Birkhoff-James ε-Orthogonality Sets in Normed Vector Spaces

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The Birkhoff-James ϵ-Orthogonality

For two elements χ and ψ of a complex normed linear space $(\mathcal{X}, \|\cdot\|)$, χ is called **Birkhoff-James orthogonal** to ψ, denoted by $\chi \perp_{BJ} \psi$, if

$$\|\chi + \lambda\psi\| \geq \|\chi\|, \quad \forall\, \lambda \in \mathbb{C}.$$

Furthermore, for any $\epsilon \in [0, 1)$, we say that χ is **Birkhoff-James ϵ-orthogonal** to ψ, denoted by $\chi \perp_{BJ}^{\epsilon} \psi$, if

$$\|\chi + \lambda\psi\| \geq \sqrt{1-\epsilon^2}\,\|\chi\|, \quad \forall\, \lambda \in \mathbb{C}.$$

In an inner product space $(\mathcal{X}, \langle\cdot,\cdot\rangle)$, a $\chi \in \mathcal{X}$ is called **ϵ-orthogonal** to a $\psi \in \mathcal{X}$, denoted by $\chi \perp^{\epsilon} \psi$, if

$$|\langle\chi, \psi\rangle| \leq \epsilon\,\|\chi\|\,\|\psi\|.$$

Furthermore, $\chi \perp^{\epsilon} \psi$ if and only if $\chi \perp_{BJ}^{\epsilon} \psi$.

The Birkhoff-James ϵ-Orthogonality Set

For any $\chi, \psi \in \mathcal{X}$ with $\psi \neq 0$, and any $\epsilon \in [0,1)$, the **Birkhoff-James ϵ-orthogonality set of χ with respect to ψ** is defined by

$$\begin{aligned} F^{\epsilon}_{\|\cdot\|}(\chi;\psi) &= \{\mu \in \mathbb{C} :\ \psi \perp^{\epsilon}_{BJ} (\chi - \mu\psi)\} \\ &= \left\{\mu \in \mathbb{C} :\ \|\chi - \lambda\psi\| \geq \sqrt{1-\epsilon^2}\,\|\psi\|\,|\mu - \lambda|,\ \forall\, \lambda \in \mathbb{C}\right\} \\ &= \bigcap_{\lambda \in \mathbb{C}} \mathcal{D}\left(\lambda, \frac{\|\chi - \lambda\psi\|}{\sqrt{1-\epsilon^2}\,\|\psi\|}\right). \end{aligned}$$

$F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$ is a **nonempty**, **compact** and **convex** subset of the complex plane that lies in $\mathcal{D}\left(0, \frac{\|\chi\|}{\sqrt{1-\epsilon^2}\,\|\psi\|}\right)$.

For $\mathcal{X} = \mathbb{C}^{n\times n}$, $\chi = A$ and $\psi = I_n$, we have $F^{0}_{\|\cdot\|_2}(A; I_n) \equiv F(A) = \{\langle Ax, x\rangle \in \mathbb{C} :\ x \in \mathbb{C}^n,\ \langle x, x\rangle = 1\}$ (numerical range of matrices).

An Equivalent Definition

Let $\mathcal{X}^*$ denote the complex normed linear space of all continuous linear functionals of $\mathcal{X}$ (using the induced operator norm).

Let $\chi, \psi \in \mathcal{X}$ with $\psi \neq 0$. For any $\varepsilon \in [0,1)$, define the set

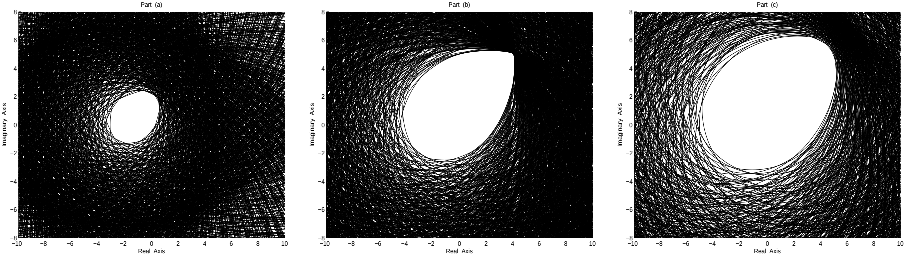
$$L_\varepsilon(\psi) = \left\{ f \in \mathcal{X}^* : \; f(\psi) = \sqrt{1-\varepsilon^2}\,\|\psi\| \text{ and } \|f\| \leqslant 1 \right\}.$$

This set $L_\varepsilon(\psi)$ is **nonempty**, **compact** and **convex**.

Moreover, it holds that

$$F^\varepsilon_{\|\cdot\|}(\chi;\psi) = \left\{ \frac{f(\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} : \; f \in L_\varepsilon(\psi) \right\}.$$

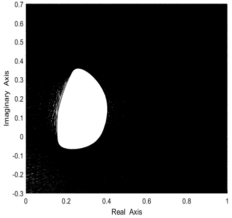
An Example

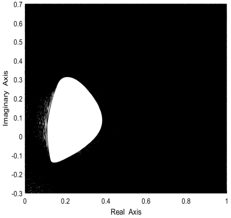


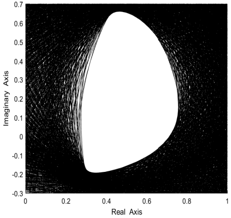
The sets $F^{0.5}_{\|\cdot\|_2}(\chi;\psi)$, $F^{\sqrt{0.5}}_{\|\cdot\|_2}(\chi;\psi)$ and $F^{\sqrt{0.6}}_{\|\cdot\|_2}(\chi;\psi)$ for

$$\chi = \begin{bmatrix} 4+i5 & 0 & i & 0 \\ 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & -i2 \end{bmatrix} \quad \text{and} \quad \psi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & -i & 0 \end{bmatrix}.$$

One More Example







The sets $F^{0.4}_{\|\cdot\|_1}(\chi_1;\psi)$ (left), $F^{0.4}_{\|\cdot\|_1}(\chi_2;\psi)$ (middle), and $F^{0.4}_{\|\cdot\|_1}(\chi_1+\chi_2;\psi)$ (right) for the sequences

$$\chi_1=\left\{1,\frac{1}{2-\mathrm{i}},\frac{1}{(2-\mathrm{i})^2},\frac{1}{(2-\mathrm{i})^3},\dots\right\},\quad \chi_2=\left\{1,\frac{1}{1-2\mathrm{i}},\frac{1}{(1-2\mathrm{i})^2},\frac{1}{(1-2\mathrm{i})^3},\dots\right\}$$

and $\psi=\left\{1,\frac{1}{1+\mathrm{i}},\frac{1}{(1+\mathrm{i})^2},\frac{1}{(1+\mathrm{i})^3},\dots\right\}$ in ℓ^1.

Some Basic Properties

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- For any $\chi_1, \chi_2 \in \mathcal{X}$, $F^{\epsilon}_{\|\cdot\|}(\chi_1 + \chi_2; \psi) \subseteq F^{\epsilon}_{\|\cdot\|}(\chi_1; \psi) + F^{\epsilon}_{\|\cdot\|}(\chi_2; \psi)$.

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- If the norm $\|\cdot\|$ is induced by the inner product of matrices $\langle\cdot,\cdot\rangle$, then

$$F^{\epsilon}_{\|\cdot\|}(\chi; \psi) = \mathcal{D}\left(\frac{\langle\chi, \psi\rangle}{\|\psi\|^2}, \left\|\chi - \frac{\langle\chi, \psi\rangle}{\|\psi\|^2}\psi\right\| \frac{\epsilon}{\sqrt{1-\epsilon^2}\,\|\psi\|}\right).$$

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- $\chi = b\psi$ for a $b \in \mathbb{C}$ if and only if

$$F^{\epsilon}_{\|\cdot\|}(\chi; \psi) = \{b\}, \quad \forall\, \epsilon \in [0, 1).$$

The Growth

Suppose χ is not a scalar multiple of ψ ($\neq 0$).
Then the following hold:

- If $0 \leq \epsilon_1 < \epsilon_2 < 1$, then $F^{\epsilon_1}_{\|\cdot\|}(\chi;\psi)$ lies in the interior of $F^{\epsilon_2}_{\|\cdot\|}(\chi;\psi)$.

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- The mappings $\chi \mapsto F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$ and $\epsilon \mapsto F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$ are continuous.

The Boundary

Suppose χ is not a scalar multiple of ψ $(\neq 0)$.
Then the following hold:

- A point μ_0 lies on $\partial F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$ if and only if

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- If $0 < \epsilon < 1$, then $\partial F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$ has no flat portions.

Vector-Valued Polynomials

Consider an vector-valued polynomial

$$p(z) = \chi_l z^l + \chi_{l-1} z^{l-1} + \cdots + \chi_1 z + \chi_0,$$

where z is a complex variable and $\chi_0, \ldots, \chi_l \in \mathcal{X}$, with $\chi_l \neq 0$.

For any nonzero vector $\psi \in \mathcal{X}$, and any $\epsilon \in [0,1)$, we define the **Birkhoff-James ϵ-orthogonality set of $p(z)$ with respect to ψ**

$$\begin{aligned} W^{\epsilon}_{\|\cdot\|}(p(z);\psi) &= \left\{\mu \in \mathbb{C} :\ 0 \in F^{\epsilon}_{\|\cdot\|}(p(\mu);\psi)\right\} \\ &= \left\{\mu \in \mathbb{C} :\ \|p(\mu) - \lambda\psi\| \geq \sqrt{1-\epsilon^2}\, \|\psi\|\, |\lambda|,\ \forall\, \lambda \in \mathbb{C}\right\} \\ &= \{\mu \in \mathbb{C} :\ \psi \perp^{\epsilon}_{BJ} p(\mu)\}. \end{aligned}$$

Some Basic Properties

- For any $\alpha \in \mathbb{C}\backslash\{0\}$, $W^{\epsilon}_{\|.\|}(p(\alpha z);\psi) = \alpha^{-1} W^{\epsilon}_{\|.\|}(p(z);\psi)$ and $W^{\epsilon}_{\|.\|}(p(z+\alpha);\psi) = W^{\epsilon}_{\|.\|}(p(z);\psi) - \alpha$.

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- If $\hat{p}(z) = \chi_0 z^l + \cdots + \chi_{l-1} z + \chi_l$ (reverse polynomial), then $W^{\epsilon}_{\|\cdot\|}(\hat{p}(z);\psi)\backslash\{0\} = \left\{\mu \in \mathbb{C} : \mu^{-1} \in W^{\epsilon}_{\|\cdot\|}(p(z);\psi)\backslash\{0\}\right\}$.

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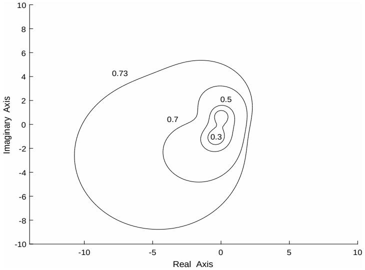
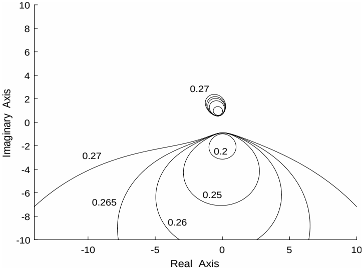
- The continuity of norms yields the closeness of $W^{\epsilon}_{\|\cdot\|}(p(z);\psi)$.
- $W^{\epsilon}_{\|\cdot\|}(p(z);\psi)$ is not necessarily bounded or connected.

Matrix Norms Induced by Inner Products





If the norm $\|\cdot\|$ is induced by the inner product $\langle\cdot,\cdot\rangle$, then

$$W^{\epsilon}_{\|\cdot\|}(p(z);\psi) = \left\{ \mu : \sum_{i,j=0}^{l} \langle \chi_i, \psi \rangle \langle \psi, \chi_j \rangle \mu^i \overline{\mu}^j - \epsilon^2 \|\psi\|^2 \sum_{i,j=0}^{l} \langle \chi_i, \chi_j \rangle \mu^i \overline{\mu}^j \leq 0 \right\}.$$

Boundedness and Boundary

- Suppose 0 is not an isolated point of $W^{\epsilon}_{\|\cdot\|}(\hat{p}(z);\psi)$. Then, $W^{\epsilon}_{\|\cdot\|}(p(z);\psi)$ is unbounded if and only if $0 \in F^{\epsilon}_{\|\cdot\|}(\chi_I;\psi)$.

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- If μ_0 lies on $\partial W^{\epsilon}_{\|\cdot\|}(p(z);\psi)$, then $0 \in \partial F^{\epsilon}_{\|\cdot\|}(p(\mu_0);\psi)$.

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- If $0 \in \partial F^{\epsilon}_{\|\cdot\|}(p(\mu_0);\psi) \backslash F^{\epsilon}_{\|\cdot\|}(P'(\mu_0);\psi)$ and $p(\mu_0) \neq 0$, then $\mu_0 \in \partial W^{\epsilon}_{\|\cdot\|}(p(z);\psi)$.

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- Suppose 0 is not an isolated point of $W_{\|\cdot\|}^{\epsilon}(\hat{p}(z);\psi)$. Then, $W_{\|\cdot\|}^{\epsilon}(p(z);\psi)$ is unbounded if and only if $0 \in F_{\|\cdot\|}^{\epsilon}(\chi_I;\psi)$.

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- If $\mu_0 \in \mathbb{C}$ such that $p(\mu_0) = 0$ and $0 \notin F_{\|\cdot\|}^{\epsilon}(P'(\mu_0);\psi)$, then μ_0 is an isolated point of $W_{\|\cdot\|}^{\epsilon}(p(z);\psi)$.

Surprise! A New Cosine

Let $\chi, \psi \in \mathcal{X}\backslash\{0\}$ be two linearly independent vectors. Then the **Birkhoff-James cosine of χ and ψ** is defined by

$$\cos_{BJ}(\widehat{\chi, \psi}) = \min\{\varepsilon \in [0,1) : \psi \perp_{BJ}^{\epsilon} \chi\} = \min\left\{\varepsilon \in [0,1) : 0 \in F_{\|\cdot\|}^{\varepsilon}(\psi; \chi)\right\}.$$

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- If norm $\|\cdot\|$ is induced by the inner product $\langle\cdot,\cdot\rangle$, then $\cos_{BJ}(\widehat{\chi,\psi}) = \frac{|\langle\chi,\psi\rangle|}{\|\chi\|\,\|\psi\|}$.

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- The norm $\|\cdot\|$ is induced by an inner product.

What about Birkhoff-James ε-orthogonality sets?

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