0^{1/2} U$ by a suitable unitary $U \in M_k$ for $j = 0, 1$, we may assume that $C_0 = I_k$ and $C_1 = \mathrm{diag}\,(c_1, \dots, c_k)$ is a real diagonal matrix. Then for all $1 \leq i \leq k$, we have $1 + c_i A \geq 0$ implying $1 + c_i B \geq 0$. Therefore, $I_{\mathcal{H}} \otimes I_k + A \otimes C_1 \geq 0$. Since this is true for all $k \in \mathbb{N}$, $\Phi$ is completely positive.

Suppose $W(A)$ is closed. Then $W(A) = [\alpha, \beta]$ such that $\alpha, \beta$ are eigenvalues of $A$. So, $A = A_0 \oplus A_1$ such that $A_0 = \mathrm{diag}\,(\alpha, \beta)$. Hence, if $W(B) \subseteq W(A) = W(A_0)$, then $B$ has a dilation of the form $A_0 \otimes I$, and thus has a dilation of the form $A \otimes I$. ∎

**Theorem 3.2** *Suppose $\mathcal{S} = \mathrm{span}\,\{I, A, A^*\} \subseteq M_n$ contains a rank one normal matrix. Then $\mathcal{S}$ is an OMAX.*

*Proof.* If $\dim \mathcal{S} = 2$, then the result follows from Proposition 3.1. Assume $\dim \mathcal{S} = 3$. We may assume that $\mathcal{S}$ has a basis $\{I, A_1, A_2\}$ with $A_1 = E_{11}$ and $A_2 = \begin{pmatrix} 0 & v^* \\ v & G \end{pmatrix}$. If $v = 0$, then $W(A) = \mathrm{conv}\,(\{1\} \cup W(iG))$ which is a line segment or a triangle depending on $W(iG)$ is a singleton or a linear segment. The result follows from [15].

Suppose $v \neq 0$. We can then replace $A_2$ by $([1] \oplus U^*)A_2([1] \oplus U)$ and assume that $v = (\gamma, 0, \ldots, 0)^t$ with $\gamma > 0$. We may replace $A_2$ by $A_2/\gamma$ and assume that $\gamma = 1$. Furthermore, by replacing $A_2$ with $A_2 - aI_n - bA_1$, we may assume that $G = \begin{pmatrix} 0 & G_{12} \\ G_{12}^* & G_{22} \end{pmatrix}$, where $G_{22} \in M_{n-2}$.

Let $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ be a unital positive map and $B_i = \Phi(A_i)$ for $i = 1, 2$. Since $A_1 \geq 0$, we have $B_1 \geq 0$. Suppose we have Hermitian matrices $C_0, C_1, C_2 \in M_k$ such that

$$I_n \otimes C_0 + A_1 \otimes C_1 + A_2 \otimes C_2 = \begin{pmatrix} C_0 + C_1 & C_2 & 0 \\ C_2 & C_0 & G_{12} \otimes C_2 \\ 0 & G_{12}^* \otimes C_2 & I_{n-2} \otimes C_0 + G_{22} \otimes C_2 \end{pmatrix} \geq 0\,.$$

Therefore, $C_0$ is positive semidefinite. Without loss of generality, we may assume that $C_0 = I_k$. We have

$$I_n \otimes I_k + A_1 \otimes C_1 + A_2 \otimes C_2 \geq 0$$

if and only if

$$Q = I_{n-1} \otimes I_k + G \otimes C_2 \geq 0 \quad \text{and} \quad I_k + C_1 \geq (C_2\ 0\ \cdots\ 0)Q^\dagger(C_2\ 0\ \cdots\ 0)^*,$$

where $X^\dagger$ denotes the Moore-Penrose inverse of $X$. For simplicity, we assume that the block matrix $Q$ is invertible and $C_2 = \operatorname{diag}(c_1, \ldots, c_k)$. Then we see that

$$C_1 \geq D = f(C_2),$$

where $D \in M_k$ is the diagonal matrix obtained by applying the rational function

$$f(x) = x^2 \det(I_{n-2} + xG_{22}) \det(I_{n-1} + xG)^{-1} - 1\,.$$

Let $D = \operatorname{diag}(d_1, \ldots, d_k)$ and $C_1 = D + P$ for some positive semidefinite $P \in M_k$. We have

$$I_n + d_iA_1 + c_iA_2 \geq 0 \quad \text{for all } 1 \leq i \leq k\,.$$

Since $\Phi$ is positive, we have

$$I_{\mathcal{H}} + d_iB_1 + c_iB_2 \geq 0 \quad \text{for all } 1 \leq i \leq k\,.$$

Therefore,

$$\begin{aligned} I_{\mathcal{H}} \otimes I_k + B_1 \otimes C_1 + B_2 \otimes C_2 &= I_{\mathcal{H}} \otimes I_k + B_1 \otimes (D + P) + B_2 \otimes C_2 \\ &\geq I_{\mathcal{H}} \otimes I_k + B_1 \otimes D + B_2 \otimes C_2 \\ &\geq 0\,. \end{aligned}$$

■

**Corollary 3.3** *If $A \in M_2$ or if $A \in M_3$ is such that the boundary of $W(A)$ contains a non-trival line segment, then* $\operatorname{span}\{I, A, A^*\}$ *is an OMAX.*

*Proof.* If $A \in M_2$, then there is $A + A^* - aI$ is a rank one normal matrix for some $a \in \mathbb{R}$. By Theorem 3.2, we get the conclusion.

If $A \in M_3$ and the boundary of $W(A)$ has a flat portion. Then we may replace $A$ by $e^{it}(A-\mu I)$ and assume that $W(A) \subseteq \{x + iy : x \geq 0, y \in \mathbb{R}\}$ and $W(A)$ contains a line segment joining 0 to $ai$ for some $a > 0$. Let $A = A_1 + iA_2$ be the Hermitian decomposition of $A$ By assumption, 0 is an eigenvalue of $A_1$ with multiplicity $\geq 2$. Therefore $A_1$ has rank $\leq 1$. By Theorem 3.2, we get the conclusion. ∎

Note that the above corollary covers all the previous results on the topic, and identify some new matrices in $M_3$ such that span $\{I, A, A^*\}$ is an OMAX. For example, if $A = E_{11} + iG \in M_3$ for any Hermitian $G$, then span $\{I, A, A^*\}$ is an OMAX.

In fact, if $A \in M_n$ with $n \geq 4$ and span $\{I, A, A^*\}$ contains a rank one normal matrix, then there is $a, b, c \in \mathbb{C}$ such that $aA + bA^* + cI = E_{11}$. Thus, we may assume that $A = E_{11} + iG$ for a Hermitian matrix $G$ with $(1, 1)$ entry equal to 0. Let $\hat{G}$ be obtained from $G$ by deleting its first row and first column. If $\hat{G}$ is a scalar matrix $gI_{n-1}$, then $A$ is unitarily similar $A_0 \oplus gI_{n-2}$ with $A_0 = \begin{pmatrix} 1 & g_{12}i \\ \bar{g}_{12}i & g\,i \end{pmatrix}$ and $W(A) = W(A_0)$. If $\hat{G}$ is not a scalar matrix, then the boundary of $W(A)$ contains a line segment conv $\{(0, y) : y \in \sigma(\hat{G})\}$. However, even if the boundary of $W(A)$ has a line segment, there does not seem to be an easy way to decide whether span $\{I, A, A^*\}$ contains a rank one normal matrix in terms of $W(A)$ if $A \in M_n$ with $n \geq 4$. Nonetheless, when $n = 4$, we can use the above analysis to determine whether span $\{I, A, A^*\}$ contains a rank one normal matrix in terms of the structure of $W(A)$, and identify another new family of $A$ such the span $\{I, A, A^*\}$ is an OMAX.

**Corollary 3.4** *Let $A \in M_4$. Suppose $W(A)$ is the convex hull of an elliptical disk $\mathcal{E}$ and two points $\alpha, \beta \in \mathbb{C} \setminus \mathcal{E}$ such that the line $L$ passing through $\alpha$ and $\beta$ is tangent to $\mathcal{E}$. Then* span $\{I, A, A^*\}$ *is an OMAX.*

*Proof.* Suppose $A \in M_4$ satisfies the hypothesis. We may replace $A$ by $e^{it}(A-\mu I)$ and assume that $L$ is the imaginary axis $\{iy : y \in \mathbb{R}\}$, $\mathcal{E} \subseteq \{x + iy : x \geq 0, y \in \mathbb{R}\}$ and $L$ is tangent to $\mathcal{E}$ at 0. Let $\alpha = ai$, $\beta = bi$ for some $a, b \in \mathbb{R}$. Then $A$ is unitarily similar to diag $(ai, bi) \oplus A_0$, where $A_0 \in M_2$ with $W(A_0) \subseteq \{x + iy : x \geq 0, y \in \mathbb{R}\}$ and $0 \in W(A_0)$. Let $A_0 = H + iG$ be the Hermitian decomposition of $A_0$. Then 0 is an eigenvalue of $H$. It follows that $A + A^*$ is a rank one matrix. By Theorem 3.2, the result follows. ∎

**Remark 3.5** Note that Corollary 3.4 also holds if we allow the elliptical disk $\mathcal{E}$ to degenerate to a line segment and $L$ intersects $\mathcal{E}$ at an endpoint. The proof also works with $\alpha = \beta$. Therefore, the corollary also covers Theorem 1.1. Furthermore, Corollary 3.4 provides new examples of OMAX such as Example 1.3 mentioned in the introduction.

Next, we consider the case when $A$ is a direct sum of matrices in $M_1$ and $M_2$ and characterize those $A$'s for which span $\{I, A, A^*\}$ is an OMAX.

**Theorem 3.6** *Let $A \in M_n$ be a direct sum of matrices in $M_1$ and $M_2$. The following conditions are equivalent.*

(a) $\operatorname{span}\{I, A, A^*\}$ *is a maximal operator system.*

(b) *Every $B \in B(H)$ with $W(B) \subseteq W(A)$ admits a dilation of the form $A \otimes I$.*

(c) *Every $B \in M_2$ with $W(B) \subseteq W(A)$ admits a dilation of the form $A \otimes I$.*

(d) *$A$ is unitarily similar to $\hat{A}_1 \oplus \hat{A}_2$ with $W(\hat{A}_2) \subseteq W(\hat{A}_1)$, and $\hat{A}_1$ satisfies one of the following:*

(d.1) $\hat{A}_1 \in M_2$,

(d.2) $\hat{A}_1 = [a_1] \oplus A_0 \in M_3$,

(d.3) $\hat{A}_1 = \operatorname{diag}(a_1, a_2) \oplus A_0 \in M_4$ *such that* $\operatorname{conv}\{a_1, a_2\} \cap W(A_0) = \{a_0\} \notin \{a_1, a_2\}$, *i.e., the line segment joining $a_1$ and $a_2$ touches a boundary point of $W(A_0)$ at $a_0 \notin \{a_1, a_2\}$.*

If we allow $a_0 = a_2$ or $a_0 = a_1 = a_2$ in (d.3), then (d.3) will cover the cases (d.2) and (d.1), respectively.

*Proof.* The implication $(d) \Rightarrow (a)$ follows from Theorem 1.1 and Corollary 3.4 that if $\hat{A}_1$ satisfies (d.1)–(d.3), then an operator $B$ satisfies $W(B) \subseteq W(A) = W(\hat{A}_1)$ will have a dilation of the form $\hat{A} \otimes I$, and hence a dilation of the form $A \otimes I$.

The implications $(a) \Rightarrow (b) \Rightarrow (c)$ follows from definition. We are going to prove $(c) \Rightarrow (d)$.

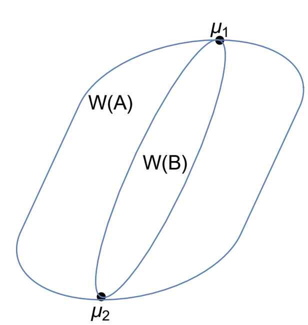
Suppose $A = \oplus_{i=1}^m A_i \in M_n$ satisfies (c), where each $A_i$ is in $M_1$ or $M_2$ and irreducible. Furthermore, we can assume that $W(A_i) \neq W(A_j)$ for $i \neq j$.

We may assume that

$$W(A_{k+1} \oplus \cdots \oplus A_m) \subseteq W(A_1 \oplus \cdots \oplus A_k) = W(A) \neq W\left(\oplus_{i=1}^{j-1} A_i \oplus_{i=j+1}^{k} A_i\right) \tag{3.1}$$

for all $1 \leq j \leq k$. By (3.1), the boundary of $W(A)$, $\partial W(A)$, consists of elliptic arcs and line segments. Consider the following cases for the boundary $\partial W(A)$ :

**Case 1** Suppose $\partial W(A)$ contains two non-degenerate elliptic arcs $S_1$ and $S_2$ coming from two summands, say, $A_1, A_2 \in M_2$. For $3 \leq i \leq m$, $W(A_i)$ can only contain a finite number of points in $S_1 \cup S_2$. Therefore, we can choose an exposed extreme point $\mu_i$ of $W(A_i)$ for $i = 1, 2$ such that $\mu_i \notin W(A_j)$ for $j > 2$. Consider the line segment joining $\mu_1$ and $\mu_2$. We can construct an elliptical disk $\mathcal{E}$ with the line segment joining $\mu_1, \mu_2$ as major axis and minor axis of length $d > 0$ with sufficiently small $d$ so that $\mathcal{E} \subseteq W(A)$. Then there exists $B \in M_2$ such that $W(B) = \mathcal{E}$. We are going to show that $B$ does not have a dilation to $A \otimes I$.




Suppose the contrary that there exist $r \geq 1$ and $X \in M_{2rn}$ such that $XX^* = I_2$ and $X(A \otimes I_r)X^* = B$. Let $\mathbf{u}_1, \mathbf{u}_2$ be unit vectors such that $\mu_i = \mathbf{u}_i^* B \mathbf{u}_i$ for $i = 1, 2$. We may further assume that $\mu_i = (A_i)_{ii}$ for $i = 1, 2$.

Let $X = [\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_{rn}]$, where $\mathbf{x}_j \in \mathbb{C}^2$. Since $\mu_1$ (respectively, $\mu_2$) is an exposed extreme point of $W(B)$ and $W(A_1)$ (respectively, $W(A_2)$) , we have

$$\mathbf{u}_1^*\mathbf{x}_j = 0 \text{ for all } r < j \le nr \text{ and } \mathbf{u}_2^*\mathbf{x}_j = 0 \text{ for all } 1 \le j \le 3r \text{ and } 4r < j \le nr\,. \tag{3.2}$$

Since $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent, we have $\mathbf{x}_j = 0$ for all $r < j \le 3r$ and $4r < j \le nr$. Also,

$$\sum_{j=1}^{r} |\mathbf{u}_1^*\mathbf{x}_j|^2 = \sum_{j=3r+1}^{4r} |\mathbf{u}_2^*\mathbf{x}_j|^2 = 1.$$

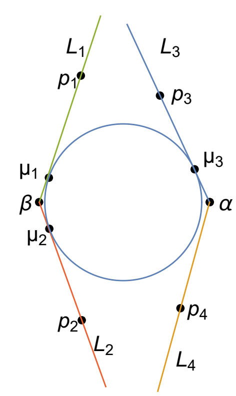
Thus, we have

$$\begin{aligned} 2 &= \sum_{j=1}^{r} |\mathbf{u}_1^*\mathbf{x}_j|^2 + \sum_{j=3r+1}^{4r} |\mathbf{u}_2^*\mathbf{x}_j|^2 \le \sum_{j=1}^{r} \|\mathbf{u}_1\|^2\|\mathbf{x}_j\|^2 + \sum_{j=3r+1}^{4r} \|\mathbf{u}_2\|^2\|\mathbf{x}_j\|^2 \\ &\le \sum_{j=1}^{nr} \mathbf{x}_j^*\mathbf{x}_j = \operatorname{tr}\left(\sum_{j=1}^{nr} \mathbf{x}_j\mathbf{x}_j^*\right) = \operatorname{tr} I_2 = 2. \end{aligned}$$

Therefore, there exist $\alpha_j, \beta_j \in \mathbb{C}$, $1 \le j \le r$ such that $\mathbf{x}_j = \alpha_j\mathbf{u}_1$ and and $\mathbf{x}_{3r+j} = \beta_j\mathbf{u}_2$ for $1 \le j \le r$. Hence, by (3.2), $\mathbf{u}_1$ is orthoginal to $\mathbf{u}_2$ and $B = \mu_1\mathbf{u}_1\mathbf{u}_1^* + \mu_2\mathbf{u}_2\mathbf{u}_2^*$ is normal, a contradiction.

From the result in Case 1, $\partial W(A)$ can only contain elliptic arcs from some $W(A_i)$ for at most one $i$, with $1 \le i \le k$. If $\partial W(A)$ does not contain any line segment, then condition (d.1) is satisfied. It remains to consider the cases when $\partial W(A)$ contains some line segments.

**Case 2** Suppose $\partial(W(A))$ has two pairs of consecutive line segments $\{L_1, L_2\}$ and $\{L_3, L_4\}$ with $L_1, L_2$ meeting at $\beta$ and $L_3, L_4$ meeting at $\alpha$ such that the open line segment $\overline{\alpha\beta}$ lies in the interior of $W(A)$. We may assume that $A = [\alpha] \oplus [\beta] \oplus_{i=3}^m A_i$ where $\alpha, \beta \notin W(\oplus_{i=3}^m A_i)$. Let $A_0 = \oplus_{i=3}^m A_i$. For $i = 1, 2$, let $p_i$ be the point on $L_i \cap W([\alpha] \oplus A_0)$ nearest to $\beta$. For $i = 3, 4$, let $p_i$ be the point on $L_i \cap W([\beta] \oplus A_0)$ nearest to $\alpha$. We may apply an affine transform to $\mathbb{R}^2$ and assume that $(\beta, \alpha) = (-1, 1)$, and $p_1, p_3, -p_2, -p_4$ have $y$-components larger than 2. We will show that there is a circular disk $\mathcal{E}$ in $W(A)$ with radius less than 1 such that the boundary is tangent to at least 3 of the lines $L_i$'s, say $L_1, L_2, L_3$ at the points $\mu_1, \mu_2, \mu_3$ respectively.




Let $B_1$ be the angular bisector at $\beta$. Then $B_1$ intersects either $L_3$ or $L_4$ at a point $\gamma$. For every $c$ on the line segment joining $\beta$ and $\gamma$, let

$$f(c) = \min\{|z - c| : z \in L_1\} = \min\{|z - c| : z \in L_2\} \text{ and } g(c) = \min\{|z - c| : z \in L_3 \cup L_4\}.$$

Then $f(\beta) < g(\beta)$ and $f(\gamma) > g(\gamma)$. Therefore, there exists $c$ on the line segment joining $\beta$ and $\gamma$ such that $f(c) = g(c) = R$. Let $\mathcal{E}$ be the circle with center $c$ and radius $R$. We may assume that $L_i$ is tangent to $\mathcal{E}$ at $\mu_i$ for $i = 1, 2, 3$.

Now, suppose $B \in M_2$ with $W(B) = \mathcal{E}$. We are going to show that $B$ does not have a dilation to $A \otimes I$ in the following.

Suppose the contrary that $B = \sum_{j=1}^r X_j A X_j^*$ for some $X_j \in M_{2\,n}$ satisfying $\sum_{j=1}^r X_j X_j^* = I_2$. Let

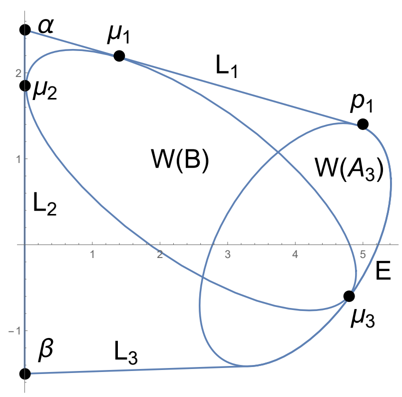
$$X(j) = [\mathbf{x}(j)_1 \mathbf{x}(j)_2 \mathbf{x}(j)_3] \quad \text{with} \quad \mathbf{x}(j)_1, \mathbf{x}(j)_2 \in \mathbb{C}^2 \quad \text{and} \quad \mathbf{x}(j)_3 \in M_{2\,(n-2)}.$$

If $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{C}^2$ are such that $\mathbf{u}_i^* B \mathbf{u}_i = \mu_i$ for $i = 1, 2$, then we must have $\mathbf{u}_i^* \mathbf{x}(j)_1 = 0$ for $i = 1, 2$ and all $1 \le j \le r$. Since $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent, we have $\mathbf{x}(j)_1 = 0$ for all $1 \le j \le r$. In this case, $\mu_3 \notin W(B)$, a contradiction.

From the results in Case 1 and 2, if $\partial W(A)$ only contains line segments, then it has to be a (possibly degenerate) triangle, which is covered by (d.1) and (d.2). If $\partial W(A)$ contains an elliptic arc and some line segments. Then the line segments have to lie consecutively on $\partial W(A)$. Furthermore, there are either two or three line segments. If $\partial W(A)$ contains an elliptic arc and two line segments, then $\hat{A}_1$ satisfies (d.2). So it remains to consider the case when $\partial W(A)$ contains an ellipic arc and three consecutive line segments.

**Case 3** Suppose $\partial(W(A))$ contains three consecutive line segments $L_1, L_2, L_3$ and an elliptic arc $E$. Suppose $L_1$ and $L_2$ meet at $\alpha$, $L_2$ and $L_3$ meet at $\beta$ and $E$ is part of the boundary of $W(A_i)$ for a summand $A_i \in M_2$ . We may assume that $A_1 = [\alpha]$, $A_2 = [\beta]$ and $i = 3$. Therefore, $W(A) = W([\alpha] \oplus [\beta] \oplus A_3)$. If $L_2$ is tangent to the boundary of $W(A_3)$, then, by Corollary 3.4, the operator system spanned by $\{I_4, [\alpha] \oplus [\beta] \oplus A_3, [\overline{\alpha}] \oplus [\overline{\beta}] \oplus A_3^*\}$ is an OMAX. Consequently, the operator system spanned by $\{I_n, A, A^*\}$ is also OMAX.

Suppose $L_2$ is not tangent to the boundary of $W(A_3)$. We may assume that $W(A_3) \not\subseteq W(A_i)$ for all $i > 3$. Hence, there exists $\mu_3$ on $E$ such that $\mu_3 \notin W(A_i)$ for all $i \ne 3$. We can construct $B \in M_2$ with $W(B) \subseteq W(A)$ and satisfies the following conditions:




1. For $i = 1, 2$, $L_i$ is tangent to $W(B)$ at $\mu_i$;
2. $W(B)$ and $W(A_3)$ have a common tangent at the point $\mu_3 \in E$.

To see that such a matrix $B \in M_2$ exists, choose a continuous family of ellipses

$$\{\mathcal{E}(\mu) : \mu \text{ on the line segment joining } \alpha \text{ and } p_1\}$$

such that $L_1$ is a tangent to $\mathcal{E}(\mu)$ at $\mu$ and $\mu_3$ is a boundary point of $\mathcal{E}(\mu)$ which has a common tangent line with $W(A_3)$ at $\mu_3$. We may further assume that $\mathcal{E}(p_1) \subseteq W(A_3)$. Since $\mathcal{E}(p_1) \cap L_2 = \emptyset$ and $\mathcal{E}(\alpha) \cap L_2$ contains more than one points, there exists $\mu_1$ on the open line segment joining $\alpha$ and $p_1$ such that $L_2$ is also tangent to $\mathcal{E}(\mu)$. Let $B \in M_2$ with boundary equal to $\mathcal{E}(\mu_1)$. Then $B$ will satisfy conditions (1) and (2) above.

We are going to prove that $B$ does not have a dilation to $A \otimes I$. Apply an affine map, if necessary, we can assume that $\alpha = b_1 i$, $\beta = b_2 i$ and $\mu_2 = ci$ with $b_1 > c > b_2$, and $W(A_3)$ lies on the open right half plane. Applying a unitary similarity to $B$, we can assume that $B = \operatorname{diag}(0, b) + iG$ for some $b > 0$ and Hermitian $G$. Then $\mu_2 = e_1^* B e_1$, where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Suppose the contrary that $B = \sum_{j=1}^r X_j A X_j^*$ for some $X_j \in M_{2\,n}$ satisfying $\sum_{j=1}^r X_j X_j^* = I_2$. Let $X_j = [x_1(j) x_2(j) \cdots x_m(j)]$, where $x_i(j) \in \mathbb{C}_2$. Applying a unitary similarity to $A_3$, we can further assume that $A_3 = \begin{pmatrix} * & * \\ * & \mu_3 \end{pmatrix}$. For $i = 1, 3$, let $u_i \in \mathbb{C}^2$ be a unit vector satisfying $u_i^* B u_i = \mu_i$. For all $1 \le j \le r$, we have

$$u_1^* x_4(j) = 0,\ e_1^* x_3(j) = e_1^* x_4(j) = 0 \text{ and } u_3^* x_3(j) = 0.$$

Since $\{u_1, e_1\}$ and $\{e_1, u_3\}$ are linearly independent sets, we have $x_3(j) = x_4(j) = 0$ for all $1 \le j \le r$. Then $\mu_3 \notin W(B)$, a contradiction. ■

We restate the above result in terms of the geometrical shape of $W(A)$ in the following.

**Theorem 3.7** *Suppose $A \in M_n$ is a direct sum of matrices in $M_1$ and $M_2$. Then the operator system spanned by $\{I_n, A, A^*\}$ is an OMAX if and only if $W(A)$ is a singleton, a line segment, a triangular region (the convex hull of 3 points that are not collinear), an elliptical disk, the convex hull of an elliptical disk $\mathcal{E}$ with a point $\mu \notin \mathcal{E}$, or the convex hull of $\mathcal{L}$ and $\mathcal{E}$, where $\mathcal{E}$ is an elliptical disk $\mathcal{L} = [z_1, z_2]$ is a line segment touching the ellipse $\mathcal{E}$.*

**Corollary 3.8** *Suppose $A \in M_n$ is normal, and $W(A)$ has four or more vertices. Then $\mathcal{S} = \operatorname{span}\{I, A, A^*\}$ is not an OMAX.*

# 4 Operator systems generated by two or more operators

Suppose $\mathcal{S}$ is an operator system with a basis $\{I, A_1, \dots, A_m\}$. In the following, we will use the algebraic properties of $A_1, \dots, A_m$ and the geometric properties of $W(A_1, \dots, A_m)$ to determine whether $\mathcal{S}$ is maximal.

Recall that a simplex in $\mathbb{R}^m$ is a convex polyhedral set with $m+1$ vertices.

**Theorem 4.1** *Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ be an operator system with a basis $\{I, H_1, \dots, H_m\}$ consisting of self-adjoint operators. If $\operatorname{conv} W(H_1, \dots, H_m)$ is a simplex in $\mathbb{R}^m$, then $\mathcal{S}$ is a maximal operator system. Equivalently, $(B_1, \dots, B_m) \subseteq \mathcal{B}(\mathcal{K})^m$ has a joint dilation of the form $(I \otimes H_1, \dots, I \otimes H_m)$ whenever $W(B_1, \dots, B_m) \subseteq W(H_1, \dots, H_m)$.*

*Proof.* Suppose $\operatorname{conv} W(H_1, \dots, H_m)$ is a simplex. Then by the result in [5], every vertex $(a_1, \dots, a_m)$ corresponding to a joint eigenvalue of $(H_1, \dots, H_m)$ such that $H_j x = a_j \mathbf{x}$ for a unit vector $\mathbf{x}$. Thus, there is a unitary $U$ such that $U^* H_j U = [a_j] \oplus \tilde{H}_j$ for $j = 1, \dots, m$. For simplicity, we will say that $(H_1, \dots, H_m)$ is untiarily similar to $\left([a_1] \oplus \tilde{H}_1, \dots [a_m] \oplus \tilde{H}_m\right)$. Then

$W(\tilde{H}_1, \ldots, \tilde{H}_m)$ will contain the other vertices of $W(H_1, \ldots, H_m)$. We can then repeat the above argument, and extract another joint eigenvalue $(b_1, \ldots, b_m)$ of $H_1, \ldots, H_m$. Thus, $(H_1, \ldots, H_m)$ is unitarily similar to $\left(\operatorname{diag}(a_1, b_1) \oplus \hat{H}_1, \ldots \operatorname{diag}(a_m, b_m) \oplus \hat{H}_m\right)$. Repeating this argument, we see that $(H_1, \ldots, H_m)$ is unitarily similar $(D_1 \oplus C_1, \ldots, D_m \oplus C_m)$ such that $D_j \in M_{m+1}$ is a diagonal matrix and $W(D_1, \ldots, D_m) = W(H_1, \ldots, H_m) = \operatorname{conv} W(H_1, \ldots, H_m)$.

Now, we show that $(B_1, \ldots, B_m)$ admits a joint dilation of the form $(D_1 \otimes I, \ldots, D_m \otimes I)$ for any self-adjoint operators $B_1, \ldots, B_m \in \mathcal{B}(\mathcal{K})$ satisfying $W(B_1, \ldots, B_m) \subseteq W(D_1, \ldots, D_m)$. Our conclusion will follow.

By Proposition 2.2, we can apply an affine transform and assume that $W(D_1, \ldots, D_m)$ is the standard simplex with vertices $\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_m \in \mathbb{R}^m$, where $\mathbf{e}_i$ has 1 at the $i^{\text{th}}$ coordinate and 0 elsewhere. Let $D_j = E_{jj} \in M_{m+1}$ for $i = 1, \ldots, m$. Then for every $k \geq 1$ and $C_0, \ldots, C_m \in M_k$, $I_{m+1} \otimes C_0 + \sum_{j=1}^m D_j \otimes C_j \geq 0$ if and only if $C_0 \geq 0$ and $C_0 + C_j \geq 0$ for all $1 \leq j \leq m$. By continuity argument, we may assume that $C_0$ is positive definite. Replacing $C_j$, with $C_0^{-1/2} C_j C_0^{-1/2}$, we may assume that $C_0 = I$ and $C_j \geq -I$ for all $1 \leq j \leq m$. If $W(B_1, \ldots, B_m) \subseteq W(D_1, \ldots, D_m)$, then $B_j \geq 0$ for all $j$ and $\sum_{j=1}^m B_j \leq I$. Thus,

$$I \otimes I + \sum_{j=1}^m B_j \otimes C_j \geq I \otimes I + \sum_{j=1}^m B_j \otimes (-I) \geq 0.$$

■

One may deduce [4, Theorem 1.1] from Theorem 4.1 above. See also [10, 17] for related results.

**Theorem 4.2** *Suppose* $\mathcal{S}_1 = \operatorname{span}\{I_A, A_1, \ldots, A_r\}$ *and* $\mathcal{S}_2 = \operatorname{span}\{I_B, B_1, \ldots, B_s\}$. *Then* $\mathcal{S} = \operatorname{span}(\{I_A \oplus 0, 0 \oplus I_B\} \cup \{A_i \oplus 0 : 1 \leq i \leq r\} \cup \{0 \oplus B_j : 1 \leq j \leq s\})$ *is maximal if and only if* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are maximal.*

*Proof.* Define $i_1 : \mathcal{S}_1 \to \mathcal{S}$, $i_2 : \mathcal{S}_2 \to \mathcal{S}$ , $\pi_1 : \mathcal{S} \to \mathcal{S}_1$ and $\pi_2 : \mathcal{S} \to \mathcal{S}_2$ by $i_1(A) = A \oplus 0$, $i_2(B) = 0 \oplus B$, $\pi_1(A \oplus B) = A$ and $\pi_2(A \oplus B) = B$.

Suppose $\mathcal{S}_1$ and $\mathcal{S}_2$ are maximal. Given $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ positive, let $\Phi_j = \Phi \circ i_j$ for $j = 1, 2$. Then $\Phi_1$ and $\Phi_2$ are positive, hence, completely positive. Therefore, $\Phi = \Phi_1 \circ \pi_1 + \Phi_2 \circ \pi_2$ is also completely positive. This proves that $\mathcal{S}$ is maximal.

Conversely, suppose $\mathcal{S}$ is maximal. Given positive maps $\Phi_j : \mathcal{S}_j \to \mathcal{B}(\mathcal{H})$, let $\Phi = \Phi_1 \circ \pi_1 + \Phi_2 \circ \pi_2$. Then $\Phi$ is positive, hence, completely positive. Therefore, $\Phi_j = \Phi \circ i_j$, $j = 1, 2$ are also completely positive. ■

By the above result, and the fact that $\operatorname{span}\{E_{11}, E_{22}, E_{12} + E_{21}\} \subseteq M_2$ is an OMAX, see [6, 14] and also Theorem 3.2 above, we have the following.

**Theorem 4.3** *Suppose* $\mathcal{S}$ *is an operator system in* $M_n$. *If, up to a unitary similarity tranform,* $\mathcal{S}$ *has a spanning set which is a subset of* $\{E_{jj} : 1 \leq j \leq n\} \cup \{E_{2j-1,2j} + E_{2j,2j-1} : 1 \leq j \leq n/2\}$, *then* $\mathcal{S}$ *is an OMAX.*

**Corollary 4.4** *Suppose $\mathcal{S} \subseteq M_3$ has a basis $\{I, A_1, A_2, A_3\}$. Then $\mathcal{S}$ is an OMAX if any one of the following equivalent conditions holds.*

(a) $W(A_1, A_2, A_3)$ *is an ice-cream cone with its interior, i.e., the convex hull of an ellipstical disk (a degenerated ellipsoid in $\mathbb{R}^3$) and a point.*

(b) *There is a unitary $U \in M_3$ such that $\mathcal{S}$ has a basis*

$$\{I, U^*(E_{11} - E_{22})U, U^*(E_{12} + E_{21})U, U^*E_{33}U\}.$$

*Proof.* By Theorem 4.3, if (b) holds, then $\mathcal{S}$ is an OMAX.

To prove (a) $\iff$ (b), note that (a) holds if and only if one can apply an affine transform to $A_1, A_2, A_3$, and assume that $W(A_1, A_2, A_3)$ is the ice-cream cone equal to the convex hull of $\{(x, y, 0) : x^2 + y^2 = 1\}$ and $\{(0, 0, 1)\}$ so that up to a unitary similarity transform, the matrices become $A_1 = E_{11} - E_{22}, A_2 = E_{12} + E_{21}, A_3 = E_{33}$. ∎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