Generalized numerical ranges, dilation, and quantum error correction

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Abstract. A survey is given to some recent results on how generalized numerical ranges relate to the study dilation and perturbation of operators. The connection to the study quantum error correction, and unital completely positive linear maps from a Calkin algebra to a matrix space is also discussed.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators on a Hilbert space $\mathcal{H}$ equipped with the inner product $\langle x, y\rangle$. If $\mathcal{H}$ has dimension n, then it is identified as $\mathbb{C}^n$ with the usual inner product $\langle x, y\rangle = y^*x$, and $\mathcal{B}(\mathcal{H})$ is identified as M_n, the algebra of $n \times n$ complex matrices. The study of quadratic forms and their applications appear in many areas of mathematics and other branches of sciences. One such form is the numerical range (a.k.a. the field of values), defined as follows.

Definition 1.1. The *numerical range* of an operator $A \in \mathcal{B}(\mathcal{H})$ is the set

$$W(A) = \{\langle Ax, x\rangle : \ x \in \mathcal{B}(\mathcal{H}) \text{ and } \langle x, x\rangle = 1\}$$

Example 1.2. Here are some simple examples, which will appear again in our subsequent discussion.

- If $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $W(A)$ is the line segment joining 0 and 1.
- If $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, then $W(A) = \{\mu \in \mathbb{C} : |\mu| \le 1\}$, the unit disk centered at the origin.
- If $A = \operatorname{diag}(a_1, a_2, a_3)$, then $W(A)$ is the triangular disk with vertices a_1, a_2, a_3.

Informally, the numerical range of an operator A can be viewed as a "picture" of A, and every point $\langle Ax, x\rangle$ in $W(A)$ can be viewed as a "pixel" of the picture. The "picture" can provide useful information of the operator. In fact, the numerical range of an operator A can be used to deduce algebraic or analytic properties, locate

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the spectrum $\sigma(A)$, and obtain norm bounds of A. The information can then be used to study useful properties such as the invertibility, stability, and convergence of the sequence $\{A^m : m = 1, \dots\}$ of the operator.

In this paper, we survey some results concerning the use of numerical range and generalized numerical ranges to study dilation and compression of operators. The connection of these results to the study quantum error correction, and unital completely positive linear maps from a Calkin algebra to a matrix space will be discussed.

We first present some basic results in Section 2. In Section 3 we describe some ideas on how one can use the inclusion relation of $W(B) \subseteq W(A)$ to ensure that B has a dilation of the $I \otimes A$. Section 4 concerns the joint numerical ranges of several operators and the joint dilation problem. In Sections 5 and 6, we discuss different kinds of generalized numerical ranges arising in the study of quantum error correction codes. The non-emptyness of such general numerical ranges associated with the error operators of a noisy quantum channel will ensure the existence of different types of quantum error correction codes. In such a case, the element in the generalized numerical range will be useful for the construction of a quantum error correction code for the given channel. It turns out that the results and insights developed in the study of quantum error correction is useful in the study of joint essential matricial ranges, and also the images of unital completely positive linear maps from the Calkin algebra associated with a Hilbert space to matrix spaces. These results will be described in Section 7.

For most results we will present the statements without proofs. Nevertheless, we will present three short proofs for the convexity of the numerical range. Also, we give several short new proofs for a few selected results that are different from those in the literature.

2. Basic results

We begin with some results which can be readily deduced from the definition.

Proposition 2.1. *Let $A = H + iG \in \mathcal{B}(\mathcal{H})$, where $H = H^*$ and $G = G^*$.*

a) $W(aH + ibG) = \{ah + ibg : h + ig \in W(A)\}$ *for any* $a, b \in \mathbb{R}$.
b) $W(aA + bI) = aW(A) + b$ *for any* $a, b \in \mathbb{C}$.
c) $W(A) = W(A^T)$ *and* $W(A^*) = \overline{W(A)} = \{\bar{\mu} : \mu \in W(A)\}$.
d) $W(X^*AX) \subseteq W(A)$ *for any subspace* $\mathcal{K}$ *of* $\mathcal{H}$ *and* $X : \mathcal{K} \to \mathcal{H}$ *satisfies* $X^*X = I_{\mathcal{K}}$. *The equality holds if* $\mathcal{K} = \mathcal{H}$ *and* X *is unitary.*

A fundamental and useful result on the numerical range is the celebrated Töplitz-Hausdorff theorem [**18, 33**], proved 100 years ago, asserting that the numerical range of an operator is always convex. There have been many different proofs of this result; see [**4**]. Here we present three short proofs.

Proof 1. Let $a = \langle Ax, x\rangle$ and $b = \langle Ay, y\rangle$ be two different elements in $W(A)$. By Proposition 2.1, we may replace A by $\frac{1}{b-a}(A-aI)$ and assume that $(a, b) = (0, 1)$. We will show that $[a, b] \subseteq W(A)$. Because $\langle Ax, x\rangle \neq \langle Ay, y\rangle$, the vectors x and y are linearly independent. Consider the family of unit vectors $z(t) = \frac{(1-t)x+te^{i\theta}y}{\|(1-t)x+te^{i\theta}y\|}$, where $\theta \in [0, 2\pi)$ satisfies $\langle Ax, e^{i\theta}y\rangle \geq 0$. Then by our choice of θ,

$$t \mapsto \mu(t) = \langle Az(t), z(t)\rangle = (1-t)^2\langle Ax, x\rangle + 2t(1-t)\langle Ax, e^{i\theta}y\rangle + t^2\langle Ay, y\rangle,$$

is a continuous real-valued function on $[0, 1]$ with $\mu(0) = 0$ and $\mu(1) = 1$. Hence, $[0, 1] \subseteq \{\mu(t) : t \in [0, 1]\} \subseteq W(A)$. $\square$

Proof 2. Let $a = \langle Ax, x\rangle$ and $b = \langle Ay, y\rangle$ be two different elements in $W(A)$. Suppose $\mathcal{K} = \mathrm{span}\,\{x, y\} \subseteq \mathcal{H}$, and $X : \mathcal{K} \to \mathcal{H}$ satisfying $X^*X = I_{\mathcal{K}}$. We may identify $B = X^*AX \in M_2$ and $\mathrm{span}\,\{x, y\} = \mathbb{C}^2$. Then

$$W(B) = \{u^*Bu : u \in \mathbb{C}^2, u^*u = 1\} = \{\mathrm{tr}\,(Buu^*) : u^*u = 1\}$$

can be viewed as the image of the "sphere"

$$\{uu^* : u \in \mathbb{C}^2, u^*u = 1\} = \left\{\frac{1}{2}\begin{pmatrix} 1+a & b-ic \\ b+ic & 1-a\end{pmatrix} : a, b, c \in \mathbb{R},\ a^2 + b^2 + c^2 = 1\right\}$$

in $\mathbb{R}^3$ under the real linear map $X \mapsto \mathrm{tr}\,(BX) \in \mathbb{C} \equiv \mathbb{R}^2$. Thus, $W(B)$ is an elliptical disk containing the two points $a = \mathrm{tr}\,(Bxx^*)$ and $b = \mathrm{tr}\,(Byy^*)$. By Proposition 2.1, $W(B) \subseteq W(A)$. The result follows. $\square$

Proof 3. We refine the second part of Proof 2, and show that for $B \in M_2$ with eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ and $b = \sqrt{\mathrm{tr}\,(BB^* - |\lambda_1|^2 - |\lambda_2|^2}$, $W(B)$ is an elliptical disk with foci λ_1, λ_2 and length of minor axis b. Replacing B by $B - \frac{\mathrm{tr}\,B}{2}I_2$, we may assume that $\mathrm{tr}\,(B) = 0$ and $B = \begin{pmatrix} \lambda & b \\ 0 & -\lambda\end{pmatrix}$. If $b = 0$, then

$$W(B) = \{|u_1|^2\lambda - |u_2|^2\lambda : u_1, u_2 \in \mathbb{C}, |u_1|^2 + |u_2|^2 = 1\}$$

is a line segment joining λ and $-\lambda$. Suppose $b > 0$. If $\lambda = 0$, then

$$W(B) = \{bu_2\bar{u}_1 : u_1, u_2 \in \mathbb{C}, |u_1|^2 + |u_2|^2 = 1\}$$

with diameter b. If $\lambda \neq 0$, we may further replace B by D^*BD/λ, where $D = \mathrm{diag}\,(1, e^{i\theta})$ satisfies $e^{i\theta}b/\lambda = c > 0$. Let $\gamma = \sqrt{(2/c)^2 + 1}$, and $B = H + iG$ with $H = H^*$, $G = G^*$. Then $\hat{B} = H + i\gamma G$ is rank one nilpotent and is unitarily similar to $\begin{pmatrix} 0 & \sqrt{4+c^2} \\ 0 & 0\end{pmatrix}$ so that $W(\hat{B}) = \{\mu \in \mathbb{C} : |\mu| \leq \sqrt{1 + (c/2)^2}\}$. Since $W(\hat{B}) = W(H + i\gamma G) = \{h + i\gamma g : h + ig \in W(B)\}$ by Proposition 2.1, $W(B)$ is an elliptical disk with major axis $[-\sqrt{1 + (c/2)^2}, \sqrt{1 + (c/2)^2}]$ and minor axis $\{ir : r \in [-c/2, c/2]\}$. $\square$

Note that Proof 1 is quite standard, and used in many textbooks, e.g., see [**35**]. Proof 2 is based on [**14**] and Proof 3 is based on [**23**]. We summarize the above results into the following.

THEOREM 2.2. *The numerical range of $A \in \mathcal{B}(\mathcal{H})$ is convex. In particular, if $A \in M_2$ has eigenvalues λ_1, λ_2, then $W(A)$ is an elliptical disk with foci λ_1, λ_2, and minor axis with length $\sqrt{\operatorname{tr} A^*A - |\lambda_1|^2 - |\lambda_2|^2}$. Consequently, if $A = \begin{pmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{pmatrix}$, then the minor axis of the elliptical disk $W(A)$ has length $|b|$.*

Next, we list some results showing that there is an interesting interplay between the algebraic and analytic properties of $A \in \mathcal{B}(\mathcal{H})$ and the geometrical properties of $W(A)$. We use $\operatorname{conv} S$ and $\operatorname{cl}(S)$ to denote the convex hull and closure of the set $S \subseteq \mathbb{C}$.

PROPOSITION 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

a) *$A = \mu I$ if and only if $W(A) = \{\mu\}$*
b) *$A = A^*$ if and only if $W(A) \subseteq \mathbb{R}$.*
c) *A is positive semi-definite if and only if $W(A) \subseteq [0, \infty)$*
d) *A is unitary if and only if $W(A)$ and $W(A^{-1})$ lie in the disk.*
e) $\operatorname{conv} \sigma(A) \subseteq \operatorname{cl}(W(A))$*; the set equality holds if A is normal.*
f) *If $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $A_1 \oplus A_2 \in \mathcal{B}(\mathcal{H}_1) \oplus \mathcal{B}(\mathcal{H}_2)$, then*

$$W(A) = \operatorname{conv}\{W(A_1) \cup W(A_2)\}.$$

g) *$W(I \otimes A) = W(A)$.*

The proof of (a) – (c) and (e) – (f) can be verified readily. The proof of (d) is more tricky, especially for the infinite dimensional case. One may see [**2**] for details.

3. Dilation and numerical range inclusion

A useful technique in studying an operator T is to dilate T to a “larger” operator $A \in \mathcal{B}(\mathcal{H})$ with “nice” structure so that one can obtain information about T using the properties of A. Formally, we have the following definition.

DEFINITION 3.1. Let $T \in \mathcal{B}(\mathcal{K})$ and $A \in \mathcal{B}(\mathcal{H})$. We say that A is a *dilation* of T, equivalently, T is a *compression* of A, if $\mathcal{K}$ can be embedded in $\mathcal{H}$, and A has operator matrix $\begin{pmatrix} T & \star \\ \star & \star \end{pmatrix}$ with respect to an orthonormal basis using the vectors in $\mathcal{K}$ and $\mathcal{K}^\perp$.

The following example illustrates how one can dilate an operator to one with nice structure.

EXAMPLE 3.2. Every contraction $T \in \mathcal{B}(\mathcal{K})$, i.e., $T \in \mathcal{B}(\mathcal{K})$ with $\|T\| \leq 1$, admits a unitary dilation of the form

$$A = \begin{pmatrix} T & \sqrt{I - TT^*} \\ \sqrt{I - TT^*} & -T^* \end{pmatrix} \in \mathcal{B}(\mathcal{K} \oplus \mathcal{K}).$$

It turns out that the numerical range can be used in studying dilation. Evidently, $A \in \mathcal{B}(\mathcal{H})$ is a dilation of $T \in \mathcal{B}(\mathcal{K})$ with $\mathcal{K} \subseteq \mathcal{H}$ if there exists $X : \mathcal{K} \to \mathcal{H}$ with $X^*X = I_{\mathcal{K}}$ such that $X^*AX = T$. By Proposition 2.1 (d), we have $W(T) \subseteq W(A)$. But the converse may not hold, i.e., $W(T) \subseteq W(A)$ does not ensure that A is a dilation of T as shown in the following.

Example 3.3. Let $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \in M_2$ and $T = 0_3 \in M_3$, then

$$W(T) = \{0\} \subset \{\mu \in \mathbb{C} : |\mu| \le 1\} = W(A),$$

but A is not a dilation of T as the dimension of A is too low.

By Proposition 2.3 (g), $W(I \otimes A) = W(A)$. This inspires the following.

Problem 3.4. Identify "good" operators $A \in \mathcal{B}(\mathcal{H})$ such that $T \in \mathcal{B}(\mathcal{K})$ has a dilation of the form $I \otimes A$ whenever $W(T) \subseteq W(A)$.

The following theorem was obtained in [**31**]; see also [**32**].

Theorem 3.5. *Let $A \in M_3$ be a normal matrix with eigenvalues $a_1, a_2, a_3 \in \mathbb{C}$. Then $T \in \mathcal{B}(\mathcal{K})$ satisfies $W(T) \subseteq W(A) = \operatorname{conv}\{a_1, a_2, a_3\}$ if and only if T has a dilation of the form $I \otimes A$.*

Note that in applying the above theorem, one does not need to fix the matrix A in advance. For a given operator T, one may choose any triangle with vertices a_1, a_2, a_3 such that $W(T)$ lies inside the triangle. Then T will admit a dilation of the form $I \otimes \operatorname{diag}(a_1, a_2, a_3)$.

Here we give a new short proof for Theorem 3.5 using the following observation, which can be extended to prove some later results in our discussion.

Lemma 3.6. *Let $A = H + iG$ with $(H, G) = (H^*, G^*)$. Suppose $a_1, a_2, b_1, b_2, c_1, c_2$ are real numbers such that $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ is invertible, and*

$$\tilde{A} = (a_1 H + b_1 G + c_1 I) + i(a_2 H + b_2 G + c_2 I).$$

Then $T = T_1 + iT_2$ with $(T_1, T_2) = (T_1^*, T_2^*)$ has a dilation of the form $I \otimes A$ if and only if $\tilde{T} = (a_1 T_1 + b_1 T_2 + c_1 I) + i(a_2 T_1 + b_2 T_2 + c_2 I)$ has a dilation of the form $I \otimes \tilde{A}$.

Proof of Theorem 3.5. If $T \in \mathcal{B}(\mathcal{K})$ is a compression of an operator of the form $I \otimes A$, then $W(T) \subseteq W(A)$. To prove the converse, we consider three cases. If $a_1 = a_2 = a_3$, then $W(A) = \{a_1\}$ and $W(T) \subseteq W(A)$ implies that $T = a_1 I_{\mathcal{K}}$. The result follows.

Suppose A is not a scalar matrix, and a_1, a_2, a_3 are collinear so that $W(A)$ is a line segment, say, with end points a_1, a_3. By Lemma 3.6, we can replace A by $\frac{1}{a_1 - a_3}(A - a_3 I)$ and assume that $(a_1, a_2, a_3) = (0, r, 1)$ for some $r \in [0, 1]$. If

$W(T) \subseteq W(A)$ then T is a positive operator with $T \le I_{\mathcal{K}}$. Then, T is a compression of the operator

$$\begin{pmatrix} \sqrt{T} \\ \sqrt{I-T} \end{pmatrix} \begin{pmatrix} \sqrt{T} & \sqrt{I-T} \end{pmatrix} = \begin{pmatrix} T & \sqrt{T-T^2} \\ \sqrt{T-T^2} & I-T \end{pmatrix},$$

which is unitarily similar to $I_{\mathcal{K}} \otimes \mathrm{diag}\,(0,1)$. So, T is a compression of a matrix of the form $I_{\mathcal{K}} \otimes \mathrm{diag}\,(0,r,1)$.

Finally, suppose a_1, a_2, a_3 are three non-collinear points in $\mathbb{C}$. We may replace A by $\frac{1}{a_2-a_1}(A - a_1 I)$ and assume that $(a_1, a_2) = (1, 0)$. Up to unitary similarity, we may assume that $A = \mathrm{diag}\,(0, 1, r + is) = H + iG$ with $(H, G) = (H^*, G^*)$. We may further replace A by $(H - \frac{r}{s}G) + i\frac{1}{s}G$, and assume that $A = \mathrm{diag}\,(0, 1, i)$. If $T = T_1 + iT_2$ with $(T_1, T_2) = (T_1^*, T_2^*)$ satisfies $W(T) \subseteq W(A)$, then for any unit vector x, the point $\langle Tx, x\rangle$ lies inside the triangle with vertices $0, 1, i$. Hence,

$$\langle T_1 x, x\rangle \ge 0, \quad \langle T_2 x, x\rangle \ge 0, \quad \langle (T_1 + T_2)x, x\rangle \le 1 \qquad \text{for all } x \in \mathcal{K} \text{ with } \|x\| = 1.$$

Thus, T_1, T_2 are positive operators such that $T_1 + T_2 \le I_{\mathcal{K}}$. Let X be such that $X = [\sqrt{I_{\mathcal{K}} - T_1 - T_2}\ \sqrt{T_1}\ \sqrt{T_2}]$ satisfies $X^*X = I_{\mathcal{K}}$ and $T = X^*(A \otimes I_{\mathcal{K}})X$. Hence T is a compression of an operator of the form $I_{\mathcal{K}} \otimes A$. □

The following result was obtained in [**11**] extending a result in [**1**] (see also [**3**]) corresponding to the special case when $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$.

Theorem 3.7. *Let $A \in M_2$ so that $W(A)$ is the elliptical disk with eigenvalues a_1, a_2 as foci and minor axis of length $b = \sqrt{\mathrm{tr}\, A^*A - |a_1|^2 - |a_2|^2}$. Then $T \in \mathcal{B}(\mathcal{H})$ satisfies $W(T) \subseteq W(A)$ if and only if T has a dilation of the form $I \otimes A$.*

In Theorem 3.7, if $A \in M_2$ is normal, then one can use the proof of Theorem 3.5 to get the conclusion. If A is not normal, one can use Lemma 3.6 to reduce the problem to the case treated in [**1**] (and also [**3**]) as follows. Replace A by $aA+bI$ and assume that $W(A)$ is a standard ellipse with major axis equal to $[-1, 1]$ and minor axis $\{ri : r \in [-b, b]\}$. Then we can further replace A by $\tilde{A} = \frac{1}{2}(A+A^*)+\frac{1}{2b}(A-A^*)$ so that $W(\tilde{A})$ is the unit disk centered at origin.

Note that in the application of Theorem 3.7, one needs not specify the matrix A in advance. For a given operator T, one may consider an ellipse $\mathcal{E}$ such that $W(T) \subseteq \mathcal{E}$. One can then construct $A = \begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} \in M_2$, where a_1, a_2 are the foci of $\mathcal{E}$ and b is the length of the minor axis of $\mathcal{E}$. Then T will admit a dilation of the form $I \otimes A$.

Theorems 3.5 and 3.7 were further extended in [**12**] to the following.

Theorem 3.8. *Let $A \in M_3$ have a reducing eigenvalue so that A is unitarily similar to $[\alpha] \oplus A_1$, with $A_1 \in M_2$, so that $W(A)$ is the convex hull of α and the elliptical disk $W(A_1)$. Then $T \in \mathcal{B}(\mathcal{H})$ satisfies $W(T) \subseteq W(A)$ if and only if T has a dilation of the form $I \otimes A$.*

Theorem 3.8 may fail for general matrices $A \in M_3$ or normal matrices $A \in M_4$ as shown in the following.

EXAMPLE 3.9. (a) Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$, then

$$W(A) = W(T) = \{\mu \in \mathbb{C} : |\mu| \leq 1/\sqrt{2}\}.$$

However $\|T\| = \sqrt{2} > 1 = \|A\|$, therefore T has no dilation of the form $I \otimes A$.

(b) Let T be as in the previous example and $A = \operatorname{diag}(1, i, -1, -i)$, then note $W(T) \subseteq \operatorname{conv}\{1, i, -1, -i\} = W(A)$. As in the previous example, $\|A\| = 1$ therefore T has no dilation of the form $I \otimes A$.

In connection to Example 3.9 (b), we have the following; see [**11**, Theorem 2.5].

THEOREM 3.10. *Let $A = \operatorname{diag}(1, i, -1, -i)$. Then $T \in B(H)$ has a dilation of the form $T \otimes A$ if and only if $W(\tilde{T})$ lies inside the unit disk, where*

$$\tilde{T} = \begin{pmatrix} 0 & T + T^* \\ i(T^* - T) & 0 \end{pmatrix}.$$

It is interesting to note that the proof of Theorem 3.8 in [**12**] relies on results of completely positive linear maps and the following theorem, which is the key to affirm a conjecture of Halmos [**17**] that we will state as Corollary 3.12.

THEOREM 3.11. *Suppose $T \in \mathcal{B}(\mathcal{H})$ is a contraction with*

$$W(T) \subseteq S = \{\mu : |\mu| \leq 1, \mu + \overline{\mu} \leq r\},$$

then T has a unitary dilation $A \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ with $W(A) \subseteq S$.

Denote by $\mathrm{cl}(\mathcal{R})$ the closure of a set $\mathcal{R} \subseteq \mathbb{C}$.

COROLLARY 3.12. Let $T \in \mathcal{B}(\mathcal{H})$ be a contraction. Then

$$\mathrm{cl}(W(T)) = \cap\{\mathrm{cl}(W(U)) : U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \text{ is a unitary dilation of } T\}.$$

There are many open problems concerning dilation and numerical range inclusion. We list a few in the following.

1. Determine $A \in \mathcal{B}(\mathcal{H})$ such that an operator $B \in \mathcal{B}(\mathcal{K})$ has a dilation of the form $I \otimes A$ whenever $W(B) \subseteq W(A)$.
2. Determine $B \in \mathcal{B}(\mathcal{K})$ such that B has a dilation of the form $I \otimes A$ for an operator $A \in \mathcal{B}(\mathcal{H})$ whenever $W(B) \subseteq W(A)$.
3. One may also consider a special region $\mathcal{R}$ in $\mathbb{C}$ such that $W(B) \subseteq \mathcal{R}$ will ensure that B has a dilation of the form $I \otimes A$ for some $A \in \mathcal{B}(\mathcal{H})$ with simple structure.

In connection to Problem 1, by Theorem 3.8 if $A \in M_3$ has a reducing eigenvalue, then for any $T \in \mathcal{B}(\mathcal{H})$ satisfying $W(T) \subseteq W(A)$ will ensure that T has dilation of the form $I \otimes A$. In a forthcoming paper, C.K. Li and Y.T. Poon show that if a matrix $A \in M_3$ is such that the boundary of $W(A)$ contains a line segment, then any operator T satisfying $W(T) \subseteq W(A)$ will have a dilation of the form $I \otimes A$. This will further extend Theorem 3.8. For Problem 2, it is clear that all normal operators B satisfy the said property. It would be nice to determine whether the converse is true. For Problem 3, it was shown in [**11**] that if $\mathcal{R}$ is a trapezoidal region in $\mathbb{R}^2 \equiv \mathbb{C}$ and $B \in M_n$ satisfies $W(B) \subseteq \mathcal{R}$, then B has a dilation of the form $A_1 \oplus \cdots \oplus A_n$ with $A_1, \ldots, A_n \in M_2$.

4. Joint numerical ranges and joint dilation

Definition 4.1. For $A_1, ..., A_k \in \mathcal{B}(\mathcal{H})$, define their *joint numerical range* by

$$W(A_1, ..., A_k) = \{(\langle A_1x, x\rangle, ..., \langle A_kx, x\rangle) : x \in \mathcal{H} \text{ and } \langle x, x\rangle = 1\}$$

Identifying $\mathbb{C}$ with $\mathbb{R}^2$, we have $W(A) = W(A_1, A_2)$ if $A = A_1 + iA_2$ with $(A_1, A_2) = (A_1^*, A_2^*)$. So we can focus on $A_1, \ldots, A_m$ lying in $\mathcal{S}(\mathcal{H})$, the real linear space of self-adjoint operators in $\mathcal{B}(\mathcal{H})$.

A natural property to consider is the convexity of the joint numerical range. The following result was obtained in [**25**]; see also [**4**].

Theorem 4.2. *Let* $\boldsymbol{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$.

a) *If the span of* $\{I, A_1, \ldots, A_m\}$ *has dimension not larger than* 3*, then* $W(\boldsymbol{A})$ *is convex.*
b) *If* $\dim \mathcal{H} \geq 3$ *and the span of* $\{I, A_1, \ldots, A_m\}$ *has dimension 4, then* $W(\boldsymbol{A})$ *is convex.*
c) *Let* $B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $B_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *Then*

$$W(B_1, B_2, B_3) = \{(b_1, b_2, b_3) : b_1, b_2, b_3 \in \mathbb{R}, b_1^2 + b_2^2 + b_3^2 = 1\}$$

is not convex.
d) *If* $\{I, A_1, A_2, A_3\}$ *are linearly independent, then there is a rank-2 orthogonal projection* $A_0 \in \mathcal{B}(\mathcal{H})$ *such that* $W(A_0, A_1, A_2, A_3)$ *is not convex.*

One can extend the result by Mirman to the joint numerical range setting. Suppose $W(B_1, B_2, B_3)$ has interior points and lies inside a simplex $\mathcal{S}$ in $\mathbb{R}^3$ with vertices

$$v_1 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \quad v_2 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \quad v_3 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \quad v_4 = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

Then (B_1, B_2, B_3) has a joint dilation (D_1, D_2, D_3) with $D_j = I \otimes \mathrm{diag}\,(a_j, b_j, c_j, d_j)$ for $j = 1, 2, 3$. In other words, there exists a unitary U such that

$$U^* D_j U = \begin{pmatrix} B_j & \star \\ \star & \star \end{pmatrix} \qquad \text{for } j = 1, 2, 3.$$

More generally, we have the following result proved in [**5**].

THEOREM 4.3. *Let $\mathbf{B} = (B_1, \ldots, B_m) \in \mathcal{S}(\mathcal{H})^m$ be such that $W(\mathbf{B})$ has non-empty interior in $\mathbb{R}^m$. That is, $\{I, B_1, \ldots, B_m\}$ is linearly independent. Suppose $\mathcal{S} \subseteq \mathbb{R}^m$ is a simplex with vertices*

$$v_1 = \begin{pmatrix} v_{1,1} \\ \vdots \\ v_{1,m} \end{pmatrix}, \ldots, \; v_{m+1} = \begin{pmatrix} v_{m+1,1} \\ \vdots \\ v_{m+1,m} \end{pmatrix} \in \mathbb{R}^m.$$

Then $W(B_1, \ldots, B_m) \subseteq S$ if and only if $B_1, \ldots, B_m$ has a joint dilation to the diagonal operators

$$I_N \otimes D_j \quad \textit{with} \quad D_j = \mathrm{diag}\,(v_{1j}, \ldots, v_{m+1,j}) \in M_{m+1}, \quad \textit{for} \;\; j = 1, \ldots, m.$$

One can extend the idea of Lemma 3.6 and give a short proof for Theorem 4.3.

Proof of Theorem 4.3. We first reduce the problem to the special case for S to the standard simplex with vertices $0, e_1, \ldots, e_m$, where $\{e_1, \ldots, e_m\}$ is the standard basis for $\mathbb{R}^{1\times m}$. To this end, consider the inveritble affine map $f : \mathbb{R}^{1\times m} \to \mathbb{R}^{1\times m}$ defined by

$$(b_1, \ldots, b_m) \mapsto (b_1, \ldots, b_m)R + v,$$

where $R \in M_m$ is a real invertible matrix and $v = (v_1, \ldots, v_m) \in \mathbb{R}^{1\times m}$. One may extend the affine map to $f : \mathcal{B}(\mathcal{H})^m \to \mathcal{B}(\mathcal{H})^m$ defined by

$$(B_1, \ldots, B_m) \mapsto (B_1, \ldots, B_m)(R \otimes I) + (v_1 I, \ldots, v_m I).$$

Then the conclusion of the theorem holds for $(\mathbf{B}, S)$ if and only if it holds for $(f(\mathbf{B}), f(S))$. Thus, one may apply a suitable invertible affine map to transform S to the standard simplex, and prove the result for this special case.

Now, suppose $\mathcal{S}$ is the standard simplex. Then $W(\mathbf{B}) \subseteq \mathcal{S}$ if and only if $B_1, \ldots, B_m$ are positive operators such that $B_1 + \cdots + B_m \le I$. Let X be such that $X^* = [\sqrt{B_0} \; \cdots \; \sqrt{B_m}]$, where $B_0 = I - (B_1 + \cdots + B_m)$. Then $X^*X = I$ and $X^*(A_j \otimes I)X = B_j$ for $A_j = A_{j0} \oplus \cdots \oplus A_{jm}$, where $A_{jj} = I_{\mathcal{H}}$ and $A_{j\ell} = 0_{\mathcal{H}}$ otherwise. $\square$

Similar to the remark after Theorem 3.5, instead of fixing a simplex in advance, one may choose any simplex $\mathcal{S}$ such that $W(\mathcal{B}) \subseteq \mathcal{S}$ and use the vertices of $\mathcal{S}$ to get an m-tuple of diagonal matrices $(D_1, \ldots, D_m)$ such that $(B_1, \ldots, B_m)$ has a joint dilation of the form $(I \otimes D_1, \ldots, I \otimes D_m)$. Thus, we have the following.

COROLLARY 4.4. *Let $\boldsymbol{A} \in \mathcal{S}(\mathcal{H})^m$. Then the closure of $\mathrm{conv}\,(W(\boldsymbol{A}))$ equals the intersection of $W(D_1, \ldots, D_n)$, where $D_1, \ldots, D_m \in \mathcal{S}(\mathcal{H})$ are mutually commuting operators such that $\mathbf{D}$ is a joint dilation of $\boldsymbol{A}$.*

In [**5**], the authors use Theorem 4.3 (or the above corollary) to define a norm on $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{B}(\mathcal{H})^m$ by

$$\|\mathbf{A}\| = \inf\{\|(\tilde{A}_1, \ldots, \tilde{A}_m)\| : \tilde{\mathbf{A}} \in \mathcal{D}(\mathbf{A})\},$$

where $\mathcal{D}(\mathbf{A})$ consists of $(\tilde{A}_1, \ldots, \tilde{A}_m)$ such that $\{\tilde{A}_1, \ldots, \tilde{A}_m\}$ is a set of mutually commuting normal operators and there is X satisfying $X^*X = I$, $X^*\tilde{A}_jX = A_j$ for $j = 1, \ldots, m$. Such a norm is invariant under any permutation of the components of $\mathbf{A}$, and the change of any of the component A_j to A_j^t, A_j^*, etc.

Recall that an operator system of $\mathcal{B}(\mathcal{H})$ is a subspace spanned by some self-adjoint operators and the identity operator. Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{B} \subseteq \mathcal{B}(\mathcal{K})$ be operators systems. A map $\phi : \mathcal{A} \to \mathcal{B}$ is positive if $\Phi(A) \in \mathcal{B}$ is positive semidefinite whenever $A \in \mathcal{A}$ is positive semidefinite. For a positive integer k, the map ϕ is k-positive if $(\phi(A_{ij})) \in M_k(\mathcal{B})$ is positive whenever $(A_{ij}) \in M_k(\mathcal{A})$ is positive. If ϕ is k-positive for all positive integers k, then ϕ is completely positive. The following results connect the notion of unital positive maps and unital completely positive maps with the inclusion relation of numerical ranges and joint dilation of operators; see [**11**].

THEOREM 4.5. *Let $B_1, \ldots, B_m \in \mathcal{S}(\mathcal{H})$ and $A_1, \ldots, A_m \in M_n$ be Hermitian matrices. Consider the map $\phi : M_n \to \mathcal{B}(\mathcal{H})$ defined by*

$$\phi(\mu_0 I + \mu_1 A_1 + \ldots + \mu_m A_m) = \mu_0 I + \mu_1 B_1 + \ldots + \mu_m B_m \quad \text{for any } \mu_0, \ldots, \mu_m \in \mathbb{C},$$

on $\operatorname{span}\{I, A_1, \ldots, A_m\}$. *Then*

- *ϕ is a positive linear map if and only if*
$$W(B_1, \ldots, B_m) \subseteq \operatorname{conv} W(A_1, \ldots, A_m).$$
- *ϕ is a completely positive (linear) map if and only if*
$$(B_1, \ldots, B_m) \text{ has joint dilation } (I \otimes A_1, \ldots, I \otimes A_m).$$

5. Quantum Channels and Higher Rank Numerical Ranges

In the mathematical setting, quantum states are density operators, i.e. positive semidefinite operators of trace 1. Quantum channels and quantum operations are trace preserving completely positive maps. In the finite dimensional case, a quantum channel Φ transforming quantum states in M_n to quantum states in M_m admits the operator sum representation:

$$\Phi(X) = F_1XF_1^* + \ldots + F_rXF_r^*$$

for some $m \times n$ and F_i satisfying $\sum_{j=1}^r F_j^*F_j = I_n$; see [**6**] and [**20**]. The matrices $F_1, \ldots, F_r$ are known as the Choi-Kraus operators or the error operators of the channel Φ.

We say that a quantum channel $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ has a quantum error code V, which is a subspace of $\mathcal{H}$, provided that there is a quantum channel $\Psi : M_n \to M_n$ satisfying $\Psi \circ \Phi(X) = X$ where $P_V X P_V = X$, where P_V is the orthogonal projection of $\mathcal{H}$ onto the coding subspace V. If $\mathcal{H} = \mathbb{C}^n$ and Φ has error operators $F_1, \ldots, F_r$, it is shown in [**19**] that the search of subspace V and P_V reduces to the search of P_V satisfying $PF_i^* F_j F = f_{ij} P$ for all $1 \le i, j \le r$. In this connection, researchers consider the *rank p-numerical range* of $\mathbf{A} = (A_1, ..., A_m) \in M_n^m$ by

$$\Lambda_p(\mathbf{A}) = \{(a_1, ..., a_m) : X^* A_j X = a_j I_p \text{ for some } X \in \mathcal{V}_p\}$$

where $\mathcal{V}_p$ is the set of linear maps $X : \mathbb{C}^p \longrightarrow \mathcal{H}$ satisfying $X^*X = I_p$; see [**7, 9, 10, 8, 34, 27**]. Note that the quantum channel Φ has a quantum error correction code of dimension p if and only if $\Lambda_p(\mathbf{A}) \neq \emptyset$ with $\mathbf{A} = (F_1^*F_1, F_1^*F_2, \ldots, F_r^*F_r) \in M_n^{r^2}$. Also, observe that $(a_1, \ldots, a_m) \in \Lambda_p(\mathbf{A})$ if and only if there is a unitary $U = [X, \tilde{X}]$ such that

$$U^* A_j U = \begin{pmatrix} a_j I_p & \star \\ \star & \star \end{pmatrix}, \quad j = 1, \ldots, m.$$

Using the higher rank numerical ranges, one can change the problem of searching for an error correct code for a quantum channel Φ to the problem of studying the non-empty-ness of the set $\Lambda_p(\mathbf{A}) \subseteq \mathbb{C}^m$, which is closely related to the *joint unitary orbit* of $\mathbf{A}$:

$$\mathcal{U}(\mathbf{A}) = \{(U^*A_1U, \ldots, U^*A_mU) : U \in M_n,\ U^*U = I_n\}.$$

Therefore, one can apply algebraic, analytic, and geometrical techniques to study the problem.

If $m = 1$ and $A_1 = A_1^*$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, and $p \le (n+1)/2$, then $\Lambda_p(A_1) = [\lambda_{n-p+1}, \lambda_p]$. To see this, assume that $\{x_1, \ldots, x_n\}$ is a set of orthonormal eigenvectors of A such that $Ax_j = \lambda_j x_j$ for $j = 1, \ldots, n$. Then for any $\mu \in [\lambda_{n-p+1}, \lambda_p]$ there is a unit vector $y_j \in \text{span}\{x_{n-p+1}, x_p\}$ such that $y_j^* A y_j = \mu$. Let $Y = [y_1 \ldots y_p]$. Then $Y^*Y = I_p$ and $Y^*AY = \mu I_p$. Conversely, if Y is $n \times p$ such that $Y^*Y = I_p$ and $Y^*AY = \mu I_p$, then by the interlacing inequalities, see [**15**], we see that $\lambda_p \ge \mu \ge \lambda_{n-p+1}$. However, $\Lambda_p(A_1)$ may be empty if $p > (n+1)/2$.

It is non-trivial to determine $\Lambda_p(A)$ even if A is a normal matrix. In [**9**], the authors conjectured the following result, which was confirmed in [**27**].

THEOREM 5.1. *Suppose $A \in M_n$ is a normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $1 \le p \le n$. Then*

$$\Lambda_p(A) = \bigcap_{1 \le j_1 < \cdots < j_{n-p+1} \le n} \text{conv}\,\{\lambda_{j_1}, \ldots, \lambda_{n-p+1}\}.$$

The authors in [**9**] also conjectured that for $A \in M_n$, the set $\Lambda_p(A)$ is convex, and they reduced the problem to the existence of the solution of certain matrix equation. This is confirmed in [**34**]. In [**29**], the authors used the theory of canonical form of matrices under $*$-congruence to give a description of the set $\Lambda_p(A)$ as the intersection of half spaces in $\mathbb{C}$. It then follows that $\Lambda_p(A)$ is convex. Here is the statement of the result.

THEOREM 5.2. *Suppose $A = H + iG \in M_n$ with $(H,G) = (H^*,G^*)$, and $1 \le p \le n$. Then*

$$\Lambda_p(A) = \bigcap_{\theta\in[0,2\pi)} \{h + ig : \cos\theta h + \sin\theta g \le \lambda_1(\cos\theta H + \sin\theta G)\},$$

where $\lambda_1(K)$ is the largest eigenvalue of the Hermitian matrix K. Consequently, $\Lambda_p(A)$ is a compact convex set.

The study of $\Lambda_p(A_1,\dots,A_m)$ for $m \ge 2$ is more intricate. Similar to the study of the joint numerical range, if we write $A_j = H_j + iG_j$ with $(H_j,G_j) = (H_j^*,G_j^*)$ for $j = 1,\dots,m$, then $\Lambda_j(A_1,\dots,A_m) \subseteq \mathbb{C}^m$ and can be identified with $\Lambda_j(H_1,G_1,\dots,H_m,G_m) \subseteq \mathbb{R}^{2m}$. So, we can focus on $\Lambda_p(\mathbf{A})$ for $\mathbf{A} = (A_1,\dots,A_m) \in \mathcal{S}(\mathcal{H})^m$. Now, suppose $R \in M_m$ is a real invertible matrix and $\tilde{\mathbf{A}} = (\tilde A_1,\dots,\tilde A_m)$ with $(\tilde A_1 \cdots \tilde A_m) = (A_1 \cdots A_m)(R\otimes I_n)$, Then $\tilde{\mathbf{a}} \in \Lambda_p(\tilde{\mathbf{A}})$ if and only if $\tilde{\mathbf{a}} = \mathbf{a}R$ with $\mathbf{a} \in \Lambda_p(A)$. Hence, we can choose a suitable $R \in M_m$ such that $\tilde{\mathbf{A}} = (\tilde A_1,\dots,\tilde A_k,0,\dots,0)$, where $\{\tilde A_1,\dots,\tilde A_k\}$ is linearly independent. Then $\Lambda_p(\mathbf{A})$ is completely determined by $\Lambda_p(\tilde A_1,\dots,\tilde A_k)$. Furthermore, if $I \in \operatorname{span}\{\tilde A_1,\dots,\tilde A_k\}$, which is the case in the study of quantum error correction, then we may assume that $\tilde A_k = I_n$ so that every $\tilde{\mathbf{a}} \in \Lambda_p(\tilde A_1,\dots,\tilde A_k)$ will have last entry equal to 1. So, we need only consider $\Lambda_p(\tilde A_1,\dots,\tilde A_{k-1})$. In the following we shall always assume that $\{A_1,\dots,A_m\}$ is linearly independent and $I \notin \operatorname{span}\{A_1,\dots,A_m\}$.

Return to the problem concerning the non-empty-ness of $\Lambda_p(A_1,\dots,A_m)$ related to the study of quantum error correction. We have the following; see [**26**].

THEOREM 5.3. *Let $\boldsymbol{A} = (A_1,\dots,A_m) \in \mathcal{S}(\mathcal{H})^m$. Then $\Lambda_p(A_1,\dots,A_m)$ is non-empty if* $\dim\mathcal{H} \ge (p-1)(m+1)^2$.

It is a challenging problem to find minimum dimension of $\mathcal{H}$ that ensure $\Lambda_p(\mathbf{A}) \ne \emptyset$ for $\mathbf{A} = (A_1,\dots,A_m) \in \mathcal{S}(\mathcal{H})^m$. In case the set is non-empty, one may ask for other properties of the set $\Lambda_p(\mathbf{A})$. The following results were also obtained in [**26**].

THEOREM 5.4. *Let $\boldsymbol{A} = (A_1,\dots,A_m) \in \mathcal{S}(\mathcal{H})^m$. Suppose $\mathcal{H}_1$ is a subspace of $\mathcal{H}$ of dimension r, where $1 \le r < p \le \dim\mathcal{H}$, and $X : \mathcal{H}_1^\perp \to \mathcal{H}$ satisfying $X^*X = I_{\mathcal{H}_1^\perp}$. Then*

$$\Lambda_p(\boldsymbol{A}) \subseteq \Lambda_{p-r}(X^*A_1X,\dots,X^*A_mX).$$

From this theorem, we can obtain the following.

THEOREM 5.5. *Let $\boldsymbol{A} = (A_1,\dots,A_m) \in \mathcal{S}(\mathcal{H})^m$. Suppose $\mathbf{K} = (K_1,\dots,K_m) \in \mathcal{S}(\mathcal{H})^m$ is an m-tuple of finite rank operators such that* $\operatorname{rank}(K_1^2 + \cdots + K_m^2) = r$, *with $1 \le r < p$. Then*

$$\Lambda_p(\boldsymbol{A}) \subseteq \Lambda_{p-r}(\boldsymbol{A} + \mathbf{K}).$$

The above two theorems provide information about $\Lambda_p(\tilde{\mathbf{A}})$ if $\tilde{\mathbf{A}}$ is a (joint) compression of $\mathbf{A}$ or perturbation of $\mathbf{A}$. These results will be extended to a more general version of numerical ranges arising in more sophisticated quantum error correction schemes.

In general, $\Lambda_p(A_1,\dots,A_m)$ is not convex if $m > 2$. It is interesting to determine the conditions on $(A_1,\dots,A_m)$ so that the set is convex. Note that if $\Lambda_p(A_1,\dots,A_m)$ is convex, one can derive efficient algorithms to find its elements (if they exist) and construct quantum error correction codes accordingly.

6. The (p,q)-matricial ranges

DEFINITION 6.1. For $\mathbf{A} = (A_1,...,A_m) \in \mathcal{B}(\mathcal{H})^m$, the *joint q-matricial range* of $\mathbf{A}$ is given by

$$W(q:\mathbf{A}) = \{(X^*A_1X,...,X^*A_mX) : X \in \mathcal{V}_q\},$$

where $\mathcal{V}_q$ denote the set of operators $X : \mathbb{C}^p \to \mathcal{H}$ satisfying $X^*X = I_q$.

In case, $m = 1$, we simply write $W(q : A_1)$. Researchers have used the set $W(q : A_1)$ to study operators $A_1 \in \mathcal{B}(\mathcal{H})$. For example, two compact operators $A_1, B_1 \in \mathcal{B}(\mathcal{H})$ are unitarily similar if and only if $W(q : A_1) = W(q : B_1)$ for all $q = 1,2,\dots$; see [**13**]. One may see the survey [**16**] and its references for more interesting results. However, not many geometrical results have been obtained from $W(q : \mathbf{A})$. Again, for the geometrical properties of the set $W(q : \mathbf{A})$, one can use the Hermitian decomposition of $A_j = H_j + iG_j$ for $j = 1,\dots,m$. and focus on the study of $W(q : \mathbf{A})$ for $A = (A_1,\dots,A_m) \in \mathcal{S}(\mathcal{H})^m$. Moreover, we can assume that $\{A_1,\dots,A_m\}$ is linearly independent and I is not in the linear span. When $m = 1$, the following was proved in [**30**].

THEOREM 6.2. *Let $A \in M_n$ be a Hermitian matrix with eigenvalues $a_1 \geq ... \geq a_n$. Then the set $W(q : \boldsymbol{A})$ consists of Hermitian matrices $B \in M_q$ with eigenvalues $b_1 \geq \cdots \geq b_q$ satisfying $a_j \geq b_j \geq a_{n-q+j}$ for $j = 1,...,q$. Consequently, the set $W(q : \boldsymbol{A})$ is convex if and only if $a_1 = a_q$ and $a_{n-q+1} = a_n$.*

By the above theorem, we see that $W(q : \mathbf{A})$ may not be convex even if $\mathbf{A} = (A_1)$ with $A_1 = A_1^*$. In general, it is difficult to determine the structure of $W(q : \mathbf{A})$. Even if $\mathbf{A} = (A_1, A_2)$ for two commuting Hermitian matrices $A_1, A_2 \in M_n$ so that $W(q : (A_1, A_2)) \equiv W(q : A)$ for the normal matrix $A = A_1 + iA_2$, it is quite difficult to determine the set $W(q : A)$ of all $q \times q$ principal submatrices of U^*AU, where $U \in M_n$ is unitary. This may be a reason why the study of $W(q : \mathbf{A})$ has not been very active in the last two decades.

In the pursuit of "better" quantum error correction codes, researchers proposed the following concept covering both $\Lambda_p(\mathbf{A})$ and $W(q : \mathbf{A})$; see [**21**] and [**28**].

DEFINITION 6.3. The *(p,q)-matricial range* of $\mathbf{A} = (A_1,...,A_m) \in \mathcal{B}(\mathcal{H})^m$, denoted by $\Lambda_{p,q}(\mathbf{A})$, is the set of m-tupe Hermitian matrices $(B_1,...,B_m) \in M_q^m$

such that

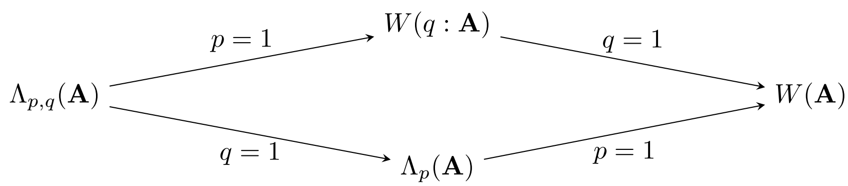
$$X^*A_jX = I_p \otimes B_j = \begin{pmatrix} B_j & & \\ & \ddots & \\ & & B_j \end{pmatrix} \quad \text{for } j = 1, ..., m$$

for some $X \in \mathcal{V}_{pq}$.

An equivalent formulation is to say there is a unitary $U = [X, \tilde{X}]$ such that

$$\begin{pmatrix} I_p \otimes B_j & \star \\ \star & \star \end{pmatrix}.$$

Observe that joint numerical range, joint rank p-numerical range and q-matricial range are all particular cases of (p, q)-*matricial range*.



There is no general convexity result for $\Lambda_{p,q}(\mathbf{A})$. In [**22**] it was shown that if $\dim \mathcal{H}$ is sufficiently large, then $\Lambda_{p,q}(\mathbf{A})$ is star-shaped and non-empty. To state the theorem we first recall the definition of star-shaped.

Definition 6.4. A set $S \subseteq \mathbb{R}^N$ is *star-shaped* if there is a star center, $v_0 \in S$, such that the line segment $[v_0, v] = \{tv_0 + (1-t)v : t \in [0, 1]\}$ lies entirely in the set S for any $v \in S$.

The following result was proved in [**22**].

Theorem 6.5. *Let $\boldsymbol{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$, and p, q be positive integers. Then*

(a) *If $\dim(\mathcal{H}) \geq (pq-1)(m+1)^2$ then $\Lambda_{pq}(\boldsymbol{A})$ and $\Lambda_{p,q}(\boldsymbol{A})$ are non-empty.*
(b) *If $\dim(\mathcal{H}) \geq (N-1)(m+1)^2$ for $N = pq(m+2)$, then every element in $\operatorname{conv}\{(a_1 I_q, ..., a_m I_q) : (a_1, ..., a_m) \in \Lambda_N(\boldsymbol{A})\}$ is a star center of $\Lambda_{p,q}(\boldsymbol{A})$.*
(c) *For any r with $1 \leq qr < p \leq \dim \mathcal{H}$, if $\boldsymbol{K} = (K_1, \ldots, K_m) \in \mathcal{S}(\mathcal{H})^m$ is such that $\operatorname{rank}(K_1^2 + \ldots + K_m^2) \leq r$, then $\Lambda_{p,q}(\boldsymbol{A}) \subseteq \Lambda_{p-qr,q}(\boldsymbol{A} + \boldsymbol{K})$.*
(d) *Suppose $\mathcal{H}_1$ is a subspace of $\mathcal{H}$ of dimension r with $1 \leq qr < p \leq \dim \mathcal{H}$, and $X : \mathcal{H}_1^\perp \to \mathcal{H}$ satisfies $X^*X = I_{\mathcal{H}_1^\perp}$. Then*

$$\Lambda_{p,q}(\boldsymbol{A}) \subseteq \Lambda_{p-qr,q}(X^*A_1X, \ldots, X^*A_mX).$$

When $q = 1$, Theorem 6.5 reduces to the results in Section 5. By Theorem 6.5 and the results in Section 5, if $\dim \mathcal{H}$ is high, then one can always find a (basic type or more sophisticated type of) quantum error correction code for a noisy quantum channel. If the dimension is much higher, then one can get a convex subset of higher rank numerical range so that there will be more choices for quantum error correction codes, and one may be able to select one with additional nice properties. Moreover, if the quantum channel is under perturbation so that the operator system associated with the error operators changed from the span$\{I, A_1, \ldots, A_m\}$ to span$\{I, A_1 + K_1, \ldots, A_m + K_m\}$ with low rank operators $K_1, \ldots, K_m$, one can still get a quantum error correction with reduced dimension based on those of the original channel. Similarly, if one considers the compression of span$\{I, A_1, \ldots, A_m\}$ to span$\{I, X^*A_1X, \ldots, X^*A_mX\}$, the quantum error correction codes for the original system will still be useful for the compressed system.

When $p = 1$, Theorem 6.5 reduces to the following.

THEOREM 6.6. *Let* $\boldsymbol{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$. *The set*

$$W(q : \boldsymbol{A}) = \{(X^*A_1X, \ldots, X^*A_mX) : X \in \mathcal{V}_q\}$$

is star-shaped if $\dim \mathcal{H} \geq (N-1)(m+1)^2$ *with* $N = q(m+2)$.

In particular, the set $\Lambda_N(\boldsymbol{A})$ *is non-empty, and* $(a_1 I_1, \ldots, a_m I_q)$ *is a star center whenever* $(a_1, \ldots, a_m) \in \operatorname{conv} \Lambda_N(\boldsymbol{A})$.

7. The joint essential numerical range

If $\dim \mathcal{H} = \infty$, we consider the following generalization of $\Lambda_q(\mathbf{A})$ and $\Lambda_{p,q}(\mathbf{A})$. Let

$$\Lambda_\infty(\mathbf{A}) = \cap_{r \in \mathbb{N}} \Lambda_r(\mathbf{A}) \subseteq \mathbb{R}^m,$$

i.e., $(a_1, \ldots, a_m) \in \Lambda_\infty(\mathbf{A})$ if there is $X \in \mathcal{V}_\infty$, where $\mathcal{V}_\infty$ consists of $X : \mathcal{K} \to \mathcal{H}$ for an infinite dimensional closed subspace of $\mathcal{H}$ satisfying $X^*X = I_\infty$, and $X^*A_jX = a_j I_\infty$ for $j = 1, ...m$;

$$\Lambda_{\infty,q}(\mathbf{A}) = \cap_{r \in \mathbb{N}} \Lambda_{r,q}(\mathbf{A}) \subseteq M_q^m,$$

i.e., $(B_1, \ldots, B_m) \in \Lambda_{\infty,q}(\mathbf{A})$ if there is $X \in \mathcal{V}_\infty$ such that $X^*A_jX = I_\infty \otimes B_j$ for $j = 1, ...m$.

THEOREM 7.1. [**22**] *Let* $\boldsymbol{A} = (A_1, \ldots, A_m) \in \mathcal{B}(\mathcal{H})$, *where* $\dim \mathcal{H} = \infty$, *then the set* $\Lambda_{\infty,q}(\boldsymbol{A})$ *is convex.*

The statement above does not guarantee that $\Lambda_{\infty,q}(\mathbf{A})$ is non-empty. For instance, if $A = \operatorname{diag}(1, 1/2, 1/3, \ldots)$ then $\Lambda_{\infty,q}(A) = \emptyset$ for any integer $q > 0$.

DEFINITION 7.2. Let $\mathcal{K}(\mathcal{H})$ denote the set of compact operators in $\mathcal{B}(\mathcal{H})$. Define the *essential* (p,q)*-matricial range* of $\mathbf{A} \in \mathcal{B}(\mathcal{H})^m$ by

$$\Lambda_{p,q}^{ess}(\mathbf{A}) = \cap\{\operatorname{cl}(\Lambda_{p,q}(\mathbf{A} + \mathbf{K})) : \mathbf{K} \in \mathcal{K}(\mathcal{H})^m\}.$$

This definition says that $\mathbf{B} = (B_1, \ldots, B_m) \in \Lambda_{p,q}^{ess}(\mathbf{A})$ if for any $\mathbf{K} \in \mathcal{K}(\mathcal{H})^m$ then $\mathbf{B} \in \operatorname{cl}(\Lambda_{p,q}(\mathbf{A}+\mathbf{K}))$. Note that if $p = 1$, we get the essential q-matricial range defined as

$$W_{ess}(q : \mathbf{A}) = \cap\{\operatorname{cl}(W(q : \mathbf{A} + \mathbf{K})) : \mathbf{K} \in \mathcal{K}(\mathcal{H})^m\}.$$

That is, $\mathbf{B} \in W_{ess}(q : \mathbf{A})$ if $\mathbf{B} \in \text{cl}(W(q : \mathbf{A} + \mathbf{K}))$ for every $\mathbf{K} \in \mathcal{K}(\mathcal{H})^m$. The following is obtained in [**22**].

Theorem 7.3. *Let* $\boldsymbol{A} = (A_1, \ldots, A_m) \in \mathcal{B}(\mathcal{H})^m$ *where* $\dim \mathcal{H} = \infty$*, then for any positive integer* p*,*

$$\Lambda_{p,q}^{ess}(\boldsymbol{A}) = W_{ess}(q : \boldsymbol{A})$$

is non-empty, compact and convex.

It turns out the the essential numerical range is connected to the algebra q-matricial range of $\mathbf{A} \in \mathcal{B}(\mathcal{H})^m$ defined as follows.

Definition 7.4. Define the *algebra q-matricial range* of $\mathbf{A} \in \mathcal{B}(\mathcal{H})^m$ by

$$V_q(\mathbf{A}) = \{(\Phi(\pi(A_1)), \ldots, \Phi(\pi(A_m))) : \Phi \text{ is a unital completely positive linear map from } \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \text{ to } M_q\},$$

where π is the canonical surjection from $\mathcal{B}(\mathcal{H})$ to the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

The following results were obtained in [**24**].

Theorem 7.5. *Let* $\boldsymbol{A} \in \mathcal{S}(\mathcal{H})^m$*, where* $\dim \mathcal{H} = \infty$*, and* p, q *be positive integers. Then there is* $\mathbf{K} \in \mathcal{K}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m$ *such that*

$$\Lambda_{p,r}^{ess}(\boldsymbol{A}) = \text{cl}(\Lambda_{p,r}(\boldsymbol{A} + \mathbf{K})) = V_r(\boldsymbol{A}) \quad \text{for all } r = 1, \ldots, p.$$

Theorem 7.6. *Let* $\boldsymbol{A} \in \mathcal{S}(\mathcal{H})^m$ *be such that* $W_{ess}(1 : \boldsymbol{A})$ *is a simplex in* $\mathbb{R}^m$*, where* $\dim \mathcal{H} = \infty$*. Then there is* $\mathbf{K} \in \mathcal{K}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m$ *such that*

$$\Lambda_{p,q}^{ess}(\boldsymbol{A}) = \text{cl}(\Lambda_{p,q}(\boldsymbol{A} + \mathbf{K})) = V_q(\boldsymbol{A}) \quad \text{for all } p, q \in \mathbb{N}.$$

It is known that Theorem 7.6 does not hold for general $\mathbf{A} \in \mathcal{B}(\mathcal{H})^m$ if $m \geq 4$. The case $m = 1$ is covered by the theorem. The cases for $m = 2, 3$ are open. One may see the references in [**24**] for more background of this problem.

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