

Numerical radius of certain two-by-two block matrices

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Dedicated to Professor Pei Yuan Wu.

Abstract

We investigate the numerical range $W(T)$ and numerical radius $w(T)$ of operators of the form $T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$. We show that $W(T)$ is the union of the numerical ranges of a family of 2×2 matrices, T_x , leading to several consequences, including improved inequalities for $w(T)$. For cases where A is a self-adjoint involution, we characterize the conditions under which $W(T)$ is an elliptical disk and determine the minimum numerical radius of $T_U = \begin{pmatrix} U^*AU & B \\ 0 & 0 \end{pmatrix}$ over all unitary operators U . Finally, we study matrices $T \in M_n$ satisfying $\|T^m x\| = \|T^m\| = \|T\|$ for a unit vector x and all positive integers m . This analysis connects these matrices to the aforementioned block form and provides a counterexample to the conjecture that if $\|T^k\| = \|T\|$ for all $k \geq 1$, then some power of the matrix has a direct summand that is a scalar multiple of an idempotent.

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1 Introduction

Let $B(H)$ be the algebra of bounded linear operators acting on the Hilbert space H equipped with the inner product $\langle \cdot, \cdot \rangle$. If H has dimension n , we identify $B(H)$ as M_n , the algebra of all $n \times n$ complex matrices, and $H = \mathbb{C}^n$ equipped with the inner product $\langle x, y \rangle = y^*x$, where y^* denotes the conjugate transpose of $y \in \mathbb{C}^n$. Let $T \in M_n$. Its operator norm is defined by

$$\|T\| = \sup\{\|Tx\| : x \in H, \|x\| = 1\},$$

where $\|x\|^2 = \langle x, x \rangle$. The numerical range and numerical radius of T are defined by

$$W(T) = \{\langle Tx, x \rangle : x \in H, \|x\| = 1\} \quad \text{and} \quad w(T) = \sup\{|\mu| : \mu \in W(T)\}.$$

The numerical range and numerical radius are useful tools for studying operators and matrices; e.g., see [6, 13]. In this paper, we study the numerical range, numerical radius, and some related results for operators of the form

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}. \tag{1.1}$$

Operators of the form (1.1) appear frequently in both pure and applied areas. For instance, if $T \in B(H)$ is idempotent, i.e., $T^2 = T$, then T has an operator matrix of the form (1.1) with respect to an orthonormal basis using an orthonormal basis of the range space of T and its orthogonal complement. More generally, if $\tilde{T} \in B(H)$ is a quadratic operator, i.e., $\tilde{T}^2 + b\tilde{T} + cI = 0$ for some $b, c \in \mathbb{C}$, then $\tilde{T} - \gamma I$ is unitarily similar to an operator T of the form (1.1). By the result in [11], $W(\tilde{T}) - \gamma = W(T)$ is an open or closed elliptical disk with foci at 0 and $(b^2 - 4c)^{1/2}$, and with minor axis of length $\|B\|$. Moreover, $W(T)$ is closed if and only if there exists a unit vector x such that $\|Bx\| = \|B\|$. There has also been interest in studying the bounds for $w(T)$ for T in the form (1.1); e.g., see [10] and its references. In section 2, we will show that if T has the form (1.1), then $W(T)$ can be written as the convex hull of elliptical disks; we also obtain upper bound of $w(T)$ improving some results in the literature. In Section 3, we focus on T in the form (1.1) with $A, B \in M_n$ such that A is a Hermitian involution, i.e., $A = A^*$ and $A^2 = I$.

Another related topic is the study of matrices $T \in M_n$ satisfying $\|T^m x\| = \|T^m\| = \|T\|$ for some unit vector x and all positive integers m that connects to matrices of the form (1.1) with $A \in M_k$ unitary. In this case, we may always assume that $B \in M_k$. In a sense, it is not surprising as $T^m = \begin{pmatrix} A^m & A^{m-1}B \\ 0 & 0 \end{pmatrix}$ for $m = 2, 3, \dots$. As we shall see in Section 4, the study leads to the class of matrices $\tilde{T}$ satisfying $\tilde{T}^* \tilde{T} = (\tilde{T}^*)^2 \tilde{T}^2$. Evidently, if $\tilde{T}$ is invertible, then $\tilde{T}$ satisfies $\tilde{T}^* \tilde{T} = I$ so that $\tilde{T}$ is unitary. If $\tilde{T}$ is singular, then $\tilde{T}$ is unitarily similar to a matrix of form (1.1). In Section 4, we will disprove the following conjecture in [4]; see Proposition 4.10.

Conjecture 1.1. *If $A \in M_n$ satisfies $\|A^k\| = \|A\|$ for all $k \geq 1$, then some power of A has a direct summand that is a scalar multiple of an idempotent with unimodular scalar.*

We conclude this section with some notations and basic facts about T in the form (1.1).

For $A \in M_n$, let A^t and A^* denote the transpose and conjugate transpose of A , respectively. The singular values of A are the nonnegative square roots of the eigenvalues of A^*A . It is known that $\|A\|$ is the largest singular value of A . If $\mathcal{F} \subseteq \{1, \dots, n\}$, we use $A(\mathcal{F})$ to denote the principal submatrix obtained by deleting the j th row and j th column of A for all $j \notin \mathcal{F}$. We write $\text{diag}(a_1, \dots, a_n)$ for the diagonal matrix with diagonal entries $a_1, \dots, a_n$, and denote by 0_n and I_n the zero and identity matrices in M_n , respectively. We use $0_{k \times m}$ to denote the $k \times m$ zero matrix. For an operator $T \in B(H)$, T is *unitarily irreducible* if it is not unitarily similar to an operator of the form $B \oplus C$; otherwise, it is *unitarily reducible*. We use $\text{Re } T$ and $\text{Im } T$ to denote the real part $(T + T^*)/2$ and imaginary part $(T - T^*)/(2i)$ of T , respectively. T is *positive semidefinite*, denoted by $T \geq 0$, if $\langle Tx, x \rangle \geq 0$ for all $x \in H$. If $T_1, T_2 \in B(H)$ are self-adjoint, then $T_1 \geq T_2$ means that $T_1 - T_2 \geq 0$. For any subset Δ of complex plane $\mathbb{C}$, $\text{cl}(\Delta)$ denotes the closure of Δ .

In our discussion, it is convenient to embed H into $H \oplus H$ and assume that $A, B \in B(H)$. We may further replace T by $U^*TU = \begin{pmatrix} U_1^*AU_1 & U_1^*BU_2 \\ 0 & 0 \end{pmatrix}$ such that $U_1^*BU_2 \in B(H)$ is positive semidefinite. In the finite dimensional case, we may assume that $U_1^*BU_2 = \text{diag}(b_1, \dots, b_n)$ with $b_1 \geq \dots \geq b_n \geq 0$.

2 Generating ellipses and consequences

It is known that for any operator $T \in B(H)$, $W(T)$ can be written as the union of $W(X^*TX)$ for $X : \mathbb{C}^2 \rightarrow H$ with $X^*X = I_2$. We will show that if T has the form (1.1), then T is the union of $W(X^*TX)$ some special X so that X^*TX has a simple form. We begin with the following easy case.

Proposition 2.1. *Let T have the form (1.1). Suppose $A, B \in M_k$ and there are unitary U, V such that $U^*AU = \text{diag}(a_1, \dots, a_k)$ and $U^*BV = \text{diag}(b_1, \dots, b_k)$. Let $T_j = \begin{pmatrix} a_j & b_j \\ 0 & 0 \end{pmatrix}$ for $j = 1, \dots, k$. Then*

$$W(T) = \text{conv}\left(\bigcup_{j=1}^k W(T_j)\right) \quad \text{and} \quad w(T) = \max\{w(T_j) : 1 \leq j \leq k\}.$$

Proof. This follows from the fact that the assumption ensures that T is unitarily similar to a direct sum of T_j so that $W(T) = \text{conv}\left(\bigcup_{j=1}^k W(T_j)\right)$ and $w(T) = \max\{w(T_j) : 1 \leq j \leq k\}$. $\square$

In general, we can write $W(T)$ as the union of $W(T_x)$ for some upper triangular matrix $T_x \in M_2$ even if T does not have the nice special form in the last proposition.

Theorem 2.2. *Let $T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \in B(H \oplus H)$. Then*

$$W(T) = \text{conv}\bigcup\{W(T_x) : x \in H, \|x\| = 1\} \quad \text{where } T_x = \begin{pmatrix} \langle Ax, x \rangle & \|B^*x\| \\ 0 & 0 \end{pmatrix}.$$

Consequently,

$$\begin{aligned} w(T) &= \sup\{w(T_x) : x \in H, \|x\| = 1\} \\ &= \sup\left\{\frac{1}{2}(|\langle Ax, x \rangle| + \sqrt{|\langle Ax, x \rangle|^2 + \|B^*x\|^2}) : x \in H, \|x\| = 1\right\} \\ &\leq \frac{1}{2}w(A) + \frac{1}{2}\min\{\sqrt{w(AA^* + BB^*)}, \sqrt{w(A)^2 + \|B\|^2}\}. \end{aligned}$$

The equality holds if and only if there is a sequence of unit vectors $\{x_m\}$ such that $|\langle Ax_m, x_m \rangle| \rightarrow w(A)$ and

- (1) $\|B^*x_m\|^2 \rightarrow w(AA^* + BB^*) - w(A)^2$, or
- (2) $\|B^*x_m\| \rightarrow \|B\|$ and $w(AA^* + BB^*) \geq w(A)^2 + \|B\|^2$.

Proof. Every unit vector $z \in H \oplus H$ can be written as $\cos \theta x \oplus \sin \theta y$ for some unit vectors $x, y \in H$. Thus, $\langle Tz, z \rangle = (\cos \theta \ \sin \theta) \tilde{T}_x (\cos \theta \ \sin \theta)^t$, where $\tilde{T}_x = \begin{pmatrix} \langle Ax, x \rangle & \langle By, x \rangle \\ 0 & 0 \end{pmatrix}$. Note that $W(\tilde{T}_x)$ is an elliptical disk with foci $\langle Ax, x \rangle$ and 0 with minor axis having length $|\langle By, x \rangle|$. We may choose $y = B^*x / \|B^*x\|$ to get T_x as described in the theorem so that $W(\tilde{T}_x) \subseteq W(T_x)$. Thus, the set equality holds.

Now, for any unit vector $x \in H$, the numerical radius of $w(T_x)$ is attained at the end point of the major axis and

$$\begin{aligned}
w(T_x) &= \frac{1}{2}(|\langle Ax, x \rangle| + \sqrt{|\langle Ax, x \rangle|^2 + \|B^*x\|^2}) \\
&\leq \frac{1}{2}(|\langle Ax, x \rangle| + \sqrt{\|A^*x\|^2 + \langle BB^*x, x \rangle}) \\
&= \frac{1}{2}(|\langle Ax, x \rangle| + \sqrt{\langle AA^*x, x \rangle + \langle BB^*x, x \rangle}) \\
&\leq \frac{1}{2}(w(A) + \sqrt{w(AA^* + BB^*)}).
\end{aligned} \tag{2.1}$$

Clearly, we also have

$$w(T_x) = \frac{1}{2}(|\langle Ax, x \rangle| + \sqrt{|\langle Ax, x \rangle|^2 + \|B^*x\|^2}) \leq \frac{1}{2}(w(A) + \sqrt{w(A)^2 + \|B\|^2}). \tag{2.2}$$

The equality case could be verified readily by (2.1) and (2.2). $\square$

Note that the two quantities $\sqrt{w(AA^* + BB^*)}$ and $\sqrt{w(A)^2 + \|B\|^2}$ are not comparable in general. For example,

- (a) if $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $w(AA^* + BB^*) = 5 \geq 2 = w(A)^2 + \|B\|^2$;
- (b) if $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then $w(AA^* + BB^*) = 1 \leq 2 = w(A)^2 + \|B\|^2$.

We have the following corollary, which is Theorem 3.1 in [10].

Corollary 2.3. Suppose $T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \in B(H \oplus H)$. Then $w(T) \leq \frac{1}{2}(\|A\| + \sqrt{\|AA^* + BB^*\|})$.

We also have the following.

Corollary 2.4. Let $T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \in B(H \oplus H)$. If $B_1, B_2 \in B(H)$ such that $B_1B_1^* \leq BB^* \leq B_2B_2^*$ and T_j is obtained from T by replacing B with B_j for $j = 1, 2$, then

$$W(T_1) \subseteq W(T) \subseteq W(T_2).$$

Proof. Note that for any unit vector x , $W(T_x)$ is an elliptical disk with foci $0, \langle Ax, x \rangle$ and minor axis of length $\|B^*x\|$. For T_j the corresponding $W((T_j)_x)$ is an elliptical disk with the same foci as $W(T_x)$ and minor axis of length $\|B_j^*x\|$. Since $B_1B_1^* \leq BB^* \leq B_2B_2^*$, we see that $\|B_1^*x\| \leq \|B^*x\| \leq \|B_2^*x\|$. Hence, $W((T_1)_x) \subseteq W(T_x) \subseteq W((T_2)_x)$. Taking the union of the sets for all unit vectors x , we get the inclusion. $\square$

Corollary 2.5. Let T have the form (1.1) be such that $B = bI$ with $b \geq 0$ and $W(A)$ is closed. Then

$$W(T) = \bigcup \{W(T_x) : x \in H, \|x\| = 1, \langle Ax, x \rangle \text{ is a boundary point of } W(A)\}.$$

Proof. Clearly, $W(T_x)$ has the same minor axis for every unit vector x . Hence, the right supporting line of $W(T)$ is determined by the endpoint of the major axis of $W(T_x)$. Therefore, we only need to focus on those T_x for which $\langle Ax, x \rangle$ is a boundary point of $W(A)$. $\square$

If $A \in M_2$ is unitary, after a rotation, then A is unitarily similar to $\cos \theta I_2 + i \sin \theta (E_{11} - E_{22})$, where $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We have the following.

Corollary 2.6. *Let T have the form (1.1) with $A, B \in M_2$ such that $A = \cos \theta I_2 + i \sin \theta (E_{11} - E_{22})$ with $\theta \in [0, \pi/2]$. Then $W(T) = \bigcup \{W(T_r) : r \in [-\sin \theta, \sin \theta]\}$, where $T_r = \begin{pmatrix} \cos \theta + ir & c \\ 0 & 0 \end{pmatrix}$, and (r, c^2) ranges over the joint numerical range*

$$W(\operatorname{Im} A, BB^*) = \{(\langle (\operatorname{Im} A)x, x \rangle, \langle BB^*x, x \rangle) : \|x\| = 1\}.$$

Next, we consider maximum $\sigma(\operatorname{Re}(e^{i\theta}T))$, which allows us to determine the support lines of $\mathbf{cl}(W(T))$, the closure of $W(T)$, and reduces the computation of $w(T)$ to an optimization problem over a parameter $\theta \in [0, 2\pi)$.

Theorem 2.7. *Suppose $T \in B(H \oplus H)$ has the form (1.1) with $A, B \in B(H)$. Let*

$$T_\theta = \operatorname{Re}(e^{i\theta}T) = \frac{1}{2} \begin{pmatrix} e^{i\theta}A + e^{-i\theta}A & e^{i\theta}B \\ e^{-i\theta}B^* & 0 \end{pmatrix}, \quad \theta \in [0, 2\pi).$$

Then

$$\begin{aligned} \lambda_{\max}(T_\theta) &= \inf\{\lambda \geq 0 : 4\lambda(\lambda I - \operatorname{Re}(e^{i\theta}A)) \geq BB^*\} \\ &\geq \frac{1}{2} \lambda_{\max}(\operatorname{Re}(e^{i\theta}A) + \sqrt{(\operatorname{Re}(e^{i\theta}A))^2 + BB^*}). \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} w(T) &= \max_{0 \leq \theta < 2\pi} \inf\{\lambda \geq 0 : 4\lambda(\lambda I - \operatorname{Re}(e^{i\theta}A)) \geq BB^*\} \\ &\geq \frac{1}{2} \max_{0 \leq \theta < 2\pi} \lambda_{\max}(\operatorname{Re}(e^{i\theta}A) + \sqrt{(\operatorname{Re}(e^{i\theta}A))^2 + BB^*}). \end{aligned}$$

Proof. Note that T_θ is unitarily similar to $\frac{1}{2} \begin{pmatrix} e^{i\theta}A + e^{-i\theta}A & B \\ B^* & 0 \end{pmatrix}$ and $0 \in W(T_\theta)$. Therefore,

$$\lambda_{\max}(T_\theta) = \inf\{\lambda \geq 0 : \lambda I - T_\theta \geq 0\}$$

and

$$\lambda I - T_\theta = \begin{pmatrix} \lambda I - \operatorname{Re}(e^{i\theta}A) & -B/2 \\ -B^*/2 & \lambda I \end{pmatrix},$$

which is positive semidefinite if and only if $\lambda I - \operatorname{Re}(e^{i\theta}A) \geq BB^*/(4\lambda)$. The first equality holds.

Note that $\lambda^2 I - \lambda \operatorname{Re}(e^{i\theta}A) - BB^*/4 \geq 0$ is equivalent to

$$\left(\lambda I - \frac{1}{2} \operatorname{Re}(e^{i\theta}A)\right)^2 \geq \frac{1}{4} \left((\operatorname{Re}(e^{i\theta}A))^2 + BB^*\right).$$

Since $\lambda I - T_\theta \geq 0$, we see that $\lambda I - \operatorname{Re}(e^{i\theta} A) \geq 0$ and $\lambda \geq 0$. So, $\lambda I - \frac{1}{2}\operatorname{Re}(e^{i\theta} A) \geq 0$. It is known that if $P \geq Q \geq 0$ then $\sqrt{P} \geq \sqrt{Q}$. So, $\lambda I - \frac{1}{2}\operatorname{Re}(e^{i\theta} A) \geq \frac{1}{2}\sqrt{(\operatorname{Re}(e^{i\theta} A))^2 + BB^*}$. This shows that

$$\{\lambda \geq 0 : \lambda I - T_\theta \geq 0\} \subseteq \{\lambda \geq 0 : \lambda I - \frac{1}{2}(\operatorname{Re}(e^{i\theta} A) + \sqrt{(\operatorname{Re}(e^{i\theta} A))^2 + BB^*}) \geq 0\}.$$

That is,

$$\inf\{\lambda \geq 0 : \lambda I - T_\theta \geq 0\} \geq \inf\{\lambda \geq 0 : \lambda I - \frac{1}{2}(\operatorname{Re}(e^{i\theta} A) + \sqrt{(\operatorname{Re}(e^{i\theta} A))^2 + BB^*}) \geq 0\}.$$

The inequality (2.3) holds. The final inequality follows from $w(T) = \max_{0 \leq \theta < 2\pi} \lambda_{\max}(T_\theta)$. $\square$

Note that Theorem 2.2 provides upper bounds for $w(T)$ whereas Theorem 2.7 provides a lower bound for $w(T)$. Moreover, in Theorem 2.2, the computation of $w(T)$ reduces to a maximization problem of $w(T_x)$ for $x \in H$ with $\|x\| = 1$. In Theorem 2.7, the computation of $w(T)$ reduced to a maximization problem

$$\max_{0 \leq \theta < 2\pi} \inf\{\lambda \geq 0 : 4\lambda(\lambda I - \operatorname{Re}(e^{i\theta} A)) \geq BB^*\}$$

with two parameters $\lambda \geq 0$ and $\theta \in [0, 2\pi)$. Furthermore, for a fixed $\theta \in [0, 2\pi)$, one may consider $\lambda > 0$ in a compact set to determine

$$\lambda_\theta = \inf\{\lambda \geq 0 : 4\lambda(\lambda I - \operatorname{Re}(e^{i\theta} A)) \geq BB^*\}$$

and then solve the original optimization by taking $\max\{\lambda_\theta : \theta \in [0, 2\pi]\}$.

If $A = \operatorname{diag}(a_1, \dots, a_k)$ and $B = \operatorname{diag}(b_1, \dots, b_k) \in M_k$, then

$$\begin{aligned} \lambda_\theta &= \inf\{\lambda \geq 0 : 4\lambda(\lambda - \operatorname{Re}(e^{i\theta} a_j)) \geq |b_j|^2, j = 1, \dots, k\} \\ &= \frac{1}{2} \max\{\operatorname{Re}(e^{i\theta} a_j) + \sqrt{(\operatorname{Re}(e^{i\theta} a_j))^2 + |b_j|^2} : j = 1, \dots, k\}. \end{aligned}$$

A similar formula remains valid for infinite-dimensional operators whenever $\operatorname{Re} A$, $\operatorname{Im} A$, and BB^* commute, by the spectral theorem.

3 When the $(1, 1)$ block is a self-adjoint involution

In this section, we consider the special case when A is a self-adjoint involution operator, i.e., $A = A^*$ and $A^2 = I$. In such a case, we may assume apply a unitary similar of the form $U = U_1 \oplus U_2$ to T , and assume $A = I_{H_1} \oplus -I_{H_2}$ so that $(\operatorname{Re}(e^{i\theta} A))^2 = \cos^2 \theta I$. By Theorem 2.7 and the discussion after it,

$$w(T) = \max_{\theta \in [0, 2\pi]} \inf\{\lambda \geq 0 : 4\lambda(\lambda I - \operatorname{Re}(e^{i\theta} A)) \geq BB^*\}.$$

In particular, if $A \in M_k$ and $B = \operatorname{diag}(b_1, \dots, b_k)$, then

$$\max \sigma(T_\theta) = \frac{1}{2} \max \sigma \left(\cos \theta A + \operatorname{diag}(\sqrt{\cos^2 \theta + |b_1|^2}, \dots, \sqrt{\cos^2 \theta + |b_k|^2}) \right).$$

Suppose T has the form (1.1) such that $A = A^*$ and $A^2 = I$. We determine the condition on T such that $W(T)$ is an elliptical disk in the following.

Theorem 3.1. Let $T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \in M_n$ such that $A = A^* \in M_k$ with $1 \leq k < n$ satisfying $A^2 = I_k$ and $B \neq 0$. The following statements are equivalent.

- (a) $W(T)$ is an elliptical disk.
- (b) The matrix T is unitarily similar to a direct sum $T_1 \oplus T_2$ with $W(T_2) \subseteq W(T_1)$, where T_1 is either

$$\begin{pmatrix} a & \|B\| \\ 0 & 0 \end{pmatrix} \text{ with } a \in \{1, -1\} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & \|B\| \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To prove Theorem 3.1, we need the following lemma.

Lemma 3.2. Let $\theta \in [-\pi, \pi]$, $\alpha_1 > \alpha_2 \geq 0$,

$$A_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}, \quad D = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \quad \text{and} \quad T_\theta = \begin{pmatrix} A_\theta & D \\ 0_2 & 0_2 \end{pmatrix}.$$

Then

$$w(T_{\pm\pi/2}) = \frac{\sqrt{4 + \alpha_1^2 + \alpha_2^2 + \sqrt{16 + 8\alpha_1^2 + 8\alpha_2^2 + (\alpha_1^2 - \alpha_2^2)^2}}}{2\sqrt{2}} < w(T_\theta) \quad \text{for all } \theta \neq \pm\frac{\pi}{2}.$$

Proof. Observe that T_θ is unitarily similar to $T_{-\theta}$, so it suffices to consider $\theta \in [0, \pi]$. For any $\mu \in [0, 2\pi]$, $e^{i\mu}T_\theta$ is unitarily similar to $\begin{pmatrix} e^{i\mu}A_\theta & D \\ 0 & 0 \end{pmatrix}$. Therefore,

$$\begin{aligned} \det(\operatorname{Re}(e^{i\mu}T_\theta) - \lambda I_4) &= \det \begin{pmatrix} \cos \theta \cos \mu - \lambda & \sin \theta \cos \mu & \frac{\alpha_1}{2} & 0 \\ \sin \theta \cos \mu & -\cos \theta \cos \mu - \lambda & 0 & \frac{\alpha_2}{2} \\ \frac{\alpha_1}{2} & 0 & -\lambda & 0 \\ 0 & \frac{\alpha_2}{2} & 0 & -\lambda \end{pmatrix} \\ &= -\lambda \det \begin{pmatrix} \cos \theta \cos \mu - \lambda & \sin \theta \cos \mu & \frac{\alpha_1}{2} \\ \sin \theta \cos \mu & -\cos \theta \cos \mu - \lambda & 0 \\ \frac{\alpha_1}{2} & 0 & -\lambda \end{pmatrix} - \frac{\alpha_2^2}{4} \det \begin{pmatrix} \cos \theta \cos \mu - \lambda & \frac{\alpha_1}{2} \\ \frac{\alpha_1}{2} & -\lambda \end{pmatrix} \\ &= \lambda^2 \det \begin{pmatrix} \cos \theta \cos \mu - \lambda & \sin \theta \cos \mu \\ \sin \theta \cos \mu & -\cos \theta \cos \mu - \lambda \end{pmatrix} - \lambda \frac{\alpha_1^2}{4} (\cos \theta \cos \mu + \lambda) - \frac{\alpha_2^2}{4} (\lambda^2 - \lambda \cos \theta \cos \mu - \frac{\alpha_1^2}{4}) \\ &= \lambda^4 - \lambda^2 (\cos^2 \mu + \frac{\alpha_1^2 + \alpha_2^2}{4}) + \lambda \frac{\alpha_2^2 - \alpha_1^2}{4} \cos \theta \cos \mu + \frac{\alpha_1^2 \alpha_2^2}{16} \\ &\equiv F_{\theta, \mu}(\lambda). \end{aligned}$$

Let $\lambda_1(F_{\theta, \mu})$ denote the largest real root of $F_{\theta, \mu}$. Then $w(T_\theta) = \max_{\mu \in [0, 2\pi]} \lambda_1(F_{\theta, \mu})$. For $\theta = \pi/2$, since $T_{\pi/2}$ is a nonnegative matrix, we get

$$\begin{aligned} \beta &\equiv w(T_{\pi/2}) = \lambda_1(F_{\pi/2, 0}) = \max\{\lambda : \lambda^4 - (1 + \frac{\alpha_1^2 + \alpha_2^2}{4})\lambda^2 + \frac{\alpha_1^2 \alpha_2^2}{16} = 0\} \\ &= \frac{\sqrt{4 + \alpha_1^2 + \alpha_2^2 + \sqrt{16 + 8\alpha_1^2 + 8\alpha_2^2 + (\alpha_1^2 - \alpha_2^2)^2}}}{2\sqrt{2}}. \end{aligned}$$

For $\theta \in [0, \pi] \setminus \{\pi/2\}$,

$$F_{\theta,\mu}(\lambda) = F_{\pi/2,0}(\lambda) + \lambda^2 \sin^2 \mu + \lambda \frac{\alpha_2^2 - \alpha_1^2}{4} \cos \theta \cos \mu.$$

Evaluating this at $\lambda = \beta$, we find

$$F_{\theta,\mu}(\beta) = \beta^2 \sin^2 \mu - \frac{\alpha_1^2 - \alpha_2^2}{4} \beta \cos \theta \cos \mu.$$

Since $\alpha_1 > \alpha_2 \geq 0$, we have $F_{\theta,0}(\beta) < 0$ for all $\theta \in [0, \pi/2)$, and, $F_{\theta,\pi}(\beta) < 0$ for all $\theta \in (\pi/2, \pi]$, that is, $\lambda_1(F_{\theta,0}) > \beta$ for all $\theta \in [0, \pi/2)$, and, $\lambda_1(F_{\theta,\pi}) > \beta$ for all $\theta \in (\pi/2, \pi]$. Therefore,

$$w(T_\theta) = \max_{\mu \in [0, 2\pi]} \lambda_1(F_{\theta,\mu}) \geq \max\{\lambda_1(F_{\theta,0}), \lambda_1(F_{\theta,\pi})\} > \beta$$

for all $\theta \in [0, \pi] \setminus \{\pi/2\}$. This completes the proof. $\square$

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. If $A = \pm I_k$, then T is a quadratic operator and the assertion follows from [11, Theorem 2.1]. Therefore, we may assume that A is unitarily similar to $I_{k_1} \oplus (-I_{k_2})$, where $1 \leq k_1 \leq k-1$ and $k_2 = k - k_1$. Let $m = n - k$, then B is a $k \times m$ matrix. Say, $B = V_0 S_B V_1^*$ is the singular value decomposition of B , where $V_0 \in M_k$ and $V_1 \in M_m$ are unitary, and $S_B = [||B||] \oplus B_1$ for some $B_1 \in M_{k-1, m-1}$. Let $V = V_0 \oplus V_1 \in M_n$ be unitary, then $V^* T V = \begin{pmatrix} V_0^* A V_0 & S_B \\ 0 & 0 \end{pmatrix}$, hence we may assume that $B = [||B||] \oplus B_1$ for some $B_1 \in M_{k-1, m-1}$.

(a) $\Rightarrow$ (b). Assume that $W(T)$ is an elliptical disk. A direct calculation gives $T^3 = T$; hence the minimal polynomial of T divides $z(z-1)(z+1)$. Thus T is diagonalizable, and the geometric and algebraic multiplicities of each eigenvalue coincide. It follows from [12, Corollary 4] that $W(T)$ is not a circular disk. Therefore, the foci of $W(T)$ must be one of $\{0, 1\}$, $\{0, -1\}$, and $\{-1, 1\}$. Moreover, since $\text{Im } T = \begin{pmatrix} 0 & B/(2i) \\ -B^*/(2i) & 0 \end{pmatrix}$, it implies that $\max \sigma(\text{Im } T) = \|B\|/2 = -\min \sigma(\text{Im } T)$, that is, the length of minor axis of $W(T)$ is $\|B\|$, we infer that either

$$W(T) = W\left(\begin{pmatrix} \pm 1 & \|B\| \\ 0 & 0 \end{pmatrix}\right) \quad \text{or} \quad W(T) = W\left(\begin{pmatrix} 1 & \|B\| \\ 0 & -1 \end{pmatrix}\right).$$

Since $A^* = A$ and $A^2 = I_k$, we may write $A = \begin{pmatrix} \cos \theta & * \\ * & A_1 \end{pmatrix}$, where $A_1 \in M_{k-1}$ is self-adjoint. By interlacing property, there exists a unitary matrix $U_1 \in M_{k-1}$ such that $U_1^* A_1 U_1 = [\alpha] \oplus I_{k_1-1} \oplus (-I_{k_2-1})$, where $\alpha \in [-1, 1]$. Write $U_1^* B_1 = [b_{ij}] \in M_{k-1, m-1}$. Set $U_2 = [1] \oplus U_1 \in M_k$ and define $U = U_2 \oplus I_m$, so that $U \in M_n$ is unitary. Then

$$U^* T U = \begin{pmatrix} U_2^* A U_2 & U_2^* B \\ 0 & 0 \end{pmatrix} \equiv T',$$

where

$$U_2^* A U_2 = \left(\begin{array}{cc|cc} \cos \theta & a_{12} & & \\ a_{21} & \alpha & & \\ \hline & & I_{k_1-1} & \\ 0 & & & -I_{k_2-1} \end{array} \right) \quad \text{and} \quad U_2^* B = \left(\begin{array}{cc|cc} \|B\| & 0 & \cdots & 0 \\ 0 & b_{11} & \cdots & b_{1,m-1} \\ \hline \vdots & \vdots & & \vdots \\ 0 & b_{k-1,1} & \cdots & b_{k-1,m-1} \end{array} \right).$$

Note that the submatrix $\begin{pmatrix} \cos \theta & a_{12} \\ a_{21} & \alpha \end{pmatrix}$ is unitarily similar to $\text{diag}(1, -1)$, thus $\alpha = -\cos \theta$ and $\bar{a}_{21} = a_{12} = e^{i\mu} \sin \theta$ for some real μ , after a unitary similarity by a diagonal unitary matrix, we may assume that $a_{12} = a_{21} = \sin \theta$. On the other hand, the horizontal line $y = \|B\|/2$ is tangent to the elliptical disk $W(T)$ at the point $c + i\|B\|/2$, where $c \in \mathbb{R}$ is the center of $W(T)$. Define $x_1 = [1 \ 0 \dots 0]^t \in \mathbb{C}^k$, $x_2 = [i \ 0 \dots 0]^t \in \mathbb{C}^m$, and $x = [x_1 \ x_2]^t / \sqrt{2} \in \mathbb{C}^n$. Then $\|x\| = 1$ and $\langle \text{Im}(T')x, x \rangle = \|B\|/2$, so the real part of $\langle T'x, x \rangle$ gives the center of the elliptical disk $W(T)$, or $c = \langle \text{Re}(T')x, x \rangle$. Compute

$$c = \langle \text{Re}(T')x, x \rangle = \left\langle \begin{pmatrix} \cos \theta & \frac{\|B\|}{2} \\ \frac{\|B\|}{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \right\rangle = \frac{\cos \theta}{2}.$$

If $W(T) = W\left(\begin{pmatrix} \pm 1 & \|B\| \\ 0 & 0 \end{pmatrix}\right)$, then $(\cos \theta)/2 = c = \pm 1/2$ and so, $\cos \theta = \pm 1$. By interchanging the 2nd and $(k+1)$ -th rows and columns of the matrix T' , we obtain $\begin{pmatrix} \pm 1 & \|B\| \\ 0 & 0 \end{pmatrix} \oplus T_2$ as asserted.

Now suppose that $W(T) = W\left(\begin{pmatrix} 1 & \|B\| \\ 0 & -1 \end{pmatrix}\right)$, then $\cos \theta = 2c = 0$. After a unitary similarity by a diagonal unitary matrix, we may assume that the 2×2 leading principal submatrix of $U_2^* A U_2$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $b_{1j} \geq 0$ for all $1 \leq j \leq m-1$. We now claim that $b_{1j} = 0$ for all $1 \leq j \leq m-1$. Indeed, for each $j = 1, \dots, m-1$, consider the 4×4 principal submatrix

$$T'(\{1, 2, k+1, k+1+j\}) = \begin{pmatrix} 0 & 1 & \|B\| & 0 \\ 1 & 0 & 0 & b_{1j} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \equiv E_j.$$

We know that $W(E_j) \subseteq W(T') = W\left(\begin{pmatrix} 1 & \|B\| \\ 0 & -1 \end{pmatrix}\right)$, hence

$$w(E_j) \leq w\left(\begin{pmatrix} 1 & \|B\| \\ 0 & -1 \end{pmatrix}\right) = \max \sigma\left(\begin{pmatrix} 1 & \frac{\|B\|}{2} \\ \frac{\|B\|}{2} & -1 \end{pmatrix}\right) = \frac{\sqrt{4 + \|B\|^2}}{2}.$$

On the other hand, Lemma 3.2 yields that

$$w^2(E_j) = \frac{4 + \|B\|^2 + b_{1j}^2 + \sqrt{16 + 8\|B\|^2 + 8b_{1j}^2 + (\|B\|^2 - b_{1j}^2)^2}}{8}.$$

If $b_{1j} \neq 0$, then

$$w^2(E_j) - \frac{4 + \|B\|^2}{4} = \frac{1}{8}(\sqrt{16 + 8\|B\|^2 + 8b_{1j}^2 + (\|B\|^2 - b_{1j}^2)^2} - (4 + \|B\|^2 - b_{1j}^2)) > 0,$$

a contradiction. We conclude that $b_{1j} = 0$ for all j . By interchanging the 3rd and $(k+1)$ -th rows and columns of T' , we obtain

$$\begin{pmatrix} 0 & 1 & \|B\| \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus T_2.$$

(b) $\Rightarrow$ (a). Let $T_1 = \begin{pmatrix} 0 & 1 & \|B\| \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We only need to show $W(T_1)$ is an elliptical disk. Since

T_1 is unitarily similar to

$$\begin{pmatrix} 1 & 0 & \|B\|/\sqrt{2} \\ 0 & -1 & \|B\|/\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix},$$

by [7, Theorem 2.2], it is easy to check that the numerical range of the above matrix is an elliptical disk. This completes the proof. $\square$

It is not hard to extend Theorem 3.1 to compact operators in $B(H)$. In fact, one can extend the result to general operators.

Theorem 3.3. *Suppose $T \in B(H \oplus H)$ has the form (1.1) such that $A = A^*$ and $A^2 = I$. Then $\mathbf{cl}(W(T))$ is an elliptical disk $\mathcal{E}$ with foci in $\{0, -1, 1\}$ if and only if one of the following holds.*

- (a) *There is a unit vector $x \in H$ such that $\|B^*x\| = \|B\|$ and T is unitarily similar to $T_1 \oplus T_2$ with $W(T_2) \subseteq W(T_1)$, where*

$$T_1 = \begin{pmatrix} a & \|B\| \\ 0 & 0 \end{pmatrix} \text{ with } a \in \{1, -1\} \quad \text{or} \quad T_1 = \begin{pmatrix} 0 & 1 & \|B\| \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- (b) *There is no unit vector $x \in H$ such that $\|B^*x\| = \|B\|$, and there is a sequence of unit vectors $\{x_m\}$ in H such that one of the following holds.*

$$(b.1) \quad \begin{pmatrix} \langle Ax_m, x_m \rangle & \|B^*x_m\| \\ 0 & 0 \end{pmatrix} \rightarrow T_0 = \begin{pmatrix} a & \|B\| \\ 0 & 0 \end{pmatrix}, \quad a \in \{1, -1\}, \text{ and } W(T_0) = \mathbf{cl}(W(T)) = \mathcal{E}.$$

$$(b.2) \quad \begin{pmatrix} \langle Ax_m, x_m \rangle & \sqrt{1 - \langle Ax_m, x_m \rangle^2} & \|B^*x_m\| \\ \sqrt{1 - \langle Ax_m, x_m \rangle^2} & -\langle Ax_m, x_m \rangle & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow T_0 = \begin{pmatrix} 0 & 1 & \|B\| \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and}$$

$$W(T_0) = \mathbf{cl}(W(T)) = \mathcal{E}.$$

Proof. If $A = \pm I$, then T is a quadratic operator and the assertion follows from [11, Theorem 2.1]. Therefore, we may assume that $A \neq \pm I$. The sufficiency proof is clear. For the necessity, assume that $\mathbf{cl}(W(T))$ is an elliptical disk, where the minor axis has end points $c \pm ib/2$, where $b = \|B\|$ and $c \in \{0, \pm 1/2, \pm 1\}$ because the foci are $\{0, 0\}, \{-1, -1\}, \{1, 1\}, \{0, 1\}, \{-1, 0\}, \{-1, 1\}$. Suppose there is a unit vector $x \in H$ such that $\|B^*x\| = \|B\|$. Consider the 2×2 matrix $T_x = \begin{pmatrix} \langle Ax, x \rangle & \|B^*x\| \\ 0 & 0 \end{pmatrix}$. Since $W(T_x) \subseteq W(T)$ and their minor axis have the same length, hence $c = \langle Ax, x \rangle/2$, that is, $|c| \leq 1/2$. It follows that $c \in \{0, \pm 1/2\}$, and hence the foci can only be $\{0, 0\}, \{0, 1\}, \{-1, 0\}, \{-1, 1\}$. If the foci are $\{0, 0\}$, then $\mathbf{cl}(W(T))$ is the circular disk centered at 0 with radius $\|B\|/2$ and thus $\langle Ax, x \rangle = 2c = 0$. It implies that $x = (1/\sqrt{2})(u_1 + u_2)$, where $u_1 \in \ker(A - I)$ and $u_2 \in \ker(A + I)$ are unit vectors. Let $y = (1/\sqrt{2})(u_1 - u_2)$, $\tilde{x} = x \oplus 0$, $\tilde{y} = y \oplus 0$, and $\tilde{z} = 0 \oplus (B^*x/\|B\|)$. Then $\{\tilde{x}, \tilde{y}, \tilde{z}\}$ is orthonormal, and the compression of T to their span is

$$\tilde{T} \equiv T|_{\{\tilde{x}, \tilde{y}, \tilde{z}\}} = \begin{pmatrix} \langle T\tilde{x}, \tilde{x} \rangle & \langle T\tilde{y}, \tilde{x} \rangle & \langle T\tilde{z}, \tilde{x} \rangle \\ \langle T\tilde{x}, \tilde{y} \rangle & \langle T\tilde{y}, \tilde{y} \rangle & \langle T\tilde{z}, \tilde{y} \rangle \\ \langle T\tilde{x}, \tilde{z} \rangle & \langle T\tilde{y}, \tilde{z} \rangle & \langle T\tilde{z}, \tilde{z} \rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 & \|B\| \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix $\tilde{T} \oplus [0]$ is the matrix $T_{\pi/2}$ in Lemma 3.2 with $\alpha_1 = \|B\|$ and $\alpha_2 = 0$. Therefore,

$$w(\tilde{T}) = \frac{\sqrt{4 + \|B\|^2}}{2} > \frac{\|B\|}{2}.$$

Since the numerical range of a compression is contained in the numerical range of the original operator, this contradicts the assumption that $\mathbf{cl}(W(T))$ is the circular disk of radius $\|B\|/2$. The pair of foci is therefore one of $\{0, 1\}$, $\{-1, 0\}$, and $\{-1, 1\}$.

Put $b = \|B\|$ and $v = B^*x/b$. Since $\|B^*x\| = b$, we have $BB^*x = b^2x$, and hence $Bv = bx$. Suppose first that the foci are 0 and $a \in \{1, -1\}$. Since $a/2 = c = \langle Ax, x \rangle/2$, the assumptions $A = A^*$ and $A^2 = I$ imply $Ax = ax$. Thus, $\text{span}\{x \oplus 0, 0 \oplus v\}$ reduces T , and T is unitarily similar to $T_1 \oplus T_2$, $T_1 = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. Now suppose that the foci are -1 and 1 . Then $\langle Ax, x \rangle = 0$. Let $y = Ax$; then x, y are orthonormal and $Ay = x$. We claim that $B^*y = 0$. Otherwise, put $d = \|B^*y\| > 0$ and $q = B^*y/d$. Since $Bv = bx$, we have $\langle q, v \rangle = 0$, and the compression of T to the span of $x \oplus 0, y \oplus 0, 0 \oplus v, \oplus q$ has the matrix

$$E_d = \begin{pmatrix} 0 & 1 & b & 0 \\ 1 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $W(E_d) \subseteq W(T)$, $w(E_d) \leq \frac{\sqrt{4+b^2}}{2}$. On the other hand, the computation in the proof of Lemma 3.2 gives

$$w^2(E_d) = \frac{4 + b^2 + d^2 + \sqrt{16 + 8b^2 + 8d^2 + (b^2 - d^2)^2}}{8} > \frac{4 + b^2}{4},$$

a contradiction. Hence $B^*y = 0$. It follows that $\text{span}\{x \oplus 0, y \oplus 0, 0 \oplus v\}$ reduces T , and T is unitarily similar to

$$T_1 \oplus T_2, \quad T_1 = \begin{pmatrix} 0 & 1 & b \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In either case, $W(T_1) = \text{cl}(W(T))$, and therefore $W(T_2) \subseteq W(T) \subseteq W(T_1)$.

Finally, suppose that no unit vector $x \in H$ satisfies $\|B^*x\| = \|B\|$, and put $b = \|B\|$. Choose unit vectors $x_m \in H$ such that $\|B^*x_m\| \rightarrow b$. Fix a free ultrafilter $\mathcal{U}$, and let K be the Hilbert space ultrapower of H , with $\langle [u_m], [v_m] \rangle = \lim_{\mathcal{U}} \langle u_m, v_m \rangle$. We identify H with the constant sequences in K , and extend every $C \in B(H)$ by $\tilde{C}[u_m] = [Cu_m]$. These extensions preserve adjoints, products, and norms. Hence $\tilde{A} = \tilde{A}^*$, $\tilde{A}^2 = I$, and, for $\xi = [x_m]$,

$$\|\xi\| = 1, \quad \|\tilde{B}^*\xi\| = \lim_{\mathcal{U}} \|B^*x_m\| = b = \|\tilde{B}\|.$$

Thus, $\tilde{B}^*$ attains its norm at ξ .

Let $\tilde{T} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ 0 & 0 \end{pmatrix}$. Since constant sequences embed $W(T)$ into $W(\tilde{T})$, while every element of $W(\tilde{T})$ is an ultralimit of elements of $W(T)$, $\text{cl}(W(\tilde{T})) = \text{cl}(W(T)) = E$. We may therefore apply the norm-attaining case to $\tilde{T}$ and ξ .

Set $a_m = \langle Ax_m, x_m \rangle$ and $r_m = \|B^*x_m\|$. If the resulting summand is 2-by-2, then $\lim_{\mathcal{U}} a_m = a$ for some $a \in \{1, -1\}$. Since this ultralimit is a cluster point, after passing to a subsequence we have $a_m \rightarrow a$, and hence

$$\begin{pmatrix} a_m & r_m \\ 0 & 0 \end{pmatrix} \rightarrow T_0 = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}.$$

This gives (b.1).

If the resulting summand is 3-by-3, then $\lim_{\mathcal{U}} a_m = 0$. Passing to a subsequence, we may assume that $a_m \rightarrow 0$, and therefore

$$\begin{pmatrix} a_m & \sqrt{1-a_m^2} & r_m \\ \sqrt{1-a_m^2} & -a_m & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow T_0 = \begin{pmatrix} 0 & 1 & b \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This gives (b.2). In either case, the norm-attaining result gives $W(T_0) = E = \text{cl}(W(T))$. This completes the proof. $\square$

Remark 3.4. *The necessity of Theorem 3.3 was proved under the assumption that $\text{cl}(W(T))$ is an elliptical disk with foci in $\{0, 1, -1\}$. We have the following conjecture.*

Conjecture 3.5. *Let $T \in B(H \oplus H)$ be of the form*

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix},$$

where $A = A^$ and $A^2 = I$. If $\text{cl}(W(T))$ is an elliptical disk, then its foci are contained in $\{0, 1, -1\}$.*

We are able to prove the conjecture in the case where one of the foci is 0; in this case, the other focus must be either 1 or -1 . The general case remains open.

Next we determine the minimum of the numerical radius of T with A such that $A = A^*$ and $A^2 = I$ for a given B . We assume that $A \neq \pm I$ to avoid trivial consideration. Therefore, we assume that up to unitary similarity, $A = I_p \oplus -I_q$ with $p \geq q \geq 1$. Among other things, suppose $A \in M_k$ and $B \in M_{k,m}$. If $k < m$, by singular value decomposition, we may assume that $B = [B_1 \ 0_{k \times (m-k)}]$ where $B_1 \in M_k$, thus T is unitarily similar to $\begin{pmatrix} U^*AU & B_1 \\ 0_k & 0_k \end{pmatrix} \oplus 0_{m-k}$ for some unitary matrix $U \in M_k$. In this case, we have $w(T) = w\left(\begin{pmatrix} U^*AU & B_1 \\ 0_k & 0_k \end{pmatrix}\right)$. On the other hand, if $k > m$, let $B' = [B \ 0_{k \times (k-m)}] \in M_k$, then the matrix $T' \equiv \begin{pmatrix} A & B' \\ 0_k & 0_k \end{pmatrix} = T \oplus 0_{k-m}$, and thus $w(T) = w(T')$. Therefore, we only need to consider that both A and B are $k \times k$ matrices.

For convenience, if $B \in M_k$ has singular values $s_1 \geq s_2 \geq \dots \geq s_k \geq 0$, let

$$S_B = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_k \end{pmatrix} \in M_k \quad \text{and} \quad J_k = \begin{pmatrix} & & 1 \\ & \ddots & \\ 1 & & \end{pmatrix} \in M_k. \quad (3.1)$$

Among other things, if $A = [a_{ij}] \in M_k$ and $B = [b_{ij}] \in M_k$ satisfy $0 \leq a_{ij} \leq b_{ij}$ for all $1 \leq i, j \leq k$, we write $A \preceq B$. In this case, we have $w(A) \leq w(B)$, because a nonnegative matrix attains its numerical radius at a nonnegative unit vector.

Here is our main theorem.

Theorem 3.6. *Let $1 \leq q \leq p$, $k = p + q$, $A = I_p \oplus (-I_q) \in M_k$, $B \in M_k$, and*

$$\alpha = \min \left\{ w\left(\begin{pmatrix} U^*AU & B \\ 0_k & 0_k \end{pmatrix}\right) : U \in M_k \text{ is unitary} \right\}.$$

Let $\tilde{A} = I_p \oplus (-I_p) \in M_{2p}$, $\tilde{B} = B \oplus s_{q+1}I_{p-q} \in M_{2p}$, and

$$\beta = \min \left\{ w\left(\begin{pmatrix} V^*\tilde{A}V & \tilde{B} \\ 0_{2p} & 0_{2p} \end{pmatrix}\right) : V \in M_{2p} \text{ is unitary} \right\}.$$

Then

$$\alpha = w\left(\left(\begin{array}{c|c} J_q & \\ \hline I_{p-q} & S_B \\ \hline J_q & \\ \hline 0_k & 0_k \end{array}\right)\right) = w\left(\begin{pmatrix} J_{2p} & S_{\tilde{B}} \\ 0_{2p} & 0_{2p} \end{pmatrix}\right) = \beta,$$

where J_q , J_{2p} , S_B , and $S_{\tilde{B}}$ are as in (3.1).

We need the following lemma to prove Theorem 3.6.

Lemma 3.7. *Let $1 \leq q \leq p$, $k = p + q$, $B = \text{diag}(s_1, \dots, s_k) \in M_k$ with $s_1 \geq s_2 \geq \dots \geq s_k \geq 0$ and $B_1 = \text{diag}(s_2, \dots, s_{k-1}) \in M_{k-2}$. Then, for any unitary $U \in M_k$, we have*

$$w\left(\begin{pmatrix} U^*(I_p \oplus (-I_q))U & B \\ 0_k & 0_k \end{pmatrix}\right) \geq w\left(\left(\begin{array}{cc|cc} 0 & 1 & s_1 & 0 \\ 1 & 0 & 0 & s_k \\ \hline 0_2 & & 0_2 & \end{array}\right) \oplus \begin{pmatrix} U'^*(I_{p-1} \oplus (-I_{q-1}))U' & B_1 \\ 0_{k-2} & 0_{k-2} \end{pmatrix}\right)$$

for some unitary matrix $U' \in M_{k-2}$.

Proof. We first show that

$$w\left(\begin{pmatrix} U^*(I_p \oplus (-I_q))U & B \\ 0_k & 0_k \end{pmatrix}\right) \geq w\left(\left(\begin{array}{cc|cc} 0 & 1 & s_1 & 0 \\ 1 & 0 & 0 & s_k \\ \hline 0_2 & & 0_2 & \end{array}\right)\right).$$

Indeed, let $U^*(I_p \oplus (-I_q))U = [a_{ij}]_{i,j=1}^k$ and $A_1 = [a_{ij}]_{i,j=2}^k \in M_{k-1}$ be the principal submatrix obtained by deleting the first row and first column of $U^*(I_p \oplus (-I_q))U$. By the interlacing property, there exists a unitary matrix $U_1 \in M_{k-1}$ such that $U_1^* A_1 U_1 = [t_1] \oplus I_{p-1} \oplus (-I_{q-1})$, where $t_1 \in [-1, 1]$. Write $U_1^* \text{diag}(s_2, \dots, s_k) U_1 = [b_{ij}]_{i,j=1}^{k-1}$, we have $b_{11} \geq s_k$ from the interlacing property. Let $W_1 = [1] \oplus U_1 \in M_k$ be unitary, then

$$W_1^*(U^*(I_p \oplus (-I_q))U)W_1 = \left(\begin{array}{cc|cc} a_{11} & a'_{12} & & 0 \\ a'_{21} & t_1 & & \\ \hline 0 & & I_{p-1} & \\ & & & -I_{q-1} \end{array}\right)$$

and

$$W_1^* B W_1 = \left(\begin{array}{cc|cc} s_1 & 0 & \cdots & 0 \\ 0 & b_{11} & \cdots & b_{1,k-1} \\ \hline \vdots & \vdots & & \vdots \\ 0 & b_{k-1,1} & \cdots & b_{k-1,k-1} \end{array}\right).$$

Note that the matrix $\begin{pmatrix} a_{11} & a'_{12} \\ a'_{21} & t_1 \end{pmatrix}$ is unitarily similar to $\text{diag}(1, -1)$ and $b_{11} \geq s_k$, by Lemma 3.2 and Corollary 2.4, we obtain

$$\begin{aligned} w\left(\begin{pmatrix} U^*(I_p \oplus (-I_q))U & B \\ 0_k & 0_k \end{pmatrix}\right) &\geq w\left(\left(\begin{array}{cc|cc} a_{11} & a'_{12} & s_1 & 0 \\ a'_{21} & t_1 & 0 & b_{11} \\ \hline 0_2 & & 0_2 & \end{array}\right)\right) \\ &\geq w\left(\left(\begin{array}{cc|cc} 0 & 1 & s_1 & 0 \\ 1 & 0 & 0 & b_{11} \\ \hline 0_2 & & 0_2 & \end{array}\right)\right) \geq w\left(\left(\begin{array}{cc|cc} 0 & 1 & s_1 & 0 \\ 1 & 0 & 0 & s_k \\ \hline 0_2 & & 0_2 & \end{array}\right)\right) \end{aligned}$$

as asserted.

We next show that

$$w\left(\begin{pmatrix} U^*(I_p \oplus (-I_q))U & B \\ 0_k & 0_k \end{pmatrix}\right) \geq w\left(\begin{pmatrix} U'^*(I_{p-1} \oplus (-I_{q-1}))U' & B_1 \\ 0_{k-2} & 0_{k-2} \end{pmatrix}\right)$$

for some unitary matrix $U' \in M_{k-2}$. Indeed, let $A_2 = [a_{ij}]_{i,j=1}^{k-1} \in M_{k-1}$ be the leading principal submatrix of $U^*(I_p \oplus (-I_q))U$. By the interlacing property, there exists a unitary matrix $U_2 \in M_{k-1}$ such that $U_2^* A_2 U_2 = [t_2] \oplus I_{p-1} \oplus (-I_{q-1})$, where $t_2 \in [-1, 1]$. Write $U_2^* \text{diag}(s_1, \dots, s_{k-1}) U_2 = [c_{ij}]_{i,j=1}^{k-1}$ and let $C = [c_{ij}]_{i,j=2}^{k-1} \in M_{k-2}$ be the principal submatrix obtained by deleting the first row and first column of $U_2^* \text{diag}(s_1, \dots, s_{k-1}) U_2$. Let $W_2 = U_2 \oplus [1] \in M_k$ be unitary. Then

$$W_2^*(U^*(I_p \oplus (-I_q))U)W_2 = \left(\begin{array}{c|cc} t_2 & 0 & a''_{1k} \\ \hline 0 & I_{p-1} & 0 \\ & -I_{q-1} & \\ \hline a''_{k1} & 0 & a_{kk} \end{array} \right)$$

and

$$W_2^* B W_2 = \left(\begin{array}{c|cc} c_{11} & \cdots c_{1,k-1} & 0 \\ \hline \vdots & C & \vdots \\ c_{k-1,1} & & 0 \\ \hline 0 & \cdots 0 & s_k \end{array} \right),$$

thus $\begin{pmatrix} I_{p-1} \oplus (-I_{q-1}) & C \\ 0_{k-2} & 0_{k-2} \end{pmatrix}$ is a principal submatrix of $\begin{pmatrix} W_2^*(U^*(I_p \oplus (-I_q))U)W_2 & W_2^* B W_2 \\ 0_k & 0_k \end{pmatrix}$. Note that C is positive semidefinite with eigenvalues $d_1 \geq d_2 \geq \cdots \geq d_{k-2} \geq 0$. Let $D = \text{diag}(d_1, \dots, d_{k-2})$, then $C = U' D U'^*$ for some unitary matrix $U' \in M_{k-2}$. Let $W_3 = U' \oplus U' \in M_{2k-4}$ be unitary, we have

$$W_3^* \begin{pmatrix} I_{p-1} \oplus (-I_{q-1}) & C \\ 0_{k-2} & 0_{k-2} \end{pmatrix} W_3 = \begin{pmatrix} U'^*(I_{p-1} \oplus (-I_{q-1}))U' & D \\ 0_{k-2} & 0_{k-2} \end{pmatrix}.$$

From the interlacing property, we deduce that

$$s_1 \geq d_1 \geq s_2 \geq d_2 \geq \cdots \geq s_{k-2} \geq d_{k-2} \geq s_{k-1},$$

thus $D \succeq B_1 = \text{diag}(s_2, \dots, s_{k-1})$. Corollary 2.4 yields that

$$w\left(\begin{pmatrix} U'^*(I_{p-1} \oplus (-I_{q-1}))U' & D \\ 0_{k-2} & 0_{k-2} \end{pmatrix}\right) \geq w\left(\begin{pmatrix} U'^*(I_{p-1} \oplus (-I_{q-1}))U' & B_1 \\ 0_{k-2} & 0_{k-2} \end{pmatrix}\right).$$

Therefore, we obtain

$$\begin{aligned} & w\left(\begin{pmatrix} U^*(I_p \oplus (-I_q))U & B \\ 0_k & 0_k \end{pmatrix}\right) \geq w\left(\begin{pmatrix} I_{p-1} \oplus (-I_{q-1}) & C \\ 0_{k-2} & 0_{k-2} \end{pmatrix}\right) \\ & = w\left(\begin{pmatrix} U'^*(I_{p-1} \oplus (-I_{q-1}))U' & D \\ 0_{k-2} & 0_{k-2} \end{pmatrix}\right) \geq w\left(\begin{pmatrix} U'^*(I_{p-1} \oplus (-I_{q-1}))U' & B_1 \\ 0_{k-2} & 0_{k-2} \end{pmatrix}\right) \end{aligned}$$

as asserted. This completes the proof. $\square$

We are now ready to prove Theorem 3.6.

Proof of Theorem 3.6. First we consider α . Let $B = Q_0 S_B Q_1^*$ be the singular value decomposition of B and $W_0 = Q_0 \oplus Q_1 \in M_{2k}$ be unitary, then, for any unitary $U \in M_k$, we have

$$W_0^* \begin{pmatrix} U^* A U & B \\ 0_k & 0_k \end{pmatrix} W_0 = \begin{pmatrix} (U Q_0)^* A (U Q_0) & S_B \\ 0_k & 0_k \end{pmatrix}.$$

Since $U Q_0 \in M_k$ is also unitary, we deduce that

$$\alpha = \min \left\{ w \left(\begin{pmatrix} U^* A U & S_B \\ 0_k & 0_k \end{pmatrix} \right) : U \in M_k \text{ is unitary} \right\} \leq w \left(\left(\begin{array}{cc|c} & & J_q \\ & I_{p-q} & \\ \hline J_q & & S_B \\ & 0_k & 0_k \end{array} \right) \right). \quad (3.2)$$

Therefore, we may assume that $B = S_B = \text{diag}(s_1, \dots, s_k)$. Let $B_j = \text{diag}(s_{j+1}, \dots, s_{k-j}) \in M_{k-2j}$ for $j = 1, \dots, q$.

Now, for any unitary $U \in M_k$, apply Lemma 3.7 repeatedly, we obtain

$$\begin{aligned} & w \left(\begin{pmatrix} U^* A U & B \\ 0_k & 0_k \end{pmatrix} \right) \\ & \geq w \left(\left(\begin{array}{cc|cc} 0 & 1 & s_1 & 0 \\ 1 & 0 & 0 & s_k \\ \hline & 0_2 & & 0_2 \end{array} \right) \oplus \begin{pmatrix} U_1^* (I_{p-1} \oplus (-I_{q-1})) U_1 & B_1 \\ 0_{k-2} & 0_{k-2} \end{pmatrix} \right) \\ & \geq w \left(\left(\begin{array}{cc|cc} 0 & 1 & s_1 & 0 \\ 1 & 0 & 0 & s_k \\ \hline & 0_2 & & 0_2 \end{array} \right) \oplus \left(\begin{array}{cc|cc} 0 & 1 & s_2 & 0 \\ 1 & 0 & 0 & s_{k-1} \\ \hline & 0_2 & & 0_2 \end{array} \right) \oplus \begin{pmatrix} U_2^* (I_{p-2} \oplus (-I_{q-2})) U_2 & B_2 \\ 0_{k-4} & 0_{k-4} \end{pmatrix} \right) \\ & \geq \dots \\ & \geq w \left(\left(\begin{array}{cc|cc} 0 & 1 & s_1 & 0 \\ 1 & 0 & 0 & s_k \\ \hline & 0_2 & & 0_2 \end{array} \right) \oplus \dots \oplus \left(\begin{array}{cc|cc} 0 & 1 & s_q & 0 \\ 1 & 0 & 0 & s_{k-q+1} \\ \hline & 0_2 & & 0_2 \end{array} \right) \oplus \begin{pmatrix} U_q^* I_{p-q} U_q & B_q \\ 0_{k-2q} & 0_{k-2q} \end{pmatrix} \right) \\ & = w \left(\left(\begin{array}{cc|cc} 0 & 1 & s_1 & 0 \\ 1 & 0 & 0 & s_k \\ \hline & 0_2 & & 0_2 \end{array} \right) \oplus \dots \oplus \left(\begin{array}{cc|cc} 0 & 1 & s_q & 0 \\ 1 & 0 & 0 & s_{p+1} \\ \hline & 0_2 & & 0_2 \end{array} \right) \oplus \begin{pmatrix} I_{p-q} & B_q \\ 0_{p-q} & 0_{p-q} \end{pmatrix} \right), \end{aligned}$$

where $U_j \in M_{k-2j}$ is unitary for $j = 1, \dots, q$. Note that the matrix

$$\left(\begin{array}{cc|c} & & J_q \\ & I_{p-q} & \\ \hline J_q & & S_B \\ & 0_k & 0_k \end{array} \right)$$

is permutationally similar to

$$\left(\begin{array}{cc|cc} 0 & 1 & s_1 & 0 \\ 1 & 0 & 0 & s_k \\ \hline & 0_2 & & 0_2 \end{array} \right) \oplus \dots \oplus \left(\begin{array}{cc|cc} 0 & 1 & s_q & 0 \\ 1 & 0 & 0 & s_{p+1} \\ \hline & 0_2 & & 0_2 \end{array} \right) \oplus \begin{pmatrix} I_{p-q} & B_q \\ 0_{p-q} & 0_{p-q} \end{pmatrix},$$

hence we conclude that

$$w\left(\begin{pmatrix} U^*AU & B \\ 0_k & 0_k \end{pmatrix}\right) \geq w\left(\begin{array}{c|c} J_q & S_B \\ \hline J_q & 0_k \end{array}\right)$$

for any unitary $U \in M_k$, that is,

$$\alpha \geq w\left(\begin{array}{c|c} J_q & S_B \\ \hline J_q & 0_k \end{array}\right) \quad \text{or} \quad \alpha = w\left(\begin{array}{c|c} J_q & S_B \\ \hline J_q & 0_k \end{array}\right)$$

by (3.2).

Now we turn to β . From the proof on α , we have

$$\begin{aligned} \alpha &= w\left(\begin{array}{c|c} J_q & S_B \\ \hline J_q & 0_k \end{array}\right) \\ &= w\left(\begin{array}{cc|cc} 0 & 1 & s_1 & 0 \\ 1 & 0 & 0 & s_k \\ \hline 0_2 & & 0_2 & \end{array}\right) \oplus \cdots \oplus \left(\begin{array}{cc|cc} 0 & 1 & s_q & 0 \\ 1 & 0 & 0 & s_{p+1} \\ \hline 0_2 & & 0_2 & \end{array}\right) \oplus \begin{pmatrix} I_{p-q} & B_q \\ 0_{p-q} & 0_{p-q} \end{pmatrix} \\ &= \max \left\{ \max_{1 \leq j \leq q} w\left(\begin{array}{cc|cc} 0 & 1 & s_j & 0 \\ 1 & 0 & 0 & s_{k-j+1} \\ \hline 0_2 & & 0_2 & \end{array}\right), \max_{q+1 \leq j \leq p} w\left(\begin{pmatrix} 1 & s_j \\ 0 & 0 \end{pmatrix}\right) \right\}. \end{aligned}$$

Since $s_{q+1} \geq s_{q+2} \geq \cdots \geq s_p$, we have $w\left(\begin{pmatrix} 1 & s_j \\ 0 & 0 \end{pmatrix}\right) \leq w\left(\begin{pmatrix} 1 & s_{q+1} \\ 0 & 0 \end{pmatrix}\right) = (1 + \sqrt{1 + s_{q+1}^2})/2$ for all $q+2 \leq j \leq p$. Therefore, we obtain

$$\alpha = \max \left\{ \max_{1 \leq j \leq q} w\left(\begin{array}{cc|cc} 0 & 1 & s_j & 0 \\ 1 & 0 & 0 & s_{k-j+1} \\ \hline 0_2 & & 0_2 & \end{array}\right), \frac{1 + \sqrt{1 + s_{q+1}^2}}{2} \right\}. \quad (3.3)$$

On the other hand, since $\tilde{B} = B \oplus s_{q+1}I_{p-q}$, from (3.1), we have

$$S_{\tilde{B}} = \text{diag}(s_1, \dots, s_q) \oplus s_{q+1}I_{p-q} \oplus \underbrace{\text{diag}(s_{q+1}, \dots, s_p)}_{p-q} \oplus \underbrace{\text{diag}(s_{p+1}, \dots, s_k)}_q.$$

From we have proved before, we obtain

$$\beta \equiv \min \left\{ w\left(\begin{pmatrix} V^* \tilde{A} V & \tilde{B} \\ 0_{2p} & 0_{2p} \end{pmatrix}\right) : V \in M_{2p} \text{ is unitary} \right\} = w\left(\begin{pmatrix} J_{2p} & S_{\tilde{B}} \\ 0_{2p} & 0_{2p} \end{pmatrix}\right).$$

Note that $\begin{pmatrix} J_{2p} & S_{\tilde{B}} \\ 0_{2p} & 0_{2p} \end{pmatrix}$ is permutationally similar to

$$\left(\sum_{j=1}^q \oplus \left(\frac{0 \ 1}{1 \ 0} \middle| \frac{s_j \ 0}{0 \ s_{k-j+1}} \right) \right) \oplus \left(\sum_{j=q+1}^p \oplus \left(\frac{0 \ 1}{1 \ 0} \middle| \frac{s_{q+1} \ 0}{0 \ s_j} \right) \right).$$

In particular, the matrix

$$\left(\frac{0 \ 1}{1 \ 0} \middle| \frac{s_{q+1} \ 0}{0 \ s_{q+1}} \right)$$

is unitarily similar to $\begin{pmatrix} 1 & s_{q+1} \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} -1 & s_{q+1} \\ 0 & 0 \end{pmatrix}$, this yields that

$$w\left(\left(\frac{0 \ 1}{1 \ 0} \middle| \frac{s_{q+1} \ 0}{0 \ s_{q+1}} \right) \right) = w\left(\begin{pmatrix} 1 & s_{q+1} \\ 0 & 0 \end{pmatrix} \right) = \frac{1 + \sqrt{1 + s_{q+1}^2}}{2}.$$

Moreover, since $s_{q+1} \geq s_{q+2} \geq \dots \geq s_p$, by Corollary 2.4, we have

$$w\left(\left(\frac{0 \ 1}{1 \ 0} \middle| \frac{s_{q+1} \ 0}{0 \ s_j} \right) \right) \leq w\left(\left(\frac{0 \ 1}{1 \ 0} \middle| \frac{s_{q+1} \ 0}{0 \ s_{q+1}} \right) \right) \text{ for all } q+2 \leq j \leq p.$$

It follows that

$$w\left(\sum_{j=q+1}^p \oplus \left(\frac{0 \ 1}{1 \ 0} \middle| \frac{s_{q+1} \ 0}{0 \ s_j} \right) \right) = w\left(\left(\frac{0 \ 1}{1 \ 0} \middle| \frac{s_{q+1} \ 0}{0 \ s_{q+1}} \right) \right) = \frac{1 + \sqrt{1 + s_{q+1}^2}}{2}.$$

Hence we obtain

$$\begin{aligned} \beta &= w\left(\begin{pmatrix} J_{2p} & S_{\tilde{B}} \\ 0_{2p} & 0_{2p} \end{pmatrix} \right) \\ &= \max \left\{ w\left(\sum_{j=1}^q \oplus \left(\frac{0 \ 1}{1 \ 0} \middle| \frac{s_j \ 0}{0 \ s_{k-j+1}} \right) \right), w\left(\sum_{j=q+1}^p \oplus \left(\frac{0 \ 1}{1 \ 0} \middle| \frac{s_{q+1} \ 0}{0 \ s_j} \right) \right) \right\} \\ &= \max \left\{ \max_{1 \leq j \leq q} w\left(\left(\frac{0 \ 1}{1 \ 0} \middle| \frac{s_j \ 0}{0 \ s_{k-j+1}} \right) \right), \frac{1 + \sqrt{1 + s_{q+1}^2}}{2} \right\} = \alpha \end{aligned}$$

from (3.3), this completes the proof. $\square$

We can extend Theorem 3.6 to infinite dimensional operators using the fact that $B(H)$ is a C^* -algebra of real rank zero; see [2]. The positive semidefinite operator B in its integral representation $\int_{\lambda} \lambda dE(\lambda)$ can be approximated by a sequence of “diagonal” operators $\{B_m\}$ each has a finite

spectrum. By a suitable choice of unitary basis, we may consider operators of the form $T_m = \begin{pmatrix} A_m & B_m \\ 0 & 0 \end{pmatrix}$ with $A_m = \begin{pmatrix} 0 & 0 & J \\ 0 & I & 0 \\ J & 0 & 0 \end{pmatrix}$, where J is the infinite dimensional extension of J_k , say, $J(e_n) = e_{-n}$ if $J \in B(H)$ with $H = \ell_2(\mathbb{Z})$, and $J(f(x)) = f(-x)$ if $J \in B(H)$ with $H = L_2(\mathbb{R})$. Then one can show that $w(T_m)$ is minimum among operators of the form $\begin{pmatrix} U A_m U^* & B_m \\ 0 & 0 \end{pmatrix}$ and the conclusion also holds for the limit.

4 Matrix powers and constant norm attaining vectors

In this section, we study matrices for which there exists a unit vector x satisfying

$$\|A^k x\| = \|A^k\| = \|A\| \quad \text{for all } k = 1, 2, \dots$$

Such a vector will be called a *common norm attaining vector* for the powers of A . This problem is motivated by a simple observation. If $A \in M_n$ is an idempotent, i.e., $A^2 = A$, then the sequence $\{\|A^k\|\}_{k \geq 1}$ is constant, and there exists a unit vector attaining these norms simultaneously. This leads naturally to the following question.

Problem 4.1. *For which matrices $A \in M_n$ does there exist a unit vector x such that*

$$\|A^k x\| = \|A^k\| = \|A\| \quad \text{for all } k = 1, 2, \dots?$$

Observe that if $\|A\| = \|A^2\| > 0$, then

$$\|A\| = \|A^2\| \leq \|A\|^2,$$

which implies that $\|A\| \geq 1$.

The cases $\|A\| = 1$ and $n = 2$ were studied in [4] and [9], respectively, leading to the following results. The next two results are from [4, Corollary 2.6 and Proposition 3.2 (b)].

Proposition 4.2. *For an $n \times n$ matrix A , the following are equivalent:*

- (a) $\|A^n\| = \|A\| = 1$.
- (b) $\|A^k\| = 1$ for all $k \geq 1$.
- (c) There exists a unit vector x such that $\|A^k x\| = \|A^k\| = \|A\| = 1$ for all $k \geq 1$.
- (d) A is unitarily similar to $[a] \oplus B$ with $\|B\| \leq |a| = 1$.

Proposition 4.3. *For a 2×2 matrix A , the following are equivalent:*

- (a) There exists a unit vector x such that $\|A^k x\| = \|A^k\| = \|A\| > 1$ for all $k \geq 1$;

(b) A is unitarily similar to

$$\lambda \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix},$$

where $|\lambda| = 1$ and $b \neq 0$.

Regarding Problem 4.1, the next result shows that the matrix $T = \begin{pmatrix} U & B \\ 0 & 0 \end{pmatrix} \in M_n$, where U is unitary, satisfies the condition in Problem 4.1.

Theorem 4.4. *Let $T \in M_n$. Then the following are equivalent.*

(a) T is unitarily similar to a matrix of the form

$$\begin{pmatrix} U & B \\ 0 & 0_{n-r} \end{pmatrix},$$

where $U \in M_r$, $0 \leq r \leq n$, is unitary.

(b) $T^*T = T^{k*}T^k$ for all $k \geq 1$.

(c) $T^*T = T^{2*}T^2$.

In this case, the following statements hold:

(i) $\|T^k x\| = \|T^k\| = \|T\|$ for some unit vector $x \in \mathbb{C}^n$ and for all $k \geq 1$.

(ii) If T is nonzero, then $\|T\| = \sqrt{1 + \|B\|^2}$.

(iii) If λ is a nonzero eigenvalue of T , then $|\lambda| = 1$.

Proof. (b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (b). Assume that $T^*T = T^{2*}T^2$. Then, for $k \geq 2$,

$$T^{k*}T^k = T^{k-2*}T^{2*}T^2T^{k-2} = T^{k-2*}T^*TT^{k-2} = T^{k-1*}T^{k-1}.$$

Continue this process, we infer that $T^{k*}T^k = T^{k-1*}T^{k-1} = T^{k-2*}T^{k-2} = \dots = T^*T$ as required.

(b) $\Rightarrow$ (a). By the Schur's triangularization theorem (cf. [6, Theorem 2.3.1]), T is unitarily similar to an upper triangular matrix of the form $\begin{pmatrix} U & B \\ 0 & N \end{pmatrix}$, where U is an $r \times r$ ($0 \leq r \leq n$) nonsingular upper triangular matrix and $N \in M_{n-r}$ is upper triangular with $\sigma(N) = \{0\}$. We first check that $N = 0$. Indeed, since N is nilpotent, then $N^n = 0$, and thus $T^n = \begin{pmatrix} U^n & * \\ 0 & 0 \end{pmatrix}$. Since U is nonsingular, it follows that

$$\text{rank}(T^n) = \text{rank}(U^n) = \text{rank}(U).$$

On the other hand, $\text{rank}(X) = \text{rank}(X^*X)$ for any matrix $X \in M_n$. Therefore, condition (b) yields that

$$\text{rank}(T) = \text{rank}(T^*T) = \text{rank}(T^{n*}T^n) = \text{rank}(T^n) = \text{rank}(U).$$

Hence $\text{rank}(N) = \text{rank}(T) - \text{rank}(U) = 0$, or $N = 0$ as asserted.

It remains to show that U is unitary for $r \geq 1$. If $r = 0$, then $T = N = 0$ as asserted form. Therefore, we assume that $r \geq 1$. The equality $T^*T = T^{2*}T^2$ yields that

$$\begin{pmatrix} U^*U & U^*B \\ B^*U & B^*B \end{pmatrix} = \begin{pmatrix} U^{2*}U^2 & U^{2*}UB \\ B^*U^*U^2 & B^*U^*UB \end{pmatrix}.$$

Comparing the $(1, 1)$ -blocks yields $U^*U = U^{2*}U^2$. Since U is invertible, we multiply by $(U^*)^{-1}$ on the left and U^{-1} on the right to obtain $I_r = U^*U$. Hence U is unitary.

(a) $\Rightarrow$ (b). After a unitary similarity, we may assume that $T = \begin{pmatrix} U & B \\ 0 & 0_{n-r} \end{pmatrix}$, where $U \in M_r$ is unitary. Let $V = U \oplus I_{n-r} \in M_n$ be unitary, then, for $k \geq 1$, we have

$$T^k = \begin{pmatrix} U^k & U^{k-1}B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} U^{k-1} & 0 \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} U & B \\ 0 & 0 \end{pmatrix} = V^{k-1}T,$$

and thus

$$T^{k*}T^k = T^*V^{k-1*}V^{k-1}T = T^*T$$

as required.

Moreover, since V is unitary, it follows that $\|T^k\| = \|V^{k-1}T\| = \|T\|$ for all $k \geq 1$. Among other things, since T is a finite matrix, then $\|Tx\| = \|T\|$ for some unit vector $x \in \mathbb{C}^n$, thus $\|T^kx\| = \|V^{k-1}Tx\| = \|Tx\| = \|T\| = \|T^k\|$ for all $k \geq 1$, hence (i) holds.

Next, it is easily seen that

$$TT^* = \begin{pmatrix} I_r + BB^* & 0 \\ 0 & 0 \end{pmatrix},$$

hence $\|T\|^2 = \|T^*\|^2 = \|TT^*\| = \|I_r + BB^*\| = 1 + \|BB^*\| = 1 + \|B^*\|^2 = 1 + \|B\|^2$, this proves (ii).

Finally, (iii) follows from $\sigma(T) \subseteq \sigma(U) \cup \{0\}$ and U is unitary. This completes the proof. $\square$

We aim to formulate a necessary and sufficient condition for a nonzero matrix $A \in M_n$ satisfying

$$\|A^kx\| = \|A^k\| = \|A\| \quad \text{for all } k \geq 1 \text{ and for some unit vector } x \in \mathbb{C}^n.$$

The following theorem is the main result of this section, it gives a complete solution of Problem 4.1. Note that the case $\|A\| = 1$ has been solved by Proposition 4.2. Therefore, we need only to consider the case $\|A\| > 1$.

Theorem 4.5. *Let $A \in M_n$ with $\|A\| > 1$, and let m be the degree of the minimal polynomial of A . The following conditions are equivalent:*

(a) *There exists a unit vector $x \in \mathbb{C}^n$ such that*

$$\|A^kx\| = \|A^k\| = \|A\| \quad \text{for all } k = 1, 2, \dots, m+1.$$

(b) A is unitarily similar to a matrix of the form

$$\begin{pmatrix} T & D \\ 0 & C \end{pmatrix},$$

where $T \in M_r$, $2 \leq r \leq m$, satisfies

$$T^*T = T^{2*}T^2 \quad \text{and} \quad \|A^k\| = \|T^k\| \quad \text{for all } k = 1, 2, \dots, m+1.$$

In particular, if $r = 2$, then $D = 0$.

In this case, A is singular.

For the proof of Theorem 4.5, we need the next lemma.

In the following, we use $\{e_1, \dots, e_n\}$ to denote the standard basis of $\mathbb{C}^n$. Moreover, for any matrix $X \in M_n$, if $\|Xx\| = \|X\|$ for some unit vector $x \in \mathbb{C}^n$, then $X^*Xx = \|X\|^2x$, because $\langle (\|X\|^2I_n - X^*X)x, x \rangle = 0$ and $\|X\|^2I_n - X^*X$ is positive semidefinite.

Lemma 4.6. For $n \geq 2$, let T be an $n \times n$ matrix of the form

$$\begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1,n-1} & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2,n-1} & t_{2n} \\ 0 & t_{32} & \cdots & t_{3,n-1} & t_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_{n,n-1} & t_{nn} \end{pmatrix},$$

where $t_{i,i-1} \neq 0$ for $i = 2, 3, \dots, n$. If $\|T^k e_1\| = \|T^k\| = \|T\|$ for $k = 1, 2, \dots, n+1$, then

$$T^*T = T^{2*}T^2.$$

Proof. Note that $Te_j \in \bigvee\{e_1, \dots, e_{j+1}\}$ for $j = 1, \dots, n-1$, it follows that

$$T^2e_j \in T(\bigvee\{e_1, \dots, e_{j+1}\}) = \bigvee\{Te_1, \dots, Te_{j+1}\} \subseteq \bigvee\{e_1, \dots, e_{j+2}\}.$$

Continue this process, we have $T^k e_j \in \bigvee\{e_1, \dots, e_{j+k}\}$. In particular,

$$T^k e_1 \in \bigvee\{e_1, \dots, e_{k+1}\} \quad \text{for } 1 \leq k \leq n-1, \quad (4.1)$$

and thus

$$\langle T^j e_1, e_{k+1} \rangle = 0 \quad \text{for } 0 \leq j < k \leq n-1. \quad (4.2)$$

Moreover, for $k = 1, 2, \dots, n-1$, since

$$\langle T^k e_1, e_{k+1} \rangle = \langle T(T^{k-1}e_1), e_{k+1} \rangle = \left\langle \sum_{j=1}^k \langle T^{k-1}e_1, e_j \rangle Te_j, e_{k+1} \right\rangle = \langle T^{k-1}e_1, e_k \rangle \langle Te_k, e_{k+1} \rangle,$$

we infer that

$$\langle T^k e_1, e_{k+1} \rangle = \langle Te_1, e_2 \rangle \langle Te_2, e_3 \rangle \cdots \langle Te_k, e_{k+1} \rangle = t_{21} \cdots t_{k+1,k} \neq 0 \quad \text{for } k = 1, 2, \dots, n-1. \quad (4.3)$$

Therefore, by (4.1), (4.2) and (4.3), we obtain successively $e_{k+1} \in \bigvee\{e_1, Te_1, \dots, T^k e_1\}$ and

$$\bigvee\{e_1, Te_1, \dots, T^k e_1\} = \bigvee\{e_1, e_2, \dots, e_{k+1}\} \quad \text{for } k = 0, 1, \dots, n-1.$$

In particular, this implies that $\{e_1, Te_1, \dots, T^{n-1}e_1\}$ is a basis of $\mathbb{C}^n$. Therefore, it suffices to prove that

$$T^*T(T^k e_1) = T^{2*}T^2(T^k e_1) \quad \text{for all } k = 0, 1, \dots, n-1. \quad (4.4)$$

We proceed by induction on k .

(i) The case $k = 0$. Since $\|Te_1\| = \|T\| = \|T^2\| = \|T^2e_1\|$, hence

$$T^*Te_1 = \|T\|^2e_1 = \|T^2\|^2e_1 = T^{2*}T^2e_1.$$

(ii) The case $1 \leq k \leq n-1$. Assume that (4.4) holds for $0, \dots, k-1$, that is,

$$T^*(T^*T - I_n)T(T^j e_1) = 0 \quad \text{for all } j = 0, 1, \dots, k-1. \quad (4.5)$$

We want to show that $T^*T(T^k e_1) = T^{2*}T^2(T^k e_1)$. Indeed, Let

$$\mu = T^{2*}T^2(T^k e_1) - T^*T(T^k e_1) = T^*(T^*T - I_n)T(T^k e_1).$$

Then, for $0 \leq j \leq k-1$, (4.5) yields that

$$\langle \mu, T^j e_1 \rangle = \langle T^*(T^*T - I_n)T(T^k e_1), T^j e_1 \rangle = \langle T^k e_1, T^*(T^*T - I_n)T(T^j e_1) \rangle = \langle T^k e_1, 0 \rangle = 0.$$

Moreover, by assumption, we have $\|T^{k+2}e_1\| = \|T^{k+2}\| = \|T\| = \|T^{k+1}\| = \|T^{k+1}e_1\|$, we infer that

$$T^{k+2*}T^{k+2}e_1 = \|T^{k+2}\|^2e_1 = \|T\|^2e_1 = \|T^{k+1}\|^2e_1 = T^{k+1*}T^{k+1}e_1,$$

or, $T^{k+1*}(T^*T - I_n)T^{k+1}e_1 = 0$. Therefore, for $k \leq j \leq n-1$, we obtain

$$\langle \mu, T^j e_1 \rangle = \langle T^*(T^*T - I_n)T(T^k e_1), T^j e_1 \rangle = \langle T^{k+1*}(T^*T - I_n)T^{k+1}e_1, T^{j-k}e_1 \rangle = \langle 0, T^{j-k}e_1 \rangle = 0.$$

Since $\{e_1, Te_1, \dots, T^{n-1}e_1\}$ is a basis of $\mathbb{C}^n$ and $\langle \mu, T^j e_1 \rangle = 0$ for all $0 \leq j \leq n-1$, we deduce that $\mu = 0$, and hence

$$T^*T(T^k e_1) = T^{2*}T^2(T^k e_1).$$

This completes the proof. □

We are now ready to prove Theorem 4.5.

Proof of Theorem 4.5. (b) $\Rightarrow$ (a). Assume that $A = \begin{pmatrix} T & D \\ 0 & C \end{pmatrix}$, where T satisfies the conditions in (b). Theorem 4.4 (i) yields that

$$\|T^k x\| = \|T^k\| = \|T\| \quad \text{for some unit vector } x \in \mathbb{C}^r \text{ and for all } k \geq 1.$$

Then

$$A^k = \begin{pmatrix} T^k & * \\ 0 & C^k \end{pmatrix} \quad \text{for all } k \geq 1. \quad (4.6)$$

Set $\tilde{x} = [x \ 0]^t \in \mathbb{C}^n$. Then $A^k \tilde{x} = [T^k x \ 0]^t$, and for $1 \leq k \leq m+1$,

$$\|A^k \tilde{x}\| = \|T^k x\| = \|T^k\| = \|A^k\| = \|T\| = \|A\|.$$

Hence (a) holds.

(a) $\Rightarrow$ (b). Let

$$K = \bigvee \{x, Ax, \dots, A^{m-1}x\} \quad \text{and} \quad r = \dim K.$$

Since m is the degree of the minimal polynomial of A , thus K is an invariant subspace of A and $r \leq m$. We first show that $r \geq 2$. If $r = 1$, then $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$. Hence

$$|\lambda| = \|Ax\| = \|A\| = \|A^2 x\| = |\lambda|^2,$$

which implies $|\lambda| = 1$, contradicting $\|A\| > 1$. Thus $r \geq 2$. Since $r = \dim K$, the vectors $\{x, Ax, \dots, A^{r-1}x\}$ are linearly independent. Extend them to an orthonormal basis $\{u_1, u_2, \dots, u_n\}$ of $\mathbb{C}^n$ such that

$$\bigvee \{x, Ax, \dots, A^{k-1}x\} = \bigvee \{u_1, u_2, \dots, u_k\}, \quad k = 1, \dots, r,$$

where $u_1 = x$. Let $U = [u_1 \ \dots \ u_n] \in M_n$ be unitary. Since $K = \bigvee \{x, Ax, \dots, A^{r-1}x\} = \bigvee \{u_1, u_2, \dots, u_r\}$ is invariant for A , U^*AU has the form

$$\tilde{A} \equiv U^*AU = \begin{pmatrix} T & D \\ 0 & C \end{pmatrix},$$

where $T = [t_{ij}] \in M_r$ is upper Hessenberg with $t_{i,i-1} \neq 0$ for $i = 2, \dots, r$. Note that $Ue_1 = u_1 = x$. Let $e'_1 = [1 \ 0 \ \dots \ 0]^t \in \mathbb{C}^r$. By assumption $\|A^k x\| = \|A^k\|$, $1 \leq k \leq m+1$, we have

$$\|T^k\| \leq \|\tilde{A}^k\| = \|A^k\| = \|A^k x\| = \|U\tilde{A}^k U^*(Ue_1)\| = \|\tilde{A}^k e'_1\| = \|T^k e'_1\| \leq \|T^k\|.$$

Thus the inequalities above are actually equalities throughout. This implies that

$$\|A^k\| = \|T^k e'_1\| = \|T^k\|$$

for $k = 1, 2, \dots, m+1$. In particular, $\|T\| = \|A\| = \|A^k\| = \|T^k\|$ for all $1 \leq k \leq m+1$. That is,

$$\|T^k e'_1\| = \|T^k\| = \|T\| \quad \text{for } k = 1, \dots, r+1.$$

By Lemma 4.6, we obtain $T^*T = T^{*2}T^2$ as required.

In particular, if $r = 2$. Since $T^*T = T^{*2}T^2$, by Theorem 4.4 and $\|T\| = \|A\| > 1$, T is unitarily similar to

$$\begin{pmatrix} \alpha & b \\ 0 & 0 \end{pmatrix}, \quad \text{where } |\alpha| = 1 \text{ and } b \neq 0.$$

Hence, without loss of generality, we may assume that T has the above form. Since $\|\tilde{A}\| = \|T\| = \sqrt{1 + |b|^2}$ and $\|\tilde{A}^*\| \geq \|\tilde{A}^* e_1\|$, we deduce that

$$D = \begin{pmatrix} 0 & \cdots & 0 \\ d_1 & \cdots & d_{n-2} \end{pmatrix},$$

and thus the first row of $\tilde{A}$ is $(\alpha \ b \ 0 \ \dots \ 0)$. It follows that the first row of $\tilde{A}^2$ is

$$(\alpha^2 \ \alpha b \ bd_1 \ \dots \ bd_{n-2}).$$

Since $\|\tilde{A}^2\| = \|\tilde{A}\| = \sqrt{1 + |b|^2}$, $|\alpha| = 1$ and $b \neq 0$, it forces that $d_j = 0$ for all $1 \leq j \leq n - 2$. Thus $D = 0$ as asserted.

Finally, if A is nonsingular, then T is also nonsingular. It follows that

$$I_r = (T^*)^{-1} T^* T (T^{-1}) = (T^*)^{-1} T^{2*} T^2 (T^{-1}) = T^* T,$$

or T is unitary. Thus $\|A\| = \|T\| = 1$, contradicting $\|A\| > 1$. Hence A is singular, this completes the proof. $\square$

We have the following consequence immediately.

Corollary 4.7. *Let A be an n -by- n nonsingular matrix, and let m be the degree of the minimal polynomial of A . The following conditions are equivalent:*

- (a) *There exists a unit vector $x \in \mathbb{C}^n$ such that $\|A^k x\| = \|A^k\| = \|A\|$ for all $1 \leq k \leq m + 1$.*
- (b) *A is unitarily similar to $[a] \oplus C$ with $\|C\| \leq |a| = 1$.*
- (c) *There exists a unit vector $x \in \mathbb{C}^n$ such that $\|A^k x\| = \|A^k\| = \|A\|$ for all $k \geq 1$.*

Proof. The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (a) are trivial. Assume (a) holds, $\|A\| = \|A^2\| \leq \|A\|^2$ implies that $\|A\| \geq 1$. On the other hand, since A is nonsingular, Theorem 4.5 yields that $\|A\| \leq 1$. Therefore, $1 = \|A\| = \|A^k\|$ for all $1 \leq k \leq m + 1$, and hence (b) follows from [9, Theorem 2.1]. This completes the proof. $\square$

For Theorem 4.5, the next proposition shows that the condition

$$\|A^k x\| = \|A^k\| = \|A\| > 1 \quad \text{for } k = 1, 2, \dots, m + 1,$$

cannot, in general, be reduced to $k \leq m$.

Proposition 4.8. *For $n \geq 2$, let $A \in M_n$ be of the form*

$$A = \begin{pmatrix} 0 & & & 1 \\ 2 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \\ & & & 1 & 0 \end{pmatrix}.$$

Then the following statements hold:

- (a) *The minimal polynomial of A has degree n .*
- (b) $\|A^k e_1\| = \|A^k\| = 2$ for $k = 1, 2, \dots, n$.
- (c) *A is not unitarily similar to a matrix of the form*

$$\begin{pmatrix} T & D \\ 0 & C \end{pmatrix},$$

where T satisfies the condition in Theorem 4.5 (b).

Proof. (a) A direct computation shows that the characteristic polynomial of A is

$$\det(xI_n - A) = x^n - 2 = \prod_{k=0}^{n-1} (x - 2^{1/n} e^{i \frac{2k\pi}{n}}).$$

Hence A has n distinct eigenvalues

$$2^{1/n}, 2^{1/n} e^{i \frac{2\pi}{n}}, \dots, 2^{1/n} e^{i \frac{2(n-1)\pi}{n}}. \quad (4.7)$$

It follows that the minimal polynomial of A is $x^n - 2$, and thus has degree n .

(b) A direct computation yields

$$\begin{aligned} A^* A &= \text{diag}(4, 1, \dots, 1), \quad A^{2*} A^2 = \text{diag}(4, 1, \dots, 1, 4), \\ \dots, \quad A^{(n-1)*} A^{n-1} &= \text{diag}(4, 1, 4, \dots, 4), \quad A^{n*} A^n = 4I_n. \end{aligned}$$

Thus $\|A^k e_1\| = \|A^k\| = 2$ for $k = 1, 2, \dots, n$.

(c) By (4.7), all eigenvalues of A have modulus $2^{1/n}$. Thus (c) follows from Theorem 4.4 (iii). $\square$

Let $T \in M_r$ satisfy $T^* T = T^{2*} T^2$, and let

$$A = \begin{pmatrix} T & D \\ 0 & C \end{pmatrix}.$$

By Theorem 4.4 (b), we have

$$\|T\| = \|T^k\| \leq \|A^k\| \quad \text{for all } k \geq 1.$$

Note that in Theorem 4.5, if $r = 2$ and $\|A^k\| = \|T\|$ for all $k \geq 1$, then $D = 0$. A natural question arises: can Theorem 4.5 be extended to the case $r \geq 3$? The answer is negative. A counterexample is given below.

Proposition 4.9. *Let*

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & \frac{1}{4} \\ 0 & -1 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and let $x = \frac{1}{\sqrt{6}}[1 \ -1 \ 2 \ 0 \ 0]^t \in \mathbb{C}^5$. Then the following hold:

- (a) $T^*T = T^{2*}T^2$.
- (b) $\|A^k x\| = \|A^k\| = \|A\| = \|T\| = \sqrt{3}$ for all $k \geq 1$.
- (c) A is unitarily irreducible.

Proof. (a) A direct computation shows that

$$T^*T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix} = T^{2*}T^2.$$

(b) For $k \geq 1$, one computes

$$A^{2k} = \begin{pmatrix} 1 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & -1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4^{2k}} & \frac{1}{4^{2k}} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A^{2k+1} = \begin{pmatrix} 1 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & -1 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4^{2k+1}} & \frac{1}{4^{2k+1}} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Moreover,

$$A^*A = \begin{pmatrix} 1 & 0 & 1 & 0 & \frac{1}{4} \\ 0 & 1 & -1 & 0 & \frac{1}{4} \\ 1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{16} & \frac{1}{16} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{16} & \frac{1}{4} \end{pmatrix}.$$

For $k \geq 1$, one verifies that

$$A^{k*}A^k x = 3x \quad \text{and} \quad T^*T x' = 3x',$$

where $x' = \frac{1}{\sqrt{6}}[1 \ -1 \ 2]^t \in \mathbb{C}^3$. Hence 3 is an eigenvalue of both $A^{k*}A^k$ and T^*T .

Furthermore,

$$\text{tr}(T^*T) = 4 < 6,$$

and

$$\text{tr}(A^{k*}A^k) \leq \text{tr}(A^{2*}A^2) = 4 + \frac{1}{2} + \frac{2}{4^4} < 6.$$

It follows that 3 is the largest eigenvalue of both $A^{k*}A^k$ and T^*T . Therefore, we obtain

$$\|T\|^2 = \|T^*T\| = 3$$

and

$$\|A^k x\|^2 = \langle A^{k*}A^k x, x \rangle = 3\|x\|^2 = 3 = \|A^{k*}A^k\| = \|A^k\|^2$$

for all $k \geq 1$.

(c) It suffices to show that the only projections commuting with A are 0_5 and I_5 . Let P be a 5×5 Hermitian projection such that $PA = AP$. Write

$$P = (p_{ij})_{i,j=1}^5.$$

Comparing the (i, j) -entries of AP and PA , we obtain successively

$$p_{15} = p_{25} = p_{35} = p_{45} = 0,$$

$$p_{14} = p_{24} = p_{34} = 0, \quad p_{55} = p_{44},$$

$$p_{13} = p_{23} = 0, \quad p_{33} = p_{55},$$

$$p_{12} = 0, \quad p_{22} = p_{55},$$

$$p_{11} = p_{55}.$$

Hence $P = p_{55}I_5$. Since $P = P^2 = P^*$, it follows that $P = 0_5$ or $P = I_5$. Therefore, A is unitarily irreducible. $\square$

Finally, we construct a 3-by-3 unitarily irreducible matrix T satisfying $T^*T = T^{2*}T^2$, which provides a counterexample to Conjecture 1.1.

Proposition 4.10. *Let*

$$T = \begin{pmatrix} e^{i\sqrt{2}\pi} & 0 & \frac{1}{\sqrt{2}} \\ 0 & e^{-i\sqrt{2}\pi} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the following statements hold:

(a) *There exists a unit vector x such that*

$$\|T^k x\| = \|T^k\| = \sqrt{2} \quad \text{for all } k \geq 1.$$

(b) *T^k is unitarily irreducible for all $k \geq 1$.*

(c) *For all $k \geq 1$, T^k is not a modulus-one multiple of an idempotent.*

Proof. (a) In Theorem 4.4, take

$$U = \begin{pmatrix} e^{i\sqrt{2}\pi} & 0 \\ 0 & e^{-i\sqrt{2}\pi} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Since $T = \begin{pmatrix} U & B \\ 0 & 0 \end{pmatrix}$ and U is unitary, the desired conclusion follows from Theorem 4.4.

(b) For each $k \geq 1$, the eigenvalues of T^k are $e^{ik\sqrt{2}\pi}$, $e^{-ik\sqrt{2}\pi}$, and 0. A direct computation shows that the corresponding eigenvectors can be chosen as e_1, e_2 , and

$$v = \begin{pmatrix} 1 \\ e^{i2\sqrt{2}\pi} \\ -\sqrt{2}e^{i\sqrt{2}\pi} \end{pmatrix}.$$

In particular, we have

$$\langle v, e_1 \rangle \neq 0 \quad \text{and} \quad \langle v, e_2 \rangle \neq 0.$$

Hence, there is no nontrivial reducing subspace spanned by a subset of eigenvectors, and therefore T^k is unitarily irreducible for all $k \geq 1$.

(c) This follows immediately from (b) and [4, Lemma 3.1]. $\square$

Here, [4, Lemma 3.1] says that if $X \in M_n$ is an idempotent and $n \geq 3$, then X is unitarily reducible.

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