LINEAR PRESERVERS OF PARALLEL/TEA VECTORS IN $L_p(\mu)$ SPACES

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Abstract. In normed vector spaces, two vectors $\mathbf{x}, \mathbf{y}$ are *parallel* (resp., *triangle equality attaining* (TEA)) if there is a scalar ξ with $|\xi| = 1$ (resp., $\xi = 1$) such that $\|\mathbf{x} + \xi\mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\|$. This paper characterizes linear maps preserving these pairs in $L_1(\mu)$ and $L_\infty(\mu)$ spaces, where non-strict convexity enables rich geometric structures absent in L_p spaces, with $p \in (0, 1) \cup (1, \infty)$ (for which all linear maps trivially preserve such pairs).

We first resolve finite-dimensional cases: ℓ_1-norm TEA pair preservers are matrices with at most one nonzero entry per row. For ℓ_∞, TEA pair preservers are scalar multiples of isometries, except in $\mathbb{R}^2$. These results extend to infinite dimensional spaces $\ell_1(\Lambda)$, $c_0(\Lambda)$, and $\ell_\infty(\Lambda)$, where TEA pair preservers are generalized permutation operators (for ℓ_1) or scalar multiples of isometries (for c_0 and ℓ_∞). In all cases, parallel pair preservers are either TEA pair preservers or rank one maps.

Crucially, we generalize to measure-theoretic settings. For $L_1(\mu)$, TEA pair preservers are automatic bounded and preserves disjointness; in many interesting cases, they are weighted compositions. Parallel pair preservers combine these with rank-one maps. For $L_\infty(\mu)$, bijective preservers are scalar isometries, establishing a dichotomy: L_1 preservers reflect sparsity, while L_∞ preservers align with isometric symmetries. These results unify finite-dimensional, sequence-space, and general L_p settings, advancing the classification of structure-preserving operators in Banach spaces.

1. Introduction

The study of linear operators preserving geometric structures, such as orthogonality, parallelism, or extremal norms, provides insights into Banach space geometry and operator theory. Of particular interest are vector pairs that satisfy norm equality conditions. Specifically, two vectors $\mathbf{x}, \mathbf{y}$ form a *triangle equality attaining* (TEA) pair if

$$\|\mathbf{x} + \mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\|,$$

and a *parallel pair* if

$$\|\mathbf{x} + \xi\mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\|$$

for some unimodular scalar ξ. Such pairs encode fundamental geometric relationships, and their preservers (linear maps maintaining these properties) are pivotal in classifying operator behavior. See, e.g., [1, 2, 5, 9–12, 14–16].

We study linear maps T that preserve TEA pairs, i.e.,

$$T\mathbf{x}, T\mathbf{y} \text{ is a TEA pair} \quad \text{whenever} \quad \mathbf{x}, \mathbf{y} \text{ is a TEA pair,} \tag{1.1}$$

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and those preserving parallel pairs, defined analogously as

$$T\mathbf{x}, T\mathbf{y} \text{ is a parallel pair} \quad \text{whenever} \quad \mathbf{x}, \mathbf{y} \text{ is a parallel pair.} \tag{1.2}$$

Let (X, μ) be a measure space. Let $L_p(\mu)$ be the Banach space of p-integrable functions f with norm $\|f\|_p = \left[\int_X |f(x)|^p \, d\mu(x)\right]^{1/p}$. When $1 < p < \infty$, the strict convexity of the p-norm forces TEA pairs as well as parallel pairs in $L_p(\mu)$ to be scalar multiples. In fact, the same conclusions hold when $0 < p < 1$, although $L_p(\mu)$ is no longer a normed space. Consequently, any linear map of $L_p(\mu)$ preserves parallel pairs and TEA pairs when $p \in (0,1) \cup (1, \infty)$.

Non-strictly convex norms like L_1-norm and L_∞-norm exhibit richer geometric structures, rendering preserver classification nontrivial. This complexity is further compounded in infinite dimensional settings by functional-analytic and measure-theoretic challenges. In this paper, we characterize TEA and parallel pair preservers in these spaces, unifying finite and infinite-dimensional settings.

We develop the characterization of such preservers in three steps. In all situations, we find that

- parallel pair preservers are either TEA pair preservers or rank one operators.

In Section 2, for $\mathbb{F}^n$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), we show that

- ℓ_1–norm TEA pair preservers are sparse matrices with at most one nonzero entry per row;
- ℓ_∞–norm TEA pair preservers are scalar multiples of isometries, except in $\mathbb{R}^2$, where an exceptional non-isometric form exists.

In Section 3, we extend these results to the infinite-dimensional spaces $\ell_1(\Lambda)$, $c_0(\Lambda)$, and $\ell_\infty(\Lambda)$ of summable, essentially null and bounded families, respectively, indexed by a (maybe uncountable) infinite set Λ. TEA pair preservers of $\ell_1(\Lambda)$ are identified as generalized permutation operators. Bijective TEA preservers of $c_0(\Lambda)$ and $\ell_\infty(\Lambda)$ are scalar multiples of isometries, provided that their inverses also preserve TEA pairs.

In Section 4, we extend the characterizations further to such linear preservers between general infinite dimensional $L_1(\mu)$ as well as $L_\infty(\mu)$ spaces. We find that a linear TEA pair preserver $T : L_1(\mu) \to L_1(\nu)$ is automatic bounded, and preserves disjointness, i.e., $TfTg = 0$ whenever $fg = 0$. In many interesting cases, such T carries a weighted composition operator form $Tf = h \cdot f \circ \psi$, where $h \in L_1(\nu)$ and $\psi : Y \to X$ is a measurable transformation from Y into X. Finally, a bijective linear map $T : L_\infty(\mu) \to L_\infty(\nu)$ is a scalar multiple of a surjective linear isometry if and only if both T and its inverse T^{-1} preserve TEA pairs, or parallel pairs.

These results reveal intrinsic geometric contrasts between L_1 and L_∞ spaces: rooted in their additive versus sup-norm structures. For L_1, preservers decompose into measure-algebraic components; for L_∞, preservers align with isometric symmetries. Our findings provide new tools and insights that may be useful in the study of preservers in Banach spaces and operator algebras; see, e.g., [9, 10].

2. The finite dimensional case

In the following, let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ denote the standard basis for $\mathbb{F}^n$, and $\mathbf{M}_n$ be the algebra of $n \times n$ matrices with entries from $\mathbb{F}$. We identify linear maps from $\mathbb{F}^n$ to $\mathbb{F}^n$ with matrices in $\mathbf{M}_n$. For $\mathbf{u} \in \mathbb{F}^n$ and $A \in \mathbf{M}_n$, we let $\mathbf{u}^{\mathrm{t}}$ and A^{t} denote their transposes. A matrix in $\mathbf{M}_n$ is a *monomial* matrix if each row and each column of it has exactly one nonzero entry, and a monomial matrix is a *generalized permutation matrix* if all its nonzero entries have modulus one.

We begin with the following observation.

Lemma 2.1. *Let* $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{t}}, \mathbf{y} = (y_1, \ldots, y_n)^{\mathrm{t}} \in \mathbb{F}^n$.

(a) $\mathbf{x}, \mathbf{y}$ *are parallel (resp. TEA) with respect to the* ℓ_1*–norm if and only if there is a unimodular scalar* ξ *(resp.* $\xi = 1$*) such that* $\xi \overline{x_k} y_k \geq 0$ *for all* $k = 1, \ldots, n$.

(b) $\mathbf{x}, \mathbf{y}$ *are parallel (resp. TEA) with respect to the* ℓ_∞*–norm if and only if* $\|\mathbf{x}\|_\infty = |x_k|$ *and* $\|\mathbf{y}\|_\infty = |y_k|$ *(resp. such that* $\overline{x_k} y_k \geq 0$*) for some* k *between* 1 *and* n.

As direct consequences of Lemma 2.1, a linear map T of $\mathbb{F}^n$ preserves ℓ_1–norm (resp. ℓ_∞–norm) TEA pairs if, and only if, PTQ does for any monomial matrices (resp. generalized permutation matrices) P, Q.

Theorem 2.2. *Let* $T : (\mathbb{F}^n, \|\cdot\|_1) \to (\mathbb{F}^n, \|\cdot\|_1)$ *be a linear map.*

(a) T *preserves TEA pairs if and only if each row of* T *has at most one nonzero entry.*

(b) T *preserves parallel pairs if and only if each row of* T *has at most one nonzero entry, or* $T = \mathbf{v}\mathbf{u}^{\mathrm{t}}$ *for some column vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{F}^n$.

Proof. (a) Suppose each row of the $n \times n$ matrix T has at most one nonzero entries. Then there are monomial matrices P, Q such that $PTQ = \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 & \cdots & \mathbf{T}_n \end{bmatrix}$ in which the column vectors $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_k$ are nonzero and satisfying that

$$\mathbf{T}_1 = \mathbf{e}_1 + \cdots + \mathbf{e}_{n_1}, \quad \mathbf{T}_2 = \mathbf{e}_{n_1+1} + \cdots + \mathbf{e}_{n_2}, \quad \ldots, \quad \mathbf{T}_k = \mathbf{e}_{n_{k-1}+1} + \cdots + \mathbf{e}_{n_k},$$

and

$$\mathbf{T}_{k+1} = \cdots = \mathbf{T}_n = 0,$$

where $k \leq n$ and $1 \leq n_1 < n_2 < \cdots < n_k \leq n$. Note that T preserves ℓ_1–norm TEA pairs exactly when PTQ does. We may replace T by PTQ, and assume that $T = \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 & \cdots & \mathbf{T}_n \end{bmatrix}$. Let $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{t}}$ and $\mathbf{y} = (y_1, \ldots, y_n)^{\mathrm{t}} \in \mathbb{F}^n$ form a TEA pair; or equivalently, $\bar{x}_j y_j \geq 0$ for $j = 1, \ldots, n$. Then

$$T\mathbf{x} = x_1 T_1 + \cdots + x_k T_k = (\underbrace{x_1, \ldots, x_1}_{n_1}, \underbrace{x_2, \ldots, x_2}_{n_2 - n_1}, \ldots, \underbrace{x_k, \ldots, x_k}_{n_k - n_{k-1}}, 0, \ldots, 0)^{\mathrm{t}},$$

$$T\mathbf{y} = y_1 T_1 + \cdots + y_k T_k = (\underbrace{y_1, \ldots, y_1}_{n_1}, \underbrace{y_2, \ldots, y_2}_{n_2 - n_1}, \ldots, \underbrace{y_k, \ldots, y_k}_{n_k - n_{k-1}}, 0, \ldots, 0)^{\mathrm{t}}$$

clearly form a TEA pair.

Conversely, suppose T preserves TEA pairs. Assume the contrary that T has a row with more than one nonzero entries. We may replace T by PTQ for suitable monomial matrices P and Q and assume that T has the $(1,1)$th and the $(1,2)$th entries equal to 1. Since $\mathbf{x} = (2,-1,0,\ldots,0)^{\mathrm{t}}$ and $\mathbf{y} = (1,-2,0,\ldots,0)^{\mathrm{t}}$ form a TEA pair, so do $T\mathbf{x}$ and $T\mathbf{y}$. But the first entries of $T\mathbf{x}$ and $T\mathbf{y}$ are 1 and -1, respectively. So, $T\mathbf{x}$ and $T\mathbf{y}$ do not form a TEA pair, a desired contradiction.

(b) If each row of T has at most one nonzero entry, then T will preserve TEA pairs. Hence, T will also preserve parallel pairs. On the other hand, if $T = \mathbf{v}\mathbf{u}^{\mathrm{t}}$ for some $\mathbf{u}, \mathbf{v} \in \mathbb{F}^n$, then $T\mathbf{x} = (\mathbf{u}^{\mathrm{t}}\mathbf{x})\mathbf{v}$ and $T\mathbf{y} = (\mathbf{u}^{\mathrm{t}}\mathbf{y})\mathbf{v}$ are both scalar multiples of $\mathbf{v}$, and thus always parallel, for any $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$.

Conversely, let T be a linear parallel pair preserver with rank larger than 1. We will show that every row of T has at most one nonzero entry. Suppose on contrary that T has a row with more than one nonzero entry. We may replace T by PTQ for some suitable monomial matrices P and Q and assume that the first row of T has the maximum number of nonzero entries among all the rows. Moreover, we may also assume that all these nonzero entries in the first row are 1 and lie in the $(1,1)$th, $(1,2)$th, $\ldots$, $(1,k)$th positions.

Since T has rank at least two, there is a row, say, the second row, which is not equal to a multiple of the first row. We consider two cases.

Case 1. The first k entries of the second row are not all equal. We may replace T by TQ for a permutation matrix of the form $Q = Q_1 \oplus I_{n-k}$ such that the $(2,1)$ entry is nonzero and different from the $(2,2)$ entry. Further replace T by PT for an invertible diagonal matrix P and assume that the leading 2×2 matrix of T equals $\begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix}$ for some $a \neq 1$.

Let $\mathbf{x} = (m, \bar{a}, 0, \ldots, 0)^{\mathrm{t}}$ and $\mathbf{y} = (1, m\bar{a}, 0, \ldots, 0)^{\mathrm{t}}$ with $m > 0$. Then $\mathbf{x}$ and $\mathbf{y}$ are parallel, and so are the vectors $T\mathbf{x}$ and $T\mathbf{y}$. The first two entries of $T\mathbf{x}$ are $m + \bar{a}$ and $m + |a|^2$, and the first two entries of $T\mathbf{y}$ are $1 + m\bar{a}$ and $1 + m|a|^2$. The second entries of $T\mathbf{x}$ and $T\mathbf{y}$ are always positive. It forces $\overline{(m+\bar{a})}(1 + m\bar{a}) = m(1 + |a|^2) + a + m^2\bar{a} \geq 0$ for all $m > 0$. Consequently, $a \geq 0$.

Furthermore, if $m > 0$, then $\mathbf{x} = (m, -1, 0, \ldots, 0)^{\mathrm{t}}$ and $\mathbf{y} = (1, -m, 0, \ldots, 0)^{\mathrm{t}}$ are parallel, and so are $T\mathbf{x}$ and $T\mathbf{y}$. The first two entries of $T\mathbf{x}$ are $m-1$ and $m-a$, and the first two entries of $T\mathbf{y}$ are $1-m$ and $1-am$. It follows that $(m-a)(1-am) \leq 0$ for all $m > 0$ with $m \neq 1$. Since $a \neq 1$ and $a \geq 0$, we see that $m = (1+a)/2 \neq 1$ and $(m-a)(1-am) = \frac{1}{4}(1-a)(2-a-a^2) = \frac{1}{4}(1-a)^2(2+a) > 0$, which is a contradiction.

Case 2. The first k entries of the (nonzero) second row of T are the same scalar γ. If $\gamma \neq 0$ then all other entries of the second row of T are zeros due to the assumption that the first row of T has maximal number of nonzero entries among all rows of T. But then the second row is γ times the first row, a contradiction. Hence, $\gamma = 0$. Suppose the $(2,j)$th entry of T equals $a \neq 0$ for some $j > k$. We may replace T by TQ for a suitable permutation matrix Q and assume that the leading 2×3 matrix of T is $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}$. Let $\mathbf{x} = (2, -1, 1, 0, \ldots, 0)^{\mathrm{t}}$ and $\mathbf{y} = (1, -2, 1, 0, \ldots, 0)^{\mathrm{t}}$. Then $\mathbf{x}$ and $\mathbf{y}$ are parallel and so are $T\mathbf{x}$ and $T\mathbf{y}$. Now, $T\mathbf{x}$ has the first two entries equal to 1 and

a, whereas $T\mathbf{y}$ has the first two entries equal to -1 and a. Thus, $T\mathbf{x}$ and $T\mathbf{y}$ cannot be parallel, which is a contradiction. ■

We now discuss the case of ℓ_∞-norm parallel/TEA pair preservers.

Lemma 2.3. *If* $\mathbf{V}$ *is a subspace of* $\mathbb{F}^n$ *such that any two elements in* $\mathbf{V}$ *are parallel with respect to* $\|\cdot\|_\infty$*, then* $\dim \mathbf{V} \le 1$.

Proof. We are going to show that any nonzero $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ are linearly dependent. To this end, we may replace $(\mathbf{u}, \mathbf{v})$ by $(\alpha\mathbf{u}, \beta\mathbf{v})$ for some nonzero scalars α, β, and assume that $\mathbf{u}, \mathbf{v}$ are unit vectors with $\|\mathbf{u}+\mathbf{v}\|_\infty = 2$. Since $\|\mathbf{u}+\mathbf{v}\|_\infty = 2$, we may further replace $\mathbf{u}, \mathbf{v}$ by $Q\mathbf{u}, Q\mathbf{v}$ for a suitable generalized permutation matrix Q and assume that $\mathbf{u} = (1, \dots, 1, u_{k+1}, \dots, u_n)^{\rm t}$ and $\mathbf{v} = (1, \dots, 1, v_{k+1}, \dots, v_n)^{\rm t}$ with $|v_j| < 1$ for $j = k+1, \dots, n$, and $k \ge 1$.

By assumption, $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are parallel with respect to the ℓ_∞–norm. There exists a unimodular scalar α such that

$$2 + \|\mathbf{u}-\mathbf{v}\|_\infty = \|\mathbf{u}+\mathbf{v}\|_\infty + \|\mathbf{u}-\mathbf{v}\|_\infty = \|(\mathbf{u}+\mathbf{v}) + \alpha(\mathbf{u}-\mathbf{v})\|_\infty = |(u_i+v_i) + \alpha(u_i - v_i)|$$

for some i. If $k+1 \le i \le n$, then $|(u_i+v_i)+\alpha(u_i-v_i)| < 2 + \|\mathbf{u}-\mathbf{v}\|_\infty$, a contradiction. Consequently, $1 \le i \le k$. Then $|(u_i+v_i)+\alpha(u_i-v_i)| = 2$, and thus $\mathbf{u} = \mathbf{v}$. ■

Lemma 2.4. *Let* $T : (\mathbb{F}^n, \|\cdot\|_\infty) \to (\mathbb{F}^n, \|\cdot\|_\infty)$ *be a nonzero linear map. Then* T *is invertible if one of the following holds.*

(a) $\mathbb{F}^n \ne \mathbb{R}^2$ *and* T *preserves TEA pairs,*
(b) T *preserves parallel pairs with range space of dimension larger than one.*

Proof. Recall that $\mathbf{e}_j$ denotes the coordinate vector with the jth coordinate 1 and all others 0 for $j = 1, \dots, n$, and let $\mathbf{e} = \sum_{j=1}^n \mathbf{e}_j$ be the constant one vector in $\mathbb{F}^n$. If J is a subset of $N = \{1, \dots, n\}$, let $\mathbf{e}_J = \sum_{j\in J} \mathbf{e}_j$. In particular, $\mathbf{e} = \mathbf{e}_N$.

(a) Suppose $T\mathbf{v} = 0$ for some unit vector $\mathbf{v} = \sum_{j=1}^n v_j \mathbf{e}_j \in \mathbb{F}^n$. We claim that T is a zero map. Replacing T by TR for a suitable general permutation matrix R, we can assume that $v_1 = 1 \ge v_2 \ge \cdots \ge v_n \ge 0$.

Suppose first that $v_1 = \cdots = v_k = 1 > v_{k+1} \ge \cdots \ge v_n \ge 0$ for some $k < n$. Since $\xi\mathbf{v} + \mathbf{e}_n$ and $\xi\mathbf{v} - \mathbf{e}_n$ form a TEA pair for large $\xi > 0$, so do $T(\xi\mathbf{v}+\mathbf{e}_n) = T(\mathbf{e}_n)$ and $T(\xi\mathbf{v}-\mathbf{e}_n) = -T(\mathbf{e}_n)$. It follows $0 = \|T(\mathbf{e}_n) + T(-\mathbf{e}_n)\|_\infty = \|T(\mathbf{e}_n)\|_\infty + \|T(-\mathbf{e}_n)\|_\infty$, and thus $T(\mathbf{e}_n) = 0$. With $\mathbf{e}_n$ taking the role of $\mathbf{v}$, we see that $T\mathbf{e}_j = 0$ for all $j = 1, \dots, n-1$, and thus $T = 0$.

Suppose next that $\mathbf{v} = \mathbf{e}$ and $n \ge 3$. For any subset J, K of N such that $J \cup K \ne N$, we have $\mathbf{e} - \mathbf{e}_J$ and $\mathbf{e} - \mathbf{e}_K$ form a TEA pair, and so do $T(\mathbf{e}-\mathbf{e}_J) = -T(\mathbf{e}_J)$ and $T(\mathbf{e}-\mathbf{e}_K) = -T(\mathbf{e}_K)$. Hence,

$$\|T(\mathbf{e}_J) + T(\mathbf{e}_K)\|_\infty = \|T(\mathbf{e}_J)\|_\infty + \|T(\mathbf{e}_K)\|_\infty.$$

An inductive argument with the fact $T(\mathbf{e}) = 0$ gives

$$\|T(\mathbf{e}_1)\|_\infty = \|T(\mathbf{e}_2) + \cdots + T(\mathbf{e}_{n-1})\|_\infty = \|T(\mathbf{e}_2)\|_\infty + \cdots + \|T(\mathbf{e}_{n-1})\|_\infty.$$

We also have similar equalities for all other $\|T(\mathbf{e}_j)\|_\infty$. Summing up these $n \geq 3$ equalities, we have

$$\sum_{j=1}^{n} \|T(\mathbf{e}_j)\|_\infty = (n-1)\sum_{j=1}^{n} \|T(\mathbf{e}_j)\|_\infty,$$

and thus all $\|T(\mathbf{e}_j)\|_\infty = 0$. This also forces $T = 0$.

Finally, suppose $n = 2$, $\mathbb{F} = \mathbb{C}$ and $\mathbf{v} = \mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2$. Since the vectors $\mathbf{e} - \frac{1+\sqrt{3}i}{2}\mathbf{e}_1$ and $\mathbf{e} - \frac{1-\sqrt{3}i}{2}\mathbf{e}_1$ attain the triangle equality, so do $T(\mathbf{e} - \frac{1+\sqrt{3}i}{2}\mathbf{e}_1) = -\frac{1+\sqrt{3}i}{2}T(\mathbf{e}_1)$ and $T(\mathbf{e} - \frac{1-\sqrt{3}i}{2}\mathbf{e}_1) = -\frac{1-\sqrt{3}i}{2}T(\mathbf{e}_1)$, which implies $T(\mathbf{e}_1) = 0$. Consequently, $T(\mathbf{e}_2) = 0$, and thus $T = 0$ again.

In conclusion, a nonzero linear map T preserving TEA pairs is invertible unless $\mathbb{F}^n = \mathbb{R}^2$.

(b) If $n = 2$ and the range space of T has dimension larger than one, then T is invertible. Suppose $n > 2$ and T is not invertible. Let $\mathbf{v}$ be a nonzero vector such that $T\mathbf{v} = 0$. We may replace T by the map $\mathbf{x} \mapsto T(\alpha Q\mathbf{x})$ for some $\alpha > 0$ and generalized permutation matrix Q, and assume that $\mathbf{v} = (v_1, \ldots, v_n)^{\mathrm{t}}$ with $v_1 = \cdots = v_k = 1 > v_{k+1} \geq \cdots \geq v_n \geq 0$ where $1 \leq k \leq n$. We are verifying that the range space of T has dimension at most one.

Case 1. Suppose $k = 1$. Then for any $\mathbf{x} = (0, x_2, \ldots, x_n)^{\mathrm{t}}, \mathbf{y} = (0, y_2, \ldots, y_n)^{\mathrm{t}}$ in the linear span E of $\mathbf{e}_2, \ldots, \mathbf{e}_n$, the vectors $r\mathbf{v} + \mathbf{x}, r\mathbf{v} + \mathbf{y}$ are parallel for sufficiently large $r > 0$. Consequently, $T(r\mathbf{v} + \mathbf{x}) = T(\mathbf{x})$ and $T(r\mathbf{v} + \mathbf{y}) = T(\mathbf{y})$ are also parallel. By Lemma 2.3, the space $T(E)$ has dimension at most one. Since $\mathbb{F}^n$ is spanned by $\mathbf{v}$ and E and $T\mathbf{v} = 0$, we conclude that $T(\mathbb{F}^n)$ has dimension at most one.

Case 2. Suppose $\mathbf{v} = \mathbf{e}_1 + \mathbf{e}_2$. In view of Case 1, we may assume $T(\mathbf{e}_1) = -T(\mathbf{e}_2) \neq 0$. We claim that $T(\mathbf{e}_1)$ and $T(\mathbf{u})$ are linearly dependent for any norm one vector $\mathbf{u} = (0, 0, u_3, \ldots, u_n)^{\mathrm{t}}$. Consequently, being the span of $T(\mathbf{e}_1), T(\mathbf{e}_2)$, and all such $T(\mathbf{u})$, the range space $T(\mathbb{F}^n)$ has dimension at most one.

Consider $\mathbf{x} = \alpha\mathbf{e}_1 + \beta\mathbf{u}$ and $\mathbf{y} = \gamma\mathbf{e}_1 + \delta\mathbf{u}$ for any scalars α, β, γ and δ. If $\mathbf{x}, \mathbf{y}$ are parallel, so are $T(\mathbf{x}), T(\mathbf{y})$. If $\mathbf{x}$ is not parallel with $\mathbf{y}$ then we can assume that $\mathbf{x} = \mathbf{e}_1 + \beta\mathbf{u}$ and $\mathbf{y} = \gamma\mathbf{e}_1 + \mathbf{u}$ with $|\beta|, |\gamma| < 1$. If $\gamma \neq 0$ then $\mathbf{x}$ is parallel with $s\gamma\mathbf{v} + \mathbf{y}$ for $s \geq \frac{1}{|\gamma|} - 1$. We see that $T(\mathbf{x})$ is parallel with $T(s\gamma\mathbf{v} + \mathbf{y}) = T(\mathbf{y})$. In the case when $\gamma = 0$, we see that $T(\mathbf{x})$ is parallel with $T(s\epsilon\mathbf{v} + \epsilon\mathbf{e}_1 + \mathbf{u}) = T(\epsilon\mathbf{e}_1 + \mathbf{u}) = \epsilon T(\mathbf{e}_1) + T(\mathbf{y})$ whenever $0 < \epsilon < 1$ and $s \geq \frac{1}{\epsilon} - 1$. In other words,

$$\|T(\mathbf{x}) + \xi_\epsilon(\epsilon T(\mathbf{e}_1) + T(\mathbf{y}))\|_\infty = \|T(\mathbf{x})\|_\infty + \|\epsilon T(\mathbf{e}_1) + T(\mathbf{y})\|_\infty$$

for some unimodular scalar ξ_ϵ. Choosing a sequence $\epsilon_n \to 0^+$ with ξ_{ϵ_n} converging to some unimodular ξ, we see that $T(\mathbf{x})$ and $T(\mathbf{y})$ are parallel. Therefore, in any case $T(\mathbf{x})$ is parallel with $T(\mathbf{y})$. Hence $T(\mathbf{e}_1)$ and $T(\mathbf{u})$ are linearly dependent by Lemma 2.3, as claimed.

Case 3. Suppose that $\mathbf{v} = \mathbf{e}_1 + \mathbf{e}_2 + a_3\mathbf{e}_3 + \cdots + a_n\mathbf{e}_n$ with $1 \geq a_3 \geq a_4 \geq \cdots \geq a_m \geq 0$. In view of Case 2, we can assume that $a_3 > 0$.

Consider $\mathbf{x} = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2$ and $\mathbf{y} = \gamma\mathbf{e}_1 + \delta\mathbf{e}_2$. We claim that $T(\mathbf{x})$ and $T(\mathbf{y})$ are parallel. If $\mathbf{x}, \mathbf{y}$ are parallel, then it is the case. Otherwise, we can assume that $\mathbf{x} = \mathbf{e}_1 + \xi\mathbf{e}_2$ and $\mathbf{y} = \nu\mathbf{e}_1 + \mathbf{e}_2$ with $|\xi|, |\nu| < 1$. If $|1 - \xi| \geq |1 + \xi|a_3$, then $\mathbf{x} - (1 + \xi)\mathbf{v}/2$ and $\mathbf{y}$ are parallel, and so are

$T(\mathbf{x}) = T(\mathbf{x} - (1+\xi)\mathbf{v}/2)$ and $T(\mathbf{y})$. If $|1-\nu| \geq |1+\nu|a_3$, then $\mathbf{x}$ and $\mathbf{y} - (1+\nu)\mathbf{v}/2$ are parallel, and so are $T(\mathbf{x})$ and $T(\mathbf{y}) = T(\mathbf{y} - (1+\nu)\mathbf{v}/2)$. If $|1-\xi| < |1+\xi|a_3$ and $|1-\nu| < |1+\nu|a_3$ then $\mathbf{x} - (1+\xi)\mathbf{v}/2$ and $\mathbf{y} - (1+\nu)\mathbf{v}/2$ are parallel, then so are $T(\mathbf{x}) = T(\mathbf{x} - (1+\xi)\mathbf{v})$ and $T(\mathbf{y}) = T(\mathbf{y} - (1+\nu)\mathbf{v})$. We see that $T(\mathbf{x})$ and $T(\mathbf{y})$ are parallel in all cases. Hence $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ are linearly dependent by Lemma 2.3. Consequently, there is a nontrivial linear combination $\mathbf{v}' = \xi\mathbf{e}_1 + \eta\mathbf{e}_2$ which belongs to the kernel of T. We can then reduce the situation to either Case 1 (if $|\xi| > |\eta|$) or Case 2 (if $|\xi| = |\eta|$).

We thus conclude that a linear parallel pair preserver is invertible if its range space has dimension larger than one. ∎

Consider the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(\mathbf{x}, \mathbf{y})^{\mathrm{t}} \mapsto (\mathbf{x} - \mathbf{y}, 0)^{\mathrm{t}}$. It is easy to see that $T(\mathbb{R}^2)$ has dimension one and T preserves parallel/TEA pairs for the ℓ_∞–norm. This example says that Lemma 2.4 does not hold in the missing cases.

Lemma 2.5. *Let $A = (a_{rs}) \in \mathbf{M}_n$. Suppose either $a_{jj} > |a_{jk}|$ whenever $j \neq k$, or $a_{jj} > |a_{kj}|$ whenever $j \neq k$. The following conditions are equivalent.*

(a) *For any $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{t}} \in \mathbb{F}^n$ with $\mathbf{y} = A^{\mathrm{t}}\mathbf{x} = (y_1, \ldots, y_n)^{\mathrm{t}}$, we have*

$$\|\mathbf{x}\|_\infty = |x_r| > |x_s| \text{ whenever } s \neq r \quad \Longrightarrow \quad \|\mathbf{y}\|_\infty = |y_r| \geq |y_s| \text{ whenever } s \neq r. \tag{2.1}$$

(b) *Either $n = 2$ and $A = A^*$ with $a_{11} = a_{22} > |a_{12}|$, or $A = a_{11}I_n$.*

Proof. The implication (b) ⇒ (a) is clear if $A = a_{11}I_n$. Suppose $n = 2$ and $A = A^*$ with $a_{11} = a_{22} > |a_{12}|$. Observe that

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A^{\mathrm{t}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & \overline{a_{12}} \\ a_{12} & a_{11} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \overline{a_{12}}x_2 \\ a_{12}x_1 + a_{11}x_2 \end{pmatrix},$$

and

$$\begin{aligned} & |y_1| \geq |y_2| \\ \Longleftrightarrow \quad & |a_{11}x_1 + \overline{a_{12}}x_2| \geq |a_{12}x_1 + a_{11}x_2| \\ \Longleftrightarrow \quad & a_{11}^2|x_1|^2 + |a_{12}|^2|x_2|^2 + 2a_{11}\operatorname{Re}(a_{12}x_1\bar{x}_2) \geq |a_{12}|^2|x_1|^2 + a_{11}^2|x_2|^2 + 2a_{11}\operatorname{Re}(a_{12}x_1\bar{x}_2) \\ \Longleftrightarrow \quad & (a_{11}^2 - |a_{12}|^2)|x_1|^2 \geq (a_{11}^2 - |a_{12}|^2)|x_2|^2 \\ \Longleftrightarrow \quad & |x_1| \geq |x_2|. \end{aligned}$$

Consequently, (b) ⇒ (a) also holds in this case.

We are going to prove (a) ⇒ (b). We may replace A by $P^{\mathrm{t}}AP$ with a suitable permutation matrix P and assume the first row $\mathbf{v}_1$ of A^{t} has the maximal ℓ_1–norm. Then further replace A by DAD^* with a suitable diagonal $D \in \mathbf{M}_n$ with $D^*D = I_n$ and assume that the first row of A^{t} has nonnegative entries.

By the continuity and an induction argument, for $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{t}}$, $\mathbf{y} = A^{\mathrm{t}}\mathbf{x} = (y_1, \ldots, y_n)^{\mathrm{t}}$ and $k = 1, \ldots, m$, we have

$$|x_1| = \cdots = |x_k| > |x_r| \text{ for all } r > k \quad \Longrightarrow \quad |y_1| = \cdots = |y_k| \geq |y_r| \text{ for all } r > k. \tag{2.2}$$

Let $\mathbf{x} = (1, \dots, 1)^{\mathrm{t}}$ and $\mathbf{y} = A^{\mathrm{t}}\mathbf{x}$. If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are the rows of A^{t}, then

$$\begin{aligned}\|\mathbf{v}_1\|_1 &= a_{11} + \cdots + a_{n1} = |y_1| = |y_j| = |a_{1j} + \cdots + a_{nj}| \\ &\le |a_{1j}| + \cdots + |a_{nj}| = \|\mathbf{v}_j\|_1 \le \|\mathbf{v}_1\|_1 \quad \text{for } j \ge 1.\end{aligned}$$

Since $a_{jj} > 0$, we see that $a_{ij} \ge 0$ for all $i \ne j$, and all row sums of A are equal, say, to $s > 0$.

Taking $\mathbf{x} = (1, 1, 0, \dots, 0)^{\mathrm{t}}$, with (2.2) we have $a_{11} + a_{21} = a_{12} + a_{22}$. Similarly, taking $\mathbf{x} = (1, -1, 0, \dots, 0)^{\mathrm{t}}$, we have $|a_{11} - a_{21}| = |a_{12} - a_{22}|$. It follows from either the assumption $a_{11} > a_{12}$ and $a_{22} > a_{21}$, or the assumption $a_{11} > a_{21}$ and $a_{22} > a_{12}$ that $a_{11} - a_{21} = a_{22} - a_{12}$. Consequently, $a_{11} = a_{22}$ and $a_{12} = a_{21}$. The assertion follows when $n = 2$.

Suppose $n \ge 3$. Apply the same argument to other pairs (i, j) with $i \ne j$ instead of $(1, 2)$, we see that

$$a_{11} = \cdots = a_{nn} \quad \text{and} \quad a_{jk} = a_{kj} \quad \text{whenever } j \ne k.$$

For a fixed $j = 1, \dots, n$, we take $\mathbf{x} = (1, \dots, 1, \underbrace{-1}_{j\text{th}}, 1, \dots, 1)^{\mathrm{t}}$ and $\mathbf{y} = A^{\mathrm{t}}\mathbf{x}$. For distinct indices j, k, l, we have $|y_l| = s - 2a_{jl} = |y_k| = s - 2a_{jk}$. It follows $a_{jl} = a_{jk} = a_{kj}$, and thus $a_{jk} = a_{12}$ are all equal for $j \ne k$. Consider $\mathbf{u} = (1, -1, \dots, -1)^{\mathrm{t}}$ and $\mathbf{v} = A^{\mathrm{t}}\mathbf{u} = (v_1, v_2, \dots, v_n)^{\mathrm{t}}$. Then $|v_1| = |v_2|$ implies either

$$a_{11} - (n-1)a_{12} = a_{11} + (n-3)a_{12} \quad \text{or} \quad (n-1)a_{12} - a_{11} = a_{11} + (n-3)a_{12}.$$

Since $n \ge 3$, either $a_{12} = 0$ or $a_{11} = a_{12}$. But $a_{11} > a_{12}$. This implies that $A = a_{11} I_n$. ∎

We are now ready to present the structure theorem of ℓ_∞-norm parallel/TEA pair preservers.

Theorem 2.6. *Let $T : (\mathbb{F}^n, \|\cdot\|_\infty) \to (\mathbb{F}^n, \|\cdot\|_\infty)$ be a linear map.*

(a) *T preserves parallel pairs if and only if there is $\gamma \ge 0$ and a generalized permutation matrix Q such that one of the following forms holds:*
   (a.1) *T has the form $\mathbf{x} \mapsto \gamma Q\mathbf{x}$,*
   (a.2) *$n = 2$ and T has the form $\mathbf{x} \mapsto \gamma CQ\mathbf{x}$, where $C = \begin{pmatrix} 1 & \beta \\ \bar{\beta} & 1 \end{pmatrix}$ for some scalar β with $|\beta| < 1$.*
   (a.3) *$T = \mathbf{v}\mathbf{u}^{\mathrm{t}}$ for some nonzero column vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}^n$.*

(b) *T preserves TEA pairs if and only if one of the following holds.*
   (b.1) *T has the form in* (a.1).
   (b.2) *$\mathbb{F}^n = \mathbb{R}^2$ and T has the form in* (a.2).
   (b.3) *$\mathbb{F}^n = \mathbb{R}^2$ and T has the form in* (a.3)*, where $\mathbf{u} = (u_1, u_2)^{\mathrm{t}}$ with $|u_1| = |u_2|$.*

Proof. (a) It is clear that T preserves parallel pairs if T has the form in (a.1) or (a.3). Suppose $n = 2$ and T has the form in (a.2). Then, T preserves parallel pairs in $(\mathbb{F}^2, \|\cdot\|_\infty)$ by the implication from (b) to (a) in Lemma 2.5; indeed, with a continuity argument the condition (2.1) implies that

$$\|\mathbf{x}\|_\infty = |x_r| \ge |x_s| \text{ whenever } s \ne r \quad \Longrightarrow \quad \|\mathbf{y}\|_\infty = |y_r| \ge |y_s| \text{ whenever } s \ne r.$$

Conversely, suppose T preserves parallel pairs. If T is not invertible then it follows from Lemma 2.4 that T is either the zero map or has the form in (a.3). Suppose from now on T is invertible.

If $T^{-1}\mathbf{e}_j = \mathbf{x}_j$, then $\mathbf{x}_i$ and $\mathbf{x}_j$ cannot be parallel for any $i \neq j$. Otherwise, $T\mathbf{x}_i = \mathbf{e}_i$ and $T\mathbf{x}_j = \mathbf{e}_j$ were parallel. Thus, the vectors $\mathbf{x}_i$ and $\mathbf{x}_j$ cannot attain the ℓ_∞-norm at a same coordinate. So, there is a permutation σ on $\{1, \ldots, n\}$ such that each $\mathbf{x}_j$ attains its norm at its $\sigma(j)$th coordinate but no other. Consequently, there is a generalized permutation matrix $Q \in \mathbf{M}_n$ such that the map L defined by $z \mapsto QT^{-1}z$ will send $\mathbf{e}_j$ to a vector $\mathbf{y}_j = (a_{j1}, \ldots, a_{jn})^{\mathrm{t}}$ such that $a_{jj} > |a_{ji}|$ for all $i \neq j$. Clearly, L^{-1} defined by $\mathbf{y} \mapsto T(Q^{-1}\mathbf{y})$ preserves parallel pairs.

Let $A = (a_{ij}) \in \mathbf{M}_n$ so that $L(\mathbf{x}) = A^{\mathrm{t}}\mathbf{x}$. If $\mathbf{x}_j = (x_{j1}, \ldots, x_{jn})^{\mathrm{t}}$ satisfies $|x_{jj}| > |x_{ji}|$ for all $i \neq j$, then we claim that $L(\mathbf{x}_j) = A^{\mathrm{t}}\mathbf{x}_j$ is parallel to $\mathbf{y}_j$ and thus $\mathbf{e}_j$, but not any other $\mathbf{y}_i$. Otherwise, the fact $L(\mathbf{x}_j)$ is parallel to $\mathbf{y}_i$ for some $i \neq j$ would imply that $\mathbf{x}_j$ is parallel to $L^{-1}(\mathbf{y}_i) = \mathbf{e}_i$, which is impossible. Thus, the matrix $A = (a_{ij})$ satisfies the hypothesis (a) of Lemma 2.5. If $n \geq 3$, we see that $A = \gamma I_n$ with $\gamma = a_{11} > 0$. If $n = 2$, we see that A is Hermitian with $a_{11} = a_{22}$. So, T has the form in (a.2).

(b) It is clear that if T assumes the form in (b.1) then it preserves TEA pairs. It is also the case if T assumes the form in (b.3) by direct verification. Suppose $\mathbb{F}^n = \mathbb{R}^2$ and $T = \gamma CQ$ as in (a.2). Clearly, T satisfies (1.1) if and only if C satisfies (1.1). There is $S = \operatorname{diag}(1, \pm 1)$ such that SCS has the form $\hat{C} = \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}$ with $0 \leq \beta < 1$. The map $\mathbf{x} \mapsto C\mathbf{x}$ satisfies (1.1) if and only if the map $\mathbf{x} \mapsto \hat{C}\mathbf{x}$ does. Now, if the nonzero vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^2$ satisfy that $\|\mathbf{x} + \mathbf{y}\|_\infty = \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty$, then we may replace $(\mathbf{x}, \mathbf{y})$ by $(\xi\mathbf{x}/\|\mathbf{x}\|_\infty, \xi\mathbf{y}/\|\mathbf{y}\|_\infty)$ with $\xi \in \{-1, 1\}$ and assume that $\mathbf{x} = (1, x_2)^{\mathrm{t}}, \mathbf{y} = (1, y_2)^{\mathrm{t}}$ with $|x_2|, |y_2| \leq 1$, or $\mathbf{x} = (x_1, 1)^{\mathrm{t}}, \mathbf{y} = (y_1, 1)^{\mathrm{t}}$ with $|x_1|, |y_1| \leq 1$. One can check that $\|\hat{C}\mathbf{x} + \hat{C}\mathbf{y}\|_\infty = \|\hat{C}\mathbf{x}\|_\infty + \|\hat{C}\mathbf{y}\|_\infty$. Thus, T preserves TEA as well.

Conversely, suppose the map T is nonzero and satisfies (1.1). Then T will preserve parallel pairs. Thus, it will be of the form (a.1), (a.2), or (a.3). We will show that (a.2) is impossible in the complex case unless it reduces to the form in (a.1), and there are additional restrictions for $\mathbf{u}$ if (a.3) holds.

Suppose $\mathbb{F}^n = \mathbb{C}^2$ and $T = \gamma CQ$ has the form in (a.2) in which $C = \begin{pmatrix} 1 & \beta \\ \overline{\beta} & 1 \end{pmatrix}$ for some complex scalar β with $|\beta| < 1$. In this case, the map $\mathbf{x} \mapsto Cx$ also preserves TEA pairs. Consider $\mathbf{x} = (1, 0)^{\mathrm{t}}$, $\mathbf{y} = (1, 1)^{\mathrm{t}}$ and $z = (1, i)^{\mathrm{t}}$, and their images $Cx = (1, \overline{\beta})^{\mathrm{t}}$, $Cy = (1 + \beta, \overline{\beta} + 1)^{\mathrm{t}}$ and $Cz = (1 + i\beta, \overline{\beta} + i)^{\mathrm{t}}$. Note that $\mathbf{x}, \mathbf{y}$ and $\mathbf{x}, \mathbf{z}$ are both TEA pairs, while Cx, Cy and Cx, Cz form TEA pairs exactly when the first coordinates of Cy and Cz assume positive values. This forces $\beta = 0$, and thus $T = \gamma Q$ reduces to the form in (a.1).

Suppose $T = \mathbf{v}\mathbf{u}^{\mathrm{t}}$ assumes the form in (a.3) for some vectors $\mathbf{u} = (u_1, u_2)^{\mathrm{t}}$ and $\mathbf{v}$ in $\mathbb{F}^n$. In this case, T is not invertible. By Lemma 2.4, $\mathbb{F}^n = \mathbb{R}^2$. Since $y_1 = (1, 1)^{\mathrm{t}}$ and $\mathbf{y}_2 = (1, -1)^{\mathrm{t}}$ form a TEA pair, so are $T\mathbf{y}_1 = (u_1 + u_2)\mathbf{v}$ and $T\mathbf{y}_2 = (u_1 - u_2)\mathbf{v}$. This forces $u_1 + u_2$ and $u_1 - u_2$ have the same sign. Similarly, $\mathbf{y}_1$ and $-\mathbf{y}_2$ also form a TEA pair, and thus $u_1 + u_2$ and $-u_1 + u_2$ also have the same sign. If $u_1 + u_2 \neq 0$ then $u_2 = u_1$. In any case, we have $|u_1| = |u_2|$ as asserted. ∎

Corollary 2.7. *Let $n \geq 3$. The following conditions are equivalent to each other for a nonzero linear map $T : (\mathbb{F}^n, \|\cdot\|_\infty) \to (\mathbb{F}^n, \|\cdot\|_\infty)$.*

(a) *T preserves TEA pairs.*
(b) *T preserves parallel pairs and its range space has dimension larger than one.*
(c) *There is $\gamma > 0$ and a generalized permutation matrix $Q \in \mathbf{M}_n$ such that T has the form $\mathbf{x} \mapsto \gamma Q\mathbf{x}$.*

3. The infinite dimensional discrete case

Let the underlying field $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$, and let Λ be a finite or an infinite index set. When $1 \leq p < \infty$, let $\ell_p(\Lambda)$ be the (real or complex) Banach space of p-summable families $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda}$ (of real or complex numbers) with ℓ_p–norm

$$\|\mathbf{x}\|_p = \left(\sum_{\lambda\in\Lambda} |x_\lambda|^p\right)^{1/p} < +\infty.$$

Note that the above sum is finite only if there are at most countably many coordinates $x_\lambda \neq 0$. Let $\ell_\infty(\Lambda)$ be the Banach space of uniformly bounded family $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda}$ with the ℓ_∞–norm

$$\|\mathbf{x}\|_\infty = \sup_{\lambda\in\Lambda} |x_\lambda| < +\infty.$$

We are also interested in the Banach subspace $c_0(\Lambda)$ of $\ell_\infty(\Lambda)$ consisting of essentially null families $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda}$ for which for any $\epsilon > 0$ there are at most finitely many coordinates x_λ with $|x_\lambda| \geq \epsilon$. Note that all $\ell_1(\Lambda), c_0(\Lambda), \ell_\infty(\Lambda)$ are Banach lattices. We write $f = (f_\lambda) \geq 0$ when all coordinates $f_\lambda \geq 0$.

It is plain that the vector spaces satisfy

$$\ell_1(\Lambda) \subseteq \ell_p(\Lambda) \subseteq c_0(\Lambda) \subseteq \ell_\infty(\Lambda), \quad \text{whenever } 1 < p < \infty.$$

When Λ is a finite set, all above spaces coincide; otherwise, all inclusions are proper. We write ℓ_p and c_0 for $\ell_p(\mathbb{N})$ and $c_0(\mathbb{N})$ as usual.

As in the finite dimensional case, the ℓ_p–norm is strictly convex when $1 < p < \infty$. Two nonzero $\mathbf{x}, \mathbf{y}$ in $\ell_p(\Lambda)$ are parallel (resp. TEA), exactly when there is a scalar t (resp. $t > 0$) such that $\mathbf{x} = t\mathbf{y}$. Therefore, any linear map of $\ell_p(\Lambda)$ preserves parallel pairs and TEA pairs when $1 < p < \infty$.

We study the cases when $p = 1$ and $p = \infty$, and Λ is an *infinite* index set below. As in Section 2, we start with the following observations. Note that $\ell_\infty(\Lambda)$ is isometrically isomorphic to the Banach space $C(\beta\Lambda)$ of continuous functions on the Stone-Cech compactification $\beta\Lambda$ of Λ, which consists of all ultrafilters of the discrete space Λ.

Lemma 3.1. (a) *$\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda}$ and $\mathbf{y} = (y_\lambda)_{\lambda\in\Lambda}$ in $\ell_1(\Lambda)$ are parallel (resp. TEA) with respect to the ℓ_1–norm if and only if there is a unimodular scalar ξ (resp. $\xi = 1$) such that $\xi\overline{x_\lambda}y_\lambda \geq 0$ for all λ in Λ.*

(b) $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda}$ *and* $\mathbf{y} = (y_\lambda)_{\lambda\in\Lambda}$ *in* $c_0(\Lambda)$ *are parallel (resp. TEA) with respect to the* ℓ_∞*–norm if and only if there is a unimodular scalar* ξ *(resp.* $\xi = 1$*) such that* $\xi\overline{x_\lambda}y_\lambda = \|\mathbf{x}\|_\infty\|\mathbf{y}\|_\infty$ *for an index* λ *in* Λ*.*

(c) $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda}$ *and* $\mathbf{y} = (y_\lambda)_{\lambda\in\Lambda}$ *in* $\ell_\infty(\Lambda)$ *are parallel (resp. TEA) with respect to the* ℓ_∞*–norm if and only if there is a unimodular scalar* ξ *(resp.* $\xi = 1$*) such that* $\lim_{\mathfrak{U}} \xi\overline{x_\lambda}y_\lambda = \|\mathbf{x}\|_\infty\|\mathbf{y}\|_\infty$ *for an ultrafilter* $\mathfrak{U}$ *on* Λ*.*

For each $\alpha \in \Lambda$, let $\mathbf{e}_\alpha = (e_{\alpha,\lambda})_{\lambda\in\Lambda}$ be the α–coordinate vector with coordinates $e_{\alpha,\lambda} = 1$ when $\alpha = \lambda$ and 0 elsewhere. We can identify any bounded linear map T of $\ell_1(\Lambda)$ or $c_0(\Lambda)$ as the infinite "matrix" $(t_{\alpha\beta})$ with $t_{\alpha\beta} = \mathbf{e}_\alpha^{\mathrm{t}}(T\mathbf{e}_\beta)$, where $\mathbf{e}_\alpha^{\mathrm{t}}$ denotes the linear functional $(x_\lambda)_{\lambda\in\Lambda} \mapsto x_\alpha$ for any $\alpha \in \Lambda$. However, there are unbounded linear maps such that their representation "matrices" are zero. For example, consider any unbounded linear functional f of c_0 vanishing on the subspace of all finite sequences, that is, $f(\mathbf{e}_n) = 0$ for all $n = 1, 2, \ldots$. Then the unbounded linear map $\mathbf{x} \mapsto f(\mathbf{x})\mathbf{e}_1$ has zero "matrix".

On the other hand, the Banach dual space of $\ell_1(\Lambda)$ is $\ell_\infty(\Lambda)$ when one identify $\mathbf{u} = (u_\lambda)_{\lambda\in\Lambda} \in \ell_\infty(\Lambda)$ with the bounded linear functional $\mathbf{u}^{\mathrm{t}} = \sum_{\lambda\in\Lambda} u_\lambda \mathbf{e}_\lambda^{\mathrm{t}}$ (converging in the weak* topology $\sigma(\ell_\infty(\Lambda), \ell_1(\Lambda))$. In this case, a nonzero bounded linear operator S of $\ell_\infty(\Lambda)$ can have zero representation "matrix". For example, let g be any nonzero bounded linear functional of ℓ_∞ vanishing on the essential null sequence space c_0, and $S\mathbf{x} = g(\mathbf{x})\mathbf{e}_1$. However, a $\sigma(\ell_\infty(\Lambda), \ell_1(\Lambda))$–$\sigma(\ell_\infty(\Lambda), \ell_1(\Lambda))$ continuous linear map is determined by its representation "matrix".

Recall also that $\ell_1(\Lambda)$ is a Banach lattice with respect to the pointwise ordering. In particular, a vector $f \geq 0$ if all its coordinates $f_\lambda = f(\lambda) \geq 0$. It is well known that any positive linear operator T between Banach lattices, that is $Tf \geq 0$ whenever $f \geq 0$, is automatic bounded.

Lemma 3.2. *Every linear map* $T : \ell_1(\Lambda) \to \ell_1(\Lambda)$ *preserves TEA pairs is automatic bounded.*

Proof. For each $\lambda \in \Lambda$, consider the linear functional T_λ of $\ell_1(\Lambda)$ defined by $T_\lambda(f) = Tf(\lambda)$. Suppose $Te(\lambda) = 0$ for all positive, and thus arbitrary, $e \in \ell_1(\Lambda)$. It is plain that the linear functional $T_\lambda = 0$. In case when $Te(\lambda) \neq 0$ for some positive vector $e \in \ell_1(\Lambda)$. Replacing T by $\overline{Te}T(\cdot)$ (pointwise product), we can assume that $Te(\lambda) > 0$. For any positive vector $f \in \ell_1(\Lambda)$, since e, f form a TEA pair, so do Te, Tf. In particular, $Tf(\lambda) \geq 0$ since $Te(\lambda) > 0$. This shows that the linear functional T_λ is positive, and thus bounded.

Let $f_n \xrightarrow{\|\cdot\|} 0$ and $Tf_n \xrightarrow{\|\cdot\|} g$. Then $Tf_n(\lambda) = T_\lambda(f_n) \longrightarrow T_\lambda(0) = 0$ implies $g = 0$. It then follows from the closed graph theorem that T is automatic bounded. ∎

Example 3.3. Let $\mathbf{e} \in \ell_1$ with all positive coordinates. Let φ be a nonzero (necessarily unbounded) linear functional of ℓ_1 such that $\varphi(\mathbf{e}_n) = 0$ for all $n = 1, 2, \ldots$, but $\varphi(\mathbf{e}) = 1$. The rank one *unbounded* linear operator $T\mathbf{x} = \varphi(\mathbf{x})\mathbf{e}_1$ sends parallel pairs to parallel pairs. Note that the representation 'matrix' of T is the zero matrix. However, T does not preserve TEA pairs. In fact, if T does then φ will be a positive linear functional, and thus bounded, an absurdity.

Problem 3.4. Does every unbounded parallel pair preserver of $\ell_1(\Lambda)$ have rank one?

Using the terminology in the finite dimensional case, we call a "matrix" $U = (u_{\alpha\beta})$ a "*monomial matrix*" if for each $\alpha \in \Lambda$ there is exactly one $\beta \in \Lambda$ such that $u_{\alpha\beta} \neq 0$. A "monomial matrix" U is a "*generalized permutation matrix*" if all its nonzero entries $|u_{\alpha\beta}| = 1$, and it is a "*diagonal unitary matrix*" if $|u_{\alpha\alpha}| = 1$ for each $\alpha \in \Lambda$. We also assume that the linear map U is bounded or $\sigma(\ell_\infty(\Lambda), \ell_1(\Lambda))$–$\sigma(\ell_\infty(\Lambda), \ell_1(\Lambda))$ continuous, depending on the context, so that the representation "*monomial matrix*" $(u_{\alpha\beta})$ determines U. It is clear that a linear map T of $\ell_1(\Lambda)$ (resp. $c_0(\Lambda)$ or $\ell_\infty(\Lambda)$) preserves parallel pairs or TEA pairs if, and only if, γUTV does whenever $\gamma > 0$, and U, V are some invertible "monomial matrices" (resp. "generalized permutation matrices").

Theorem 3.5. *Let $T = (t_{\alpha\beta}) : \ell_1(\Lambda) \to \ell_1(\Lambda)$ be a bounded linear map.*

(a) *T preserves TEA pairs if and only if for each $\alpha \in \Lambda$ there is at most one $\beta \in \Lambda$ such that $t_{\alpha\beta} \neq 0$.*

(b) *T preserves parallel pairs if and only if T preserves TEA, or $T = \mathbf{v}\mathbf{u}^{\mathrm{t}}$ for some $\mathbf{u} \in \ell_\infty(\Lambda)$ and $\mathbf{v} \in \ell_1(\Lambda)$.*

Proof. (a) Suppose for each $\alpha \in \Lambda$ there is at most one $\beta \in \Lambda$ such that $t_{\alpha\beta} \neq 0$. Write such $\beta = \alpha'$ in this case. Let $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda}, \mathbf{y} = (y_\lambda)_{\lambda\in\Lambda}$ be a TEA pair in $\ell_1(\Lambda)$. By Lemma 3.1, $\overline{x_\lambda} y_\lambda \geq 0$ for all $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, the λ-coordinate of $T\mathbf{x}$ and $T\mathbf{y}$ are $t_{\lambda\lambda'} x_{\lambda'}$ and $t_{\lambda\lambda'} y_{\lambda'}$, respectively. Since $\overline{t_{\lambda\lambda'} x_{\lambda'}} t_{\lambda\lambda'} y_{\lambda'} = \overline{x_{\lambda'}} y_{\lambda'} |t_{\lambda\lambda'}|^2 \geq 0$ for all $\lambda \in \Lambda$, we see that $T\mathbf{x}, T\mathbf{y}$ form a TEA pair. The converse follows from exactly the same arguments for the finite dimensional case given in the proof of Theorem 2.2(a).

(b) It suffices to verify the necessity for the case when T does not preserve TEA. Suppose T preserves parallel pairs, and the α_1-row $(t_{\alpha_1\beta})_{\beta\in\Lambda}$ of its matrix representation has more than one nonzero entry. Suppose the range of T has dimension at least two, and thus there is another α_2–row $(t_{\alpha_2\beta})_{\beta\in\Lambda}$ of T linearly independent from the α_1-row. We are going to derive a contradiction.

Since the α_1–row and the α_2–row of T are linearly independent, there are distinct indices β_1, β_2 such that the 2×2 matrix

$$\begin{pmatrix} t_{\alpha_1\beta_1} & t_{\alpha_1\beta_2} \\ t_{\alpha_2\beta_1} & t_{\alpha_2\beta_2} \end{pmatrix}$$

is invertible. If both $t_{\alpha_1\beta_1}, t_{\alpha_2\beta_1}$, or both $t_{\alpha_1\beta_2}, t_{\alpha_2\beta_2}$, are nonzero, then by replacing T with PTQ for some suitable invertible "monomial matrix" P, Q, we can assume that the above matrix assumes the form

$$\begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix}$$

for some scalar $a \neq 1$. Then the argument in Case 1 of the proof of Theorem 2.2(b) derives a desired contradiction. If $t_{\alpha_1\beta_2} = t_{\alpha_2\beta_1} = 0$, say, then we search for an other index β_3 with $t_{\alpha_1\beta_3} \neq 0$, and such β_3 exists by assumption. If $t_{\alpha_2\beta_3} \neq 0$, then it comes back to the first case above, and we are done. Suppose $t_{\alpha_2\beta_3} = 0$. By replacing T with $P'TQ'$ for some suitable invertible "monomial matrices" P', Q', we may assume that T has a "submatrix" of the form

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}$$

with $a \neq 0$ for the indices α_1, α_2, and $\beta_1, \beta_2, \beta_3$. Then the argument in Case 2 of the proof of Theorem 2.2(b) provides us a contradiction, as well. ■

Lemma 3.6. *Let Λ be an infinite index set.*

(a) *If $\mathbf{V}$ is a subspace of $\ell_\infty(\Lambda)$ such that any two elements in $\mathbf{V}$ are parallel with respect to $\|\cdot\|_\infty$, then $\dim \mathbf{V} \leq 1$.*

(b) *Let $T : \ell_\infty(\Lambda) \to \ell_\infty(\Lambda)$ or $T : c_0(\Lambda) \to c_0(\Lambda)$ be a nonzero linear map. Then T is injective if one of the following holds.*
   (i) *T preserves TEA pairs.*
   (ii) *T preserves parallel pairs with range space of dimension larger than one.*

Proof. (a) It follows from the proof of Lemma 2.3.

(b) We discuss only the case T is a linear map of $\ell_\infty(\Lambda)$, since the other case is similar.

(i) Let T be a linear TEA pair preserver of $\ell_\infty(\Lambda)$ such that $T(\mathbf{v}) = 0$ for some $\mathbf{v} = (v_\lambda)_{\lambda\in\Lambda}$ with $\|\mathbf{v}\|_\infty = \sup_{\lambda\in\Lambda} |v_\lambda| = 1$. We will show that $T = 0$.

Suppose $\Lambda' = \{\lambda \in \Lambda : |v_\lambda| < 1\} \neq \emptyset$. For any $\lambda' \in \Lambda'$, since $\xi\mathbf{v} + \mathbf{e}_{\lambda'}$ and $\xi\mathbf{v} - \mathbf{e}_{\lambda'}$ form a TEA pair for large $\xi > 0$, so do $T(\xi\mathbf{v} + \mathbf{e}_{\lambda'}) = T(\mathbf{e}_{\lambda'})$ and $T(\xi\mathbf{v} - \mathbf{e}_{\lambda'}) = -T(\mathbf{e}_{\lambda'})$. It forces $T(\mathbf{e}_{\lambda'}) = 0$. For any $\mathbf{u} = \sum_{\lambda\neq\lambda'} v_\lambda \mathbf{e}_\lambda$ with zero λ'–coordinate, we have $\xi\mathbf{e}_{\lambda'} + \mathbf{u}$ and $\xi\mathbf{e}_{\lambda'} - \mathbf{u}$ form a TEA pair for large $\xi > 0$, and so do $T(\xi\mathbf{e}_{\lambda'} + \mathbf{u}) = T(\mathbf{u})$ and $T(\xi\mathbf{e}_{\lambda'} - \mathbf{u}) = -T(\mathbf{u})$. It follows $T(\mathbf{u}) = 0$. In general, for any $\mathbf{x} = x_{\lambda'}\mathbf{e}_{\lambda'} + \mathbf{u}$ such that $\mathbf{u} = \mathbf{x} - x_{\lambda'}\mathbf{e}_{\lambda'}$ has zero λ'–coordinate,

$$T(\mathbf{x}) = x_{\lambda'}T(\mathbf{e}_{\lambda'}) + T(\mathbf{u}) = 0.$$

Hence $T = 0$. Therefore, we may assume $\Lambda' = \emptyset$.

Replacing T with TQ for some suitable "generalized permutation matrix" Q, we may further assume that $\mathbf{v} = \mathbf{e}_\Lambda$, that is all coordinates of $\mathbf{v}$ is 1. For any distinct $\lambda_1, \lambda_2 \in \Lambda$, since $\mathbf{e}_\Lambda - \mathbf{e}_{\lambda_2}$ and $\mathbf{e}_{\lambda_1} + \mathbf{e}_{\lambda_2}$ form a TEA pair, so do $T(\mathbf{e}_\Lambda - \mathbf{e}_{\lambda_2}) = -T(\mathbf{e}_{\lambda_2})$ and $T(\mathbf{e}_{\lambda_1} + \mathbf{e}_{\lambda_2}) = -T(\mathbf{e}_\Lambda - \mathbf{e}_{\lambda_1} - \mathbf{e}_{\lambda_2})$, since $T(\mathbf{e}_\Lambda) = 0$. Therefore,

$$\|T(\mathbf{e}_{\lambda_1})\|_\infty = \|T(\mathbf{e}_\Lambda - \mathbf{e}_{\lambda_1})\|_\infty = \|T(\mathbf{e}_{\lambda_2})\|_\infty + \|T(\mathbf{e}_\Lambda - \mathbf{e}_{\lambda_1} - \mathbf{e}_{\lambda_2})\|_\infty$$

for any distinct $\lambda_1, \lambda_2 \in \Lambda$. Exchanging the roles of λ_1 and λ_2, we see that

$$T(\mathbf{e}_\Lambda - \mathbf{e}_{\lambda_1} - \mathbf{e}_{\lambda_2}) = 0.$$

Replacing $\mathbf{v} = \mathbf{e}_\Lambda$ with $\mathbf{v}' = \mathbf{e}_\Lambda - \mathbf{e}_{\lambda_1} - \mathbf{e}_{\lambda_2} \neq 0$ (since Λ has more than two elements), and arguing as above, we see that $\Lambda' = \{\lambda_1, \lambda_2\} \neq \emptyset$, and then $T = 0$.

(ii) Suppose T preserves parallel pairs. Assume T is not injective, and $\mathbf{v} = (v_\lambda)_{\lambda\in\Lambda}$ is a norm one element such that $T\mathbf{v} = 0$. Let $\Lambda' = \{\lambda \in \Lambda : |v_\lambda| < 1\}$. For any $\mathbf{x}, \mathbf{y} \in \ell_\infty(\Lambda)$ such that their λ–coordinates $\mathbf{e}_\lambda^{\mathrm{t}}\mathbf{x} = \mathbf{e}_\lambda^{\mathrm{t}}\mathbf{y} = 0$ for all λ outside Λ', we see that $\xi\mathbf{v} + \mathbf{x}$ and $\xi\mathbf{v} + \mathbf{y}$ are parallel for large $\xi > 0$, and so are $T(\xi\mathbf{v} + \mathbf{x}) = T(\mathbf{x})$ and $T(\xi\mathbf{v} + \mathbf{y}) = T(\mathbf{y})$. It follows from part (a) that the space $\{T(\mathbf{x}) : \mathbf{e}_\lambda^{\mathrm{t}}\mathbf{x} = 0 \text{ for all } \lambda \text{ outside } \Lambda'\}$ has dimension at most one.

If $\Lambda = \Lambda'$ then we are done. If $\Lambda'' = \Lambda \setminus \Lambda'$ is nonempty, then by replacing T with TQ for some "generalized permutation matrix" Q, we may assume that $v_\lambda = 1$ for all $\lambda \in \Lambda''$. Then the proof of Lemma 2.4(b) shows that the range of T has dimension at most one. ∎

We note that unlike the finite dimensional case, an injective parallel/TEA pair linear preserver of $c_0(\Lambda)$ or $\ell_\infty(\Lambda)$ can be non-surjective. For an example, consider the isometric right shift operator L of ℓ_∞ or c_0 by sending $\mathbf{e}_n$ to $\mathbf{e}_{n+1}$ for $n = 1, 2, \ldots$. However, such preservers are automatical continuous as shown in the following result.

Theorem 3.7. *Let Λ be an infinite index set. Let $T : c_0(\Lambda) \to c_0(\Lambda)$ be a nonzero linear map. The following conditions are equivalent to each other.*

(a) *$T(\mathbf{u}), T(\mathbf{v})$ is a TEA pair if and only if $\mathbf{u}, \mathbf{v}$ is a TEA pair, for any $\mathbf{u}, \mathbf{v} \in c_0(\Lambda)$.*
(b) *$T(\mathbf{u}), T(\mathbf{v})$ is a parallel pair if and only if $\mathbf{u}, \mathbf{v}$ is a parallel pair, for any $\mathbf{u}, \mathbf{v} \in c_0(\Lambda)$.*
(c) *T is a scalar multiple of a (not necessarily surjective) linear isometry.*

In this case, there is $\gamma > 0$, a subset Λ_1 of Λ, a family $\{\xi_\lambda : \lambda \in \Lambda_1\}$ of unimodular scalars, and a surjective map $\tau : \Lambda_1 \to \Lambda$ such that for any $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda} \in c_0(\Lambda)$, the image $\mathbf{y} = T(\mathbf{x}) = (y_\lambda)_{\lambda\in\Lambda}$ has coordinates

$$y_\beta = \gamma \xi_\beta x_{\tau(\beta)} \quad \textit{for all } \beta \in \Lambda_1, \tag{3.1}$$

and

$$|y_{\beta'}| \leq \gamma \quad \textit{when } \beta' \in \Lambda \setminus \Lambda_1. \tag{3.2}$$

Proof. The implications (c) $\implies$ (a) $\implies$ (b) are plain. We are verifying (b) $\implies$ (c). Note that T is injective. Indeed, if $T(\mathbf{x}) = 0$ for some nonzero $\mathbf{x} \in c_0(\Lambda)$, then the fact $T(\mathbf{x})$ and $T(\mathbf{e}_\lambda)$ are parallel would imply that $\mathbf{x}$ and $\mathbf{e}_\lambda$ are parallel for every $\lambda \in \Lambda$. But this contradicts to the fact that Λ is infinite and $\mathbf{x}$ is essentially null. In particular, for any nonzero $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda} \in c_0(\Lambda)$, its peak set

$$\mathrm{Pk}(\mathbf{x}) = \{\lambda \in \Lambda : |x_\lambda| = \|\mathbf{x}\|_\infty\}.$$

is a nonempty proper subset of Λ.

For any $\alpha, \beta \in \Lambda$, we claim that

$$\mathrm{Pk}(T(\mathbf{e}_\alpha)) \cap \mathrm{Pk}(T(\mathbf{e}_\beta)) = \emptyset \quad \text{whenever } \alpha \neq \beta.$$

In fact, if $\lambda \in \mathrm{Pk}(T(\mathbf{e}_\alpha)) \cap \mathrm{Pk}(T(\mathbf{e}_\beta))$ then both $T(\mathbf{e}_\alpha)$ and $T(\mathbf{e}_\beta)$ attain their norms at the λ-coordinate, and thus they are parallel. This forces $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$ are parallel, a contradiction.

Consider the disjoint union

$$\Lambda_1 = \bigcup_{\lambda\in\Lambda} \mathrm{Pk}(T(\mathbf{e}_\lambda)).$$

We define a surjective map $\tau : \Lambda_1 \to \Lambda$ such that

$$\tau(\beta) = \alpha \quad \text{if and only if} \quad \beta \in \mathrm{Pk}(T(\mathbf{e}_\alpha)).$$

There is a unimodular scalar ξ_λ such that the norm attaining λ–coordinate of $\overline{\xi_\lambda}T(\mathbf{e}_{\tau(\lambda)})$ is positive for every λ in Λ_1. Replacing T with QT for a suitable "diagonal unitary matrix" Q, we can assume that all $\xi_\lambda = 1$.

Let $\alpha_1 = \tau(\beta_1), \alpha_2 = \tau(\beta_2)$ and $\alpha_3 = \tau(\beta_3)$ be three distinct indices for β_1, β_2 and β_3 in Λ_1. Consider the linear map $L : \operatorname{span}(\mathbf{e}_{\alpha_1}, \mathbf{e}_{\alpha_2}, \mathbf{e}_{\alpha_3}) \to \operatorname{span}(\mathbf{e}_{\beta_1}, \mathbf{e}_{\beta_2}, \mathbf{e}_{\beta_3})$ defined by taking only the β_1–, β_2– and β_3–coordinates of $T(\mathbf{x})$ when $\mathbf{x} \in \operatorname{span}(\mathbf{e}_{\alpha_1}, \mathbf{e}_{\alpha_2}, \mathbf{e}_{\alpha_3})$. We can identify L with a 3×3 matrix A satisfying the assumption in Lemma 2.5, from which we have a positive γ such that $L(\mathbf{e}_{\alpha_j}) = \gamma \mathbf{e}_{\beta_j}$ for $j = 1, 2, 3$.

The above argument shows that for any α in Λ, the β-coordinate of $T(\mathbf{e}_\alpha)$ is a fixed nonzero scalar γ whenever $\tau(\beta) = \alpha$, and all the other β'-coordinates with $\beta' \in \Lambda_1$ are zero. In other words,

$$T(\mathbf{e}_\alpha) = \gamma \sum_{\tau(\beta)=\alpha} \mathbf{e}_\beta + \mathbf{t}_\alpha \quad \text{for every } \alpha \in \Lambda, \tag{3.3}$$

where $\mathbf{t}_\alpha \in c_0(\Lambda \setminus \Lambda_1)$, that is, all λ-coordinate of $\mathbf{t}_\alpha$ with $\lambda \in \Lambda_1$ are zero. Since the peak set $\operatorname{Pk}(T(\mathbf{e}_\alpha)) \subseteq \Lambda_1$, we have $\|\mathbf{t}_\alpha\|_\infty < \gamma$. Note that the above sum must be finite, as $T(\mathbf{e}_\alpha)$ is essentially null. Replacing T by T/γ, we can assume that $\gamma = 1$, and thus all $\|T(\mathbf{e}_\alpha)\|_\infty = 1$.

We claim that $\|T(\mathbf{x})\|_\infty \leq 1$ whenever $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda}$ in $c_0(\Lambda)$ has norm one. To see this, we first assume that $x_{\alpha_1} = 1$ and $x_{\alpha_2} = x_{\alpha_3} = 0$ with $\alpha_1 = \tau(\beta_1)$, $\alpha_2 = \tau(\beta_2)$, $\alpha_3 = \tau(\beta_3)$ for distinct indices $\alpha_1, \alpha_2, \alpha_3 \in \Lambda$ and $\beta_1, \beta_2, \beta_3 \in \Lambda_1$. Since $\mathbf{x}$ and $\mathbf{e}_{\alpha_1}$ are parallel, so are $T(\mathbf{x})$ and $T(\mathbf{e}_{\alpha_1}) = \mathbf{e}_{\beta_1}$. In particular, $\|T(\mathbf{x})\|_\infty$ is attained at the β_1-coordinate of $T(\mathbf{x})$. On the other hand, $\mathbf{x}$ is not parallel with $\mathbf{e}_{\alpha_2}, \mathbf{e}_{\alpha_3}$, and thus $T(\mathbf{x})$ is not parallel with $T(\mathbf{e}_{\alpha_2}) = \mathbf{e}_{\beta_2}, T(\mathbf{e}_{\alpha_3}) = \mathbf{e}_{\beta_3}$. With $\mathbf{x}$ playing the role of $\mathbf{e}_{\alpha_1}$, and $T(\mathbf{x})$ playing the role of $\mathbf{e}_{\beta_1}$, the above argument shows that the β_1-coordinate of $T(\mathbf{x})$ is $\|T(\mathbf{x})\|_\infty = 1$. In general, for any norm one $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda}$ in $c_0(\Lambda)$, we may choose distinct indices $\alpha_1 = \tau(\beta_1)$, $\alpha_2 = \tau(\beta_2)$, $\alpha_3 = \tau(\beta_3)$, and assume its α_1-coordinate equals 1. Then

$$\Big| \|T(\mathbf{x})\|_\infty - \|T(\mathbf{x} - x_{\alpha_2}\mathbf{e}_{\alpha_2} - x_{\alpha_3}\mathbf{e}_{\alpha_3})\|_\infty \Big| \leq \|T(x_{\alpha_2}\mathbf{e}_{\alpha_2} + x_{\alpha_3}\mathbf{e}_{\alpha_3})\|_\infty$$

implies

$$\begin{aligned} \Big| \|T(\mathbf{x})\|_\infty - 1 \Big| &\leq \left\| x_{\alpha_2} \left(\sum_{\tau(\beta_2)=\alpha_2} \mathbf{e}_{\beta_2} + \mathbf{t}_{\alpha_2} \right) + x_{\alpha_3} \left(\sum_{\tau(\beta_3)=\alpha_3} \mathbf{e}_{\beta_3} + \mathbf{t}_{\alpha_3} \right) \right\|_\infty \\ &\leq |x_{\alpha_2}| + |x_{\alpha_3}|. \end{aligned}$$

Since $\mathbf{x}$ is essentially null, $|x_{\alpha_2}|$ and $|x_{\alpha_3}|$ can be any small numbers. It follows that $\|T(\mathbf{x})\|_\infty = 1$, as claimed. In particular, T is an isometry.

Going back to the original bounded linear map T, the formula (3.3) becomes

$$T(\mathbf{e}_\alpha) = \gamma \sum_{\tau(\beta)=\alpha} \xi_\beta \mathbf{e}_\beta + \mathbf{t}_\alpha,$$

where $\mathbf{t}_\alpha \in c_0(\Lambda \setminus \Lambda_1)$ with $\|\mathbf{t}_\alpha\|_\infty < \gamma$ for every $\alpha \in \Lambda$. For any $\mathbf{x} = (x_\alpha)_{\alpha\in\Lambda} = \sum_{\alpha\in\Lambda} x_\alpha \mathbf{e}_\alpha$ in $c_0(\Lambda)$, the boundedness of T ensures that

$$T(\mathbf{x}) = \sum_{\alpha\in\Lambda} x_\alpha T(\mathbf{e}_\alpha) = \sum_{\alpha\in\Lambda} x_\alpha \left(\gamma \sum_{\tau(\beta)=\alpha} \xi_\beta \mathbf{e}_\beta + \mathbf{t}_\alpha\right) = \sum_{\alpha\in\Lambda} \left(\Big(\sum_{\tau(\beta)=\alpha} \gamma \xi_\beta x_{\tau(\beta)} \mathbf{e}_\beta \Big) + x_\alpha \mathbf{t}_\alpha \right).$$

It follows (3.1).

To verify (3.2), we may assume that there is $\beta' \in \Lambda \setminus \Lambda_1$ and a norm one $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda}$ in $c_0(\Lambda)$ such that $\mathbf{y} = T(\mathbf{x})$ has β'–coordinate with $|y_{\beta'}| > \gamma$. Since the β–coordinates of $\mathbf{y}$ are bounded by γ for all $\beta \in \Lambda_1$ due to (3.1), we see that $\mathbf{y} = T(\mathbf{x})$ and $T(\mathbf{e}_\alpha)$ have disjoint peak sets, and thus they are not parallel to each other for any α in $\Lambda = \tau(\Lambda_1)$. However, if $\mathbf{x}$ attains its norm at the α_1–coordinate, then $\mathbf{x}$ is parallel with $\mathbf{e}_{\alpha_1}$, and thus $T(\mathbf{x})$ is parallel with $T(\mathbf{e}_{\alpha_1})$, a contradiction. We thus establish (3.2). It is now clear that T/γ is an into isometry.

Finally, we note that a (scalar multiple of an) into isometry of $c_0(\Lambda)$ assumes the stated forms (3.1) and (3.2), due to Holsztynski's Theorem (see, e.g., [3]). ∎

The case for linear parallel/TEA pair preservers of $\ell_\infty(\Lambda)$ seems to be more complicated, as an element $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda}$ in $\ell_\infty(\Lambda)$ might have empty peak set, that is, all its coordinates $|x_\lambda| < \|x\|_\infty$. This makes the argument in the proof of Theorem 3.7 cannot be transported directly. However, $\ell_\infty(\Lambda) \cong C(\beta\Lambda)$, and we can apply the following result for abelian C*-algebras.

Theorem 3.8 ([10])**.** *Let X, Y be compact Hausdorff spaces, each of which has at least three elements. Let $T : C(X) \to C(Y)$ be a bijective linear map such that both T, T^{-1} preserve parallel pairs. Then there is a homeomorphism $\sigma : Y \to X$, a scalar $\gamma > 0$, and a unimodulus function $h \in C(Y)$ such that*

$$Tf(y) = \gamma h(y) f(\sigma(y)) \quad \text{for any } f \in C(X) \text{ and } y \in Y.$$

Below is an infinite dimensional analog of Corollary 2.7.

Corollary 3.9. *Let Λ be an infinite index set. Let $T : \ell_\infty(\Lambda) \to \ell_\infty(\Lambda)$ be a bijective linear map. The following conditions are equivalent to each other.*

(a) *Both T and T^{-1} send TEA pairs to TEA pairs.*
(b) *Both T and T^{-1} send parallel pairs to parallel pairs.*
(c) *T is a scalar multiple of a surjective linear isometry.*

In this case, there is $\gamma > 0$, a family $\{\xi_\lambda : \lambda \in \Lambda\}$ of unimodular scalars, and a bijective map $\tau : \Lambda \to \Lambda$ such that for any $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda} \in \ell_\infty(\Lambda)$, the image $\mathbf{y} = T(\mathbf{x}) = (y_\lambda)_{\lambda\in\Lambda}$ has coordinates

$$y_\lambda = \gamma \xi_\lambda x_{\tau(\lambda)} \quad \text{for all } \lambda \in \Lambda. \tag{3.4}$$

In other words, T is a scalar multiple of a "generalized permutation matrix".

Proof. While the implications (c) $\implies$ (a) $\implies$ (b) are plain, Theorem 3.8 establishes (b) $\implies$ (c) when we identify $\ell_\infty(\Lambda)$ with $C(\beta\Lambda)$. We note that Λ consists of all isolated points of $\beta\Lambda$. Thus the homeomorphism σ given in Theorem 3.8 induces a bijective map τ from Λ onto itself to implement (3.4). ∎

Problem 3.10. Can one replace the two direction preserving conditions by that the linear map T sends parallel/TEA pairs to parallel/TEA pairs in Theorem 3.7 and Corollary 3.9, as in the finite dimensional case?

4. The infinite dimensional general case

Let $(X, \mu), (Y, \nu)$ be measure spaces. Two functions $f, g \in L_1(\mu)$ are parallel if there is a unimodular scalar α such that $\alpha \bar{g} f \geq 0$. They form a TEA pair if $\bar{g} f \geq 0$. In this section, all equalities and inequalities of measurable functions are understood as the ones hold modulo a set of zero measure. Recall

$$\text{supp } f = \{x \in X : f(x) \neq 0\} = \bigcap_{n \geq 1} \{x \in X : |f(x)| > 1/n\}.$$

Recall that a linear operator $T : L_p(\mu) \to L_p(\nu)$ is called a *Lamperti operator* if T is bounded and sends disjoint functions to disjoint functions, that is, $TfTg = 0$ whenever $fg = 0$. Surjective isometries of L_p spaces are Lamperti operators when $p \in [1, \infty) \setminus \{2\}$, or when $p = 2$ and T is also positive, that is, $Tf \geq 0$ whenever $f \geq 0$ ([4, 6]). The following result describes the structure of Lamperti operators.

Recall that a *regular set homomorphism* ([4, 6]) Ψ from (X, μ) to (Y, ν) is a transformation between the measure algebras, modulo sets of measure zero, satisfying

- $\Psi(X \setminus E) = \Psi(X) \setminus \Psi(E)$,
- $\Psi\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty} \Psi(E_n)$ for disjoint E_n, and
- $\nu(\Psi(E)) = 0$ if $\mu(E) = 0$.

A regular set homomorphism Ψ induces a linear map between the sets of measurable functions sending $\mathbf{1}_E$ to $\mathbf{1}_{\Psi(E)}$.

Theorem 4.1. *Let $(X, \mu), (Y, \nu)$ be σ-finite measure spaces, and $1 \leq p < \infty$. Let $T : L_p(\mu) \to L_p(\nu)$ be a Lamperti operator.*

(a) ([4, Theorem 4.1]) *There is a regular set homomorphism Ψ sending μ-measurable sets to ν-measurable sets, and a ν-measurable function h such that*

$$T(\mathbf{1}_E) = h \cdot \mathbf{1}_{\Psi(E)}, \quad \textit{for every } \mu\textit{-measurable subset } E \textit{ of } X \textit{ of finite measure.}$$

(b) ([7, Theorem 5.4]) *Suppose further that μ is a tight Baire measure on a topological space X, and (Y, ν) is a complete measure space. Then there is a measurable transformation $\psi : Y \to X$ such that*

$$Tf = h \cdot f \circ \psi, \quad \textit{for all } f \in L_p(\mu).$$

Theorem 4.2. *Let $(X, \mu), (Y, \nu)$ be measure spaces, and let $T : L_1(\mu) \to L_1(\nu)$ be a linear map. Then T sends TEA pairs to TEA pairs if and only if T is a Lamperti operator.*

Proof. Let T be a linear TEA pair preserver. We show that T is a Lamperti operator, that is, bounded and sends disjoint pairs to disjoint pairs.

Suppose first that $\nu(Y) < +\infty$. Let

$$\eta = \sup\{\nu(\text{supp } Tk) : k \in L_1(\mu), k \geq 0\}.$$

If $\eta = 0$ then $T = 0$. Otherwise, let $h_n \geq 0$ in $L_1(\mu)$ such that $\nu(\text{supp } Th_n) \geq \eta - 1/n$ for $n = 1, 2, \ldots$. Since h_m, h_n form a TEA pair, so do Th_m, Th_n for all $m, n \geq 1$. Hence there is a unimodular function $w \in L_\infty(\nu)$ such that all $wTh_n \geq 0$. Replacing T with wT, we can assume that all $Th_n \geq 0$. Let $g \geq 0$ in $L_1(\mu)$ and $n \geq 1$. Since g, h_n form a TEA pair, so do Tg, Th_n. Consequently, $Tg \geq 0$ on supp Th_n (modulo a set of ν-measure zero). It follows

$$\nu(\{y \in Y : T(g + h_n)(y) > 0\}) \geq \nu(\text{supp } Th_n) \geq \eta - 1/n \quad \text{for } n = 1, 2, \ldots.$$

This implies that $\{y \in Y : Tg(y) \not\geq 0\}$, which is disjoint from $\text{supp}\, Th_n$, has zero measure, for else $\nu(\text{supp}\, T(g + h_n)) > \eta$ for some big n. This says that $Tg \geq 0$ in $L_1(\nu)$. Being a positive linear operator, T is automatic bounded. In the original setting, wT is bounded, and thus $T = \overline{w}(wT)$ is bounded.

Next, if ν is σ-finite then we write $Y = \bigcup_m Y_m$, a measurable partition of Y, such that all $\nu(Y_m) < +\infty$. The above argument shows that the linear TEA preserver $T_m : L_1(\mu) \to L_1(\nu_{|Y_m})$ defined by $T_m = T(\cdot)_{|Y_m}$ is bounded for $m = 1, 2, \ldots$. Let $f_n \to \mathbf{0}_X$ in $L_1(\mu)$ and $Tf_n \to k$ in $L_1(\nu)$. Then $T_m f_n \to T_m \mathbf{0}_X = \mathbf{0}_{Y_m}$ in norm, and thus $k_{|Y_m} = \mathbf{0}_{|Y_m}$ for $m = 1, 2, \ldots$. It follows from the σ-finiteness of ν that $k = \mathbf{0}_Y$. Consequently, the linear map T has closed graph, and thus T is bounded.

Finally, we consider the case when (Y, ν) is not necessarily σ-finite. Let $f_n \to f$ in $L_1(\mu)$. There is a σ-finite ν-measurable subset Y' of Y such that all Tf_n, Tf vanish outside Y'. The induced map $T' : L_1(\mu) \to L_1(\nu_{|Y'})$ of T is again a TEA pair preserver, and thus bounded by above. Consequently,

$$\begin{aligned}\|Tf - Tf_n\|_{L_1(\nu)} &= \int_Y |Tf(y) - Tf_n(y)|\, d\nu(y) = \int_{Y'} |Tf(y) - Tf_n(y)|\, d\nu_{|Y'}(y) \\ &= \|T'f - T'f_n\|_{L_1(\nu_{|Y'})} \to 0.\end{aligned}$$

It follows the boundedness of T.

To see that T preserves disjointness, let $f, g \in L_1(\mu)$ such that $fg = 0$. For any nonzero real scalar α, the pair $f + \alpha g$ and $f + \frac{1}{\alpha} g$ is TEA, and so is the pair $Tf + \alpha Tg$ and $Tf + \frac{1}{\alpha} Tg$. In other words, $|Tf|^2 + |Tg|^2 + \frac{1}{\alpha} Tf\overline{Tg} + \alpha \overline{Tf} Tg \geq 0$. Letting $\alpha \to \pm\infty$, we see that $\overline{Tf}Tg = 0$, and thus $TfTg = 0$. Therefore, T is a Lamperti operator.

Conversely, assume that $T : L_1(\mu) \to L_1(\nu)$ is a Lamperti operator. Let $f, g \in L_1(\mu)$ form a TEA pair, namely, $\overline{g}f \geq 0$. We verify that Tf, Tg form a TEA pair, too.

By restricting μ and ν, respectively, to σ-finite measurable subsets of X and Y, outside which f, g and Tf, Tg vanish, we can assume that both X, Y are σ-finite. By Theorem 4.1(a), there is a regular set homomorphism Ψ sending μ-measurable sets to ν-measurable sets, and a ν-measurable function h such that $T(\mathbf{1}_E) = h \cdot \mathbf{1}_{\Psi(E)}$ for all μ-measurable sets E of finite measure.

Modulo a set of measure zero, we can assume f, g, h are pointwise functions (rather than equivalence classes of $L_1(\mu)$ functions) and $\overline{g}f \geq 0$ everywhere. Let $X = X_1 \cup \cdots \cup X_n \cup X_{n+1}$ be any μ-measurable partition of X such that $\mu(X_i) < +\infty$ for $i = 1, \ldots, n$, and $\int_{X_{n+1}} |f| + |g| \, d\mu < \epsilon$ for some small $\epsilon > 0$. Let $x_i \in X_i$ be arbitrarily chosen for $i = 1, \ldots, n$. Then

$$T\left(\sum_{i=1}^n f(x_i)\mathbf{1}_{X_i}\right) = \sum_{i=1}^n f(x_i)h\mathbf{1}_{\Psi(X_i)},$$
$$T\left(\sum_{i=1}^n g(x_i)\mathbf{1}_{X_i}\right) = \sum_{i=1}^n g(x_i)h\mathbf{1}_{\Psi(X_i)}$$

form a TEA pair since the disjointness of $\Psi(X_i)$'s implies that

$$\left(\sum_{i=1}^n \overline{g(x_i)h}\mathbf{1}_{\Psi(X_i)}\right)\left(\sum_{i=1}^n f(x_i)h\mathbf{1}_{\Psi(X_i)}\right) = \sum_{i=1}^n \overline{g(x_i)}f(x_i)|h|^2\mathbf{1}_{\Psi(X_i)} \geq 0.$$

By the continuity of T and the fact that the norm limits of TEA pairs form a TEA pair, we see that Tf, Tg form a TEA pair as well. ∎

Theorem 4.3. *Let $(X, \mu), (Y, \nu)$ be measure spaces. Let $T : L_1(\mu) \to L_1(\nu)$ be a bounded linear map sending parallel pairs to parallel pairs. Then either T is a Lamperti operator, or T has the form $f \mapsto \varphi(f)h$ for a bounded linear functional φ of $L_1(\mu)$ and a fixed function h in $L_1(\nu)$.*

Proof. In view of Theorem 4.2, it suffices to show that if T does not preserve disjointness then T has rank one. Suppose $f, g \in L_1(\mu)$ with $fg = 0$ but $TfTg \neq 0$.

Claim 1. Tf, Tg are linearly dependent.

Observe that for any scalar β, the pair $f + \beta g$ and g is parallel, and so is $Tf + \beta Tg$ and Tg. There is a unimodular scalar α_β such that

$$\alpha_\beta\overline{Tg}(Tf + \beta Tg) \geq 0.$$

In particular, $\alpha_0\overline{Tg}Tf \geq 0$. Replacing g with $\overline{\alpha_0}g$, we can assume

$$\overline{Tg}Tf \geq 0. \tag{4.1}$$

In general,

$$\alpha_\beta\overline{Tg}Tf + \alpha_\beta\beta|Tg|^2 \geq 0, \quad \forall \beta \in \mathbb{F}. \tag{4.2}$$

If the set Y_1 on which $Tf = 0$ but $Tg \neq 0$ has positive ν-measure, we see that $\alpha_\beta\beta \geq 0$ by (4.2). Then $\alpha_{-\epsilon} = -1$ whenever $\epsilon > 0$. Letting $\beta = -\epsilon \to 0^-$, we have $\overline{Tg}Tf = 0$, contradicting to $TfTg \neq 0$. This shows that $\nu(Y_1) = 0$. Exchanging the roles of f and g, we see that

$$Tf(y) = 0 \quad \text{if and only if} \quad Tg(y) = 0, \tag{4.3}$$

modulo a set of zero measure.

It follows from (4.2) that

$$\frac{\alpha_{-\beta}\overline{Tg}Tf}{\beta} - \alpha_{-\beta}|Tg|^2 \geq 0 \quad \text{whenever } \beta > 0.$$

Hence, $\alpha_{-\beta} = -1$, and thus

$$\overline{Tg}Tf \leq \beta|Tg|^2 \quad \text{for big enough } \beta > 0. \tag{4.4}$$

Let

$$r = \inf\{\beta > 0 : \overline{Tg}Tf \leq \beta|Tg|^2\} > 0.$$

For any $0 < s < r$, the set $A_s = \{y \in Y : \overline{Tg}Tf > s|Tg|^2\}$ has positive measure. To keep (4.2) hold for $\beta = -s$, we need $\alpha_{-s} = 1$, and thus $\overline{Tg}Tf \geq s|Tg|^2$ on Y. This forces $\overline{Tg}Tf \geq r|Tg|^2$. Hence, $\overline{Tg}Tf = r|Tg|^2$, and thus $Tf = rTg$ by (4.3), as claimed.

Claim 2. If $k \in L_1(\mu)$ such that $fk = gk = 0$ and $TfTk = 0$, then $Tk = 0$.

Claim 1 shows that $Tf = rTg$ for some nonzero scalar r, and thus $TgTk = 0$ as well. Observe that $mf - g + k$ and $f - mg + k$ are parallel, and so are $mTf - Tg + Tk = (mr-1)Tg + Tk$ and $Tf - mTg + Tk = (r-m)Tg + Tk$ for any scalar $m \geq 0$. In other words, there is a unimodular scalar α_m such that $\alpha_m\overline{((mr-1)Tg + Tk)}((r-m)Tg + Tk) \geq 0$, and thus

$$\alpha_m\overline{(mr-1)}(r-m)|Tg|^2 + \alpha_m|Tk|^2 \geq 0 \quad \text{for any } m \geq 0,$$

since Tg, Tk are disjoint. If $Tk \neq 0$ then $\alpha_m|Tk|^2 \geq 0$ implies $\alpha_m = 1$, and thus

$$\overline{(mr-1)}(r-m)|Tg|^2 \geq 0 \quad \text{for all } m \geq 0.$$

This is impossible. This forces $Tk = 0$.

Let C, D be the supports of the disjoint functions f, g, respectively, and $E = X\setminus(C \cup D)$. For any $u \in L_1(\mu)$, write $u = u_C + u_D + u_E$ as disjoint sum of functions supported in C, D, E, respectively. Since $fu_E = gu_E = 0$, Claims 1 and 2 imply $Tu_E = \gamma_E Tf$ for some (maybe zero) scalar γ_E. On the other hand, $fu_D = f(g - u_D) = 0$. If $TfT(g - u_D) = 0$, then $TfTu_D = TfTg \neq 0$, and Claim 1 implies that Tu_D, Tf are linearly dependent. If $TfT(g - u_D) \neq 0$, then Claim 1 implies that $Tg - Tu_D, Tf$ are linearly dependent. Since Tf, Tg are linearly dependent, Tu_D, Tf are linearly dependent, too. Thus $Tu_D = \gamma_D Tf$ for some scalar γ_D. Similarly, $Tu_C = \gamma_C Tf$ for some scalar γ_C. Consequently, $Tu = (\gamma_C + \gamma_D + \gamma_E)Tf$. This shows that the range of T is spanned by Tf, and thus T has rank one. ■

Theorem 4.3 says that *bounded* linear parallel pair preservers are either TEA pair preservers or rank one maps. In Example 3.3, we have an unbounded rank one parallel pair preserver of ℓ_1, while every linear TEA pair preserver of L_1 spaces is automatic bounded due to Theorem 4.2. The open problem is whether an unbounded linear parallel pair preserver T can have rank more than one (see Problem 3.4).

Theorem 4.4. *Let $L_\infty(\mu_1), L_\infty(\mu_2)$ have dimension not 2. Let $T : L_\infty(\mu_1) \to L_\infty(\mu_2)$ be a bijective linear map. The following conditions are equivalent to each other.*

(a) *Both T and T^{-1} send TEA pairs to TEA pairs.*
(b) *Both T and T^{-1} send parallel pairs to parallel pairs.*
(c) *T is a scalar multiple of a surjective linear isometry.*

Proof. Being commutative unital C^*-algebras, $L_\infty(\mu_1) \cong C(X)$ and $L_\infty(\mu_2) \cong C(Y)$ for some hyperstonian spaces X, Y. We can assume both X, Y have at least three points. While the implications (c) $\implies$ (a) $\implies$ (b) are plain, Theorem 3.8 establishes (b) $\implies$ (c). ∎

For $\ell_1(\Lambda), \ell_\infty(\Lambda)$ and general $L_1(\mu), L_\infty(\mu)$ spaces, the structure of (surjective) linear isometries have a concrete description. Therefore, we can connect invertible parallel/TEA pair preservers with linear isometries. In particular, in the $\ell_\infty(\Lambda)$ and $L_\infty(\mu)$ cases, with dimension not two (see the part (a.2) in Theorem 2.6 for a counter example), we see that invertible parallel/TEA pair preservers are exactly positive multiple of isometries.

In the $\ell_1(\Lambda)$ case, invertible bounded parallel/TEA pair preservers are monomial matrices. In the general $L_1(\mu)$ case, invertible bounded parallel/TEA pair preservers are Lamperti operators. Therefore, the group of positive multiples of surjective isometries of L_1 spaces is a proper subgroup of invertible bounded parallel/TEA pair preservers.

It is interesting to ask

Problem 4.5. For a non-strictly convex normed space, what properties characterize that the group of invertible bounded parallel/TEA pair preservers consists of positive multiples of surjective isometries?

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