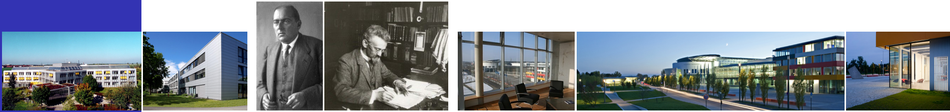
Some Remarks on Quantum Systems Theory as Pertaining to Numerical Ranges

A Unified Lie-Geometric Viewpoint

Thomas Schulte-Herbrüggen

relating to (joint) work with

G. Dirr, R. Zeier, C.K. Li, Y.T. Poon, F. vom Ende



14th Workshop on Numerical Ranges and Radii WONRA

– 100th anniversary of the TOEPLITZ-HAUSDORFF Theorem (1918/1919) –

MPQ and TUM-IAS, Munich-Garching, June 2018

Motivation: Numerical Ranges as Measuring Sticks for Optimising Quantum Dynamics

I Symmetry Approach to Quantum Systems Theory

II Generalising C-Numerical Ranges

III Gradient Flows

Outline

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Expectation values of observables $B = B^\dagger \in \mathcal{B}(\mathcal{H})$:

- pure quantum states $|\psi(t)\rangle = U(t)|\psi_0\rangle \in \mathcal{H}$:

  $$\langle B \rangle_t := \langle \psi(t) B | \psi(t) \rangle \quad \in \quad W(B) := \{\langle \phi B | \phi \rangle, \,|\, \|\phi\| = 1\}$$

- mixed states $\rho(t) \in U\rho_0 U^\dagger$:

  $$\operatorname{tr}\left(B^\dagger \rho(t)\right) \in W(B, \rho_0) = \{\operatorname{tr}\left(B^\dagger \, U\rho_0 U^{-1}\right) \,|\, U \in \mathcal{U}(\mathcal{H})\}$$

- C numerical range:
  generalisation to non-Hermitian operators A, C

  $$W(C, A) := \{\operatorname{tr}(C^\dagger \, UAU^{-1}) | U \in \mathcal{U}(\mathcal{H})\}$$

Significance of Numerical and C-Numerical Ranges

Generalising Expectation Values

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Generalise from $B = B^\dagger$ to non-Hermitian operator C:

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$$W(C,A) := \{\operatorname{tr}(C^\dagger\; UAU^{-1}) | U \in \mathcal{U}(\mathcal{H})\}$$

Find points on unitary orbit of initial state A with

- minimal Euclidean distance to target C

$$\min_U \|C - UAU^{-1}\|_2^2 \;\Leftrightarrow\; \max_U \operatorname{Re}\operatorname{tr}\{C^\dagger\, UAU^{-1}\}$$

$\Leftrightarrow$ find max. real part of C num. range

- minimal angle to target C

$$\max_U \cos^2_{A,C}(U) = \max_U \frac{|\operatorname{tr}\{C^\dagger UAU^{-1}\}|^2}{\|A\|_2^2 \cdot \|C\|_2^2}$$

$\Leftrightarrow$ find: C num. radius $r_C(A) = \max_U |\operatorname{tr}\{C^\dagger\, UAU^{-1}\}|$

pro memoria: $\|C - UAU^{-1}\|_2^2 = \|A\|_2^2 + \|C\|_2^2 - 2\operatorname{Re}\operatorname{tr}\{C^\dagger\, UAU^{-1}\}$

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Find points on unitary orbit of initial state A with

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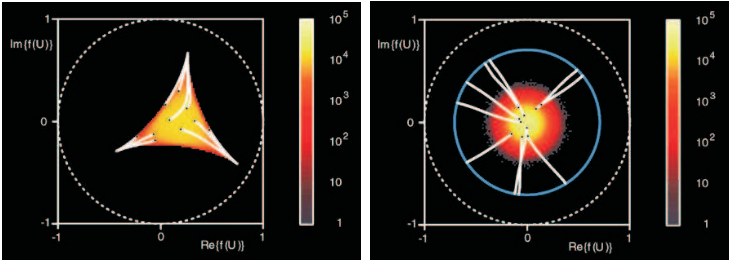
- find $r_C(A)$ by gradient flow on unitary group

$A, C \in \mathrm{Mat}_3(\mathbb{C})$ $\qquad$ $A, C \in \mathrm{Mat}_8(\mathbb{C})$



Glaser, T.S.H., Sieveking, Schedletzky, Nielsen, Sørensen, Griesinger, *Science* **280** (1998), 421

Early Connections

Trip from ETH to ILAS 1996

7004

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Sixth Conference of the

International Linear Algebra Society

Hosted by
The International Linear Algebra Society
and
Technische Universität Chemnitz–Zwickau

Böttcher Bau
TU Chemnitz–Zwickau
Chemnitz, Germany

August 14-17, 1996

Program

List of Participants

Early Connections
Trip from ETH to ILAS 1996

Friday, August 16

Invited Talks

9.00 – 9.50	Johannes Grabmeier *Computer Algebra — a Part of the Foundation of Scientific Computing*	HS 201 → p. 37
9.50 – 10.20	**Coffee / Tea**	
10.20 – 10.50	Chi-Kwong Li *Isomorphisms Between Normed Spaces*	HS 201 → p. 53
10.20 – 10.50	Michael Eiermann *Field of Values and Iterative Methods*	HS 305 → p. 25

Minisymposia

11.00 – 13.00	**Bernd Silbermann: C^*-Algebra Techniques in Computational Linear Algebra**	HS 201
11.00 – 12.30	**R. Horn: Canonical Forms**	HS 305
13.00 – 14.00	**Lunch**	

Minisymposium

14.00 – 15.00	**R. Horn: Canonical Forms**	HS 305

Contributed Talks

14.00 – 16.00	**Combinatorial Linear Algebra**	HS 201
14.00 – 16.00	**Control, Signals and Systems**	SR 367 A
14.00 – 16.00	**General Linear Algebra**	SR 367
15.00 – 15.40	**Numerical Methods: Linear Systems**	HS 305
15.40 – 16.00	**Numerical Methods: Eigenvalue Problems**	HS 305
16.30	**Excursion**	

Early Connections

Trip from ETH to ILAS 1996

Thursday, August 15

9.00 – 9.50	**Olga Taussky–Todd Lecture:** R. Guralnick *Traces and Generation of Matrix Algebras*	HS 201 → p. 38
9.50 – 10.20	**Coffee / Tea**	
	Invited Talks	
10.20 – 11.00	U. Helmke *Geometric Optimization Methods Solving Matrix Eigenvalue Problems*	HS 201 → p. 41
11.05 – 11.35	Thomas H. Pate *Node Diagrams, Row Adding, and Immanant Inequalities for Hermitian Positive Semi-definite Matrices*	HS 201 → p. 66
11.05 – 11.35	Roy Mathias *Relative Perturbation Theory for the Eigenvalue Problem*	HS 305 → p. 59
	Minisymposia	
11.40 – 12.40	**G. Michler: Parallel Computations in Algebra**	HS 201
11.40 – 12.40	**Nicholas Higham: Perturbation Theory**	HS 305
12.40 – 14.00	**Lunch**	
	Minisymposia	
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16.00 – 17.20	**Numerical Methods: Linear Systems**	SR 367
16.00 – 17.20	**General Linear Algebra**	HS 305
16.00 – 17.20	**Structured Matrices and Fast Algorithms**	HS 201
16.00 – 17.20	**Numerical Methods: Eigenvalue Problems**	SR 367 A
17.30 – 18.30	**ILAS Business Meeting**	
19.00	**Banquet**	

Generalised task:
approximate A_0 by elements on the sum of orbits

- unitary similarity: $$\min_{U_j \in SU(n)} \Big|\Big| \sum_{j=1}^{N} U_j A_j U_j^\dagger - A_0 \Big|\Big|_2^2$$
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- unitary t-congruence: $$\min_{U_j \in SU(n)} \Big|\Big| \sum_{j=1}^{N} U_j A_j U_j^t - A_0 \Big|\Big|_2^2$$
- $\dagger$-congruence: $$\min_{S_j \in SL(n)} \Big|\Big| \sum_{j=1}^{N} S_j A_j S_j^\dagger - A_0 \Big|\Big|_2^2$$

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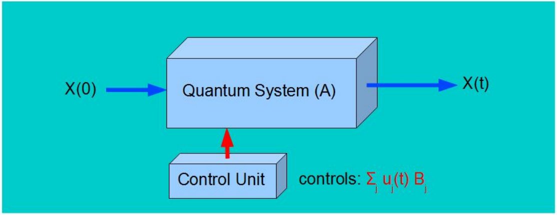
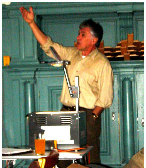
Systems Theory





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I. Systems Theory
Simulability
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Consider

1. linear control system: $\dot{x}(t) = Ax(t) + Bv$
2. bilinear control system: $\dot{X}(t) = (A + \sum_j u_j B_j) X(t)$

Conditions for Full Controllability $\Leftrightarrow$ Universality

1. in linear systems: $\mathrm{rank}\,[B, AB, A^2 B, \ldots, A^{N-1} B] = N$
2. in bilinear systems: $\langle A, B_j \,|\, j = 1, 2, ..., m \rangle_{\text{Lie}} = \mathfrak{su}(N)$

key: system algebra $\mathfrak{k} := \langle A, B_j \,|\, j = 1, 2, ..., m \rangle_{\text{Lie}}$

reachable set $\mathrm{Reach}(\rho_0) = \{ K \rho_0 K^\dagger \,|\, K \in \exp \mathfrak{k} \}$

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symmetries $\text{ad}'_{\mathfrak{k}} := \{S \in \mathfrak{gl}(N^2) \,|\, [S, \text{ad}_A] = [S, \text{ad}_{B_j}] = 0,\ \forall j\}$

$$\dot{X}(t) = -\big(A + \Sigma_j u_j(t) B_j\big) X(t)$$ as operator lift of
$$\dot{x}(t) = -\big(A + \Sigma_j u_j(t) B_j\big) x(t)$$

$X(t)$ or $x(t)$: 'state'; A: drift; B_j: control Hamiltonians; u_j: control amplitudes

Setting and Task	'State' $X(t)$	Drift A	Controls B_j
closed systems:			
pure-state transfer	$X(t) = \lvert\psi(t)\rangle$	iH_0	iH_j
gate synthesis (fixed global phase)	$X(t) = U(t)$	iH_0	iH_j
state transfer	$X(t) = \lvert\rho(t)\rangle$	$i\widehat{H}_0$	$i\widehat{H}_j$
gate synthesis (free global phase)	$X(t) = \widehat{U}(t)$	$i\widehat{H}_0$	$i\widehat{H}_j$
open systems:			
state transfer I	$X(t) = \lvert\rho(t)\rangle$	$i\widehat{H}_0 + \widehat{\Gamma}$	$i\widehat{H}_j$
quantum-map synthesis I	$X(t) = F(t)$	$i\widehat{H}_0 + \widehat{\Gamma}$	$i\widehat{H}_j$
state transfer II	$X(t) = \lvert\rho(t)\rangle$	$i\widehat{H}_0$	$i\widehat{H}_j, \widehat{\Gamma}_j$
map synthesis II	$X(t) = F(t)$	$i\widehat{H}_0$	$i\widehat{H}_j, \widehat{\Gamma}_j$

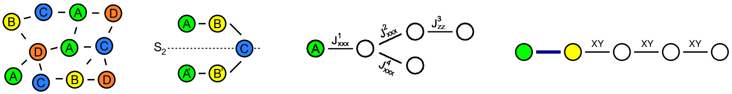
$\widehat{H}$ is Hamiltonian commutator superoperator generating $\widehat{U} := U(\cdot)U^\dagger$.

Control system Σ with algebra $\mathfrak{k} = \langle iH_\nu \,|\, \nu = d; 1, 2, \dots, m \rangle_{\text{Lie}}$.

Theorem (Simplicity)

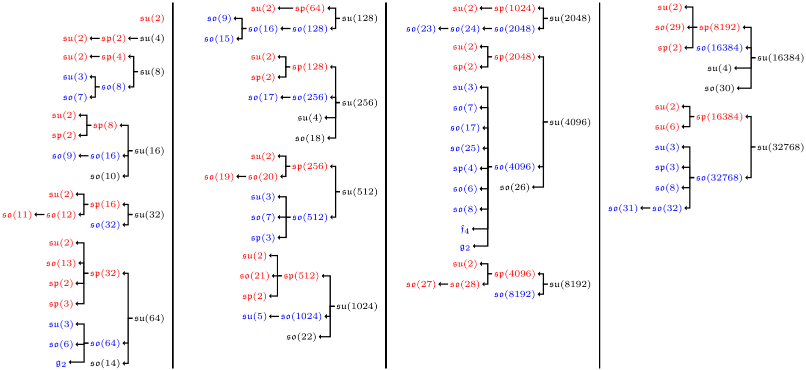
The above system algebra $\mathfrak{k}$ is an irreducible simple subalgebra of $\mathfrak{su}(N)$, if both

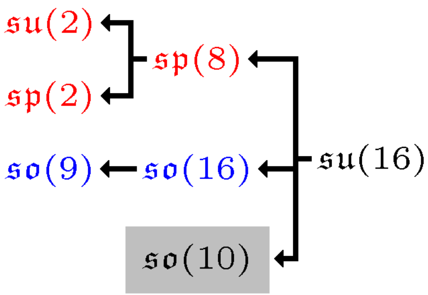
1. *the commutant is trivial, i.e. $\mathfrak{k}' = \text{span}\{\mathbb{1}\}$,*
2. *the coupling graph to H_d is connected.*



Irreducible Simple Subalgebras to $\mathfrak{su}(N)$ up to $N = 2^{15}$

J. Math. Phys. **52**, 113510 (2011)




$\mathfrak{su}(2)$
$\mathfrak{sp}(8)$
$\mathfrak{sp}(2)$
$\mathfrak{so}(9)$
$\mathfrak{so}(16)$
$\mathfrak{su}(16)$
$\mathfrak{so}(10)$

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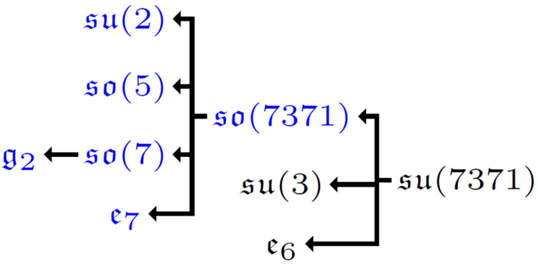

Theorem (Trivial Second-Order Symmetries)

Let $\{H_\nu \,|\, \nu = d; 1, 2, \dots, m\}$ *be drift and control Hamiltonians of control system* Σ *with system algebra* $\mathfrak{k}$.

Define $\Phi_\Theta := \{(\mathbb{1} \otimes iH_\nu + \Theta(iH_\nu \otimes \mathbb{1})) \,|\, \nu = d, 1, \dots, m\}$.

Then Σ *is fully controllable, i.e.* $\mathfrak{k} = \mathfrak{su}(2^n)$, *iff*

- *joint commutant to* Φ_Θ *is two-dimensional, i.e.* $\Phi'_\Theta = \mathrm{span}\{\mathbb{1}^{\otimes 2}, \Theta(SWAP)\}$.

Symmetry vs. Controllability II

Single Symmetry Condition

J. Math. Phys. **52**, 113510 (2011)

Theorem

Let $\{H_\nu \,|\, \nu = d; 1, 2, \dots, m\}$ *be drift and control Hamiltonians of control system* Σ *with system algebra* $\mathfrak{k}$.

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  i.e. $\Phi'_{AB} = \text{span}\{\mathbb{1}, \text{SWAP}_{AB}\}$.

  $[\Phi_{AB}] = [\text{symmetric}]_{\text{'bosonic'}} \oplus [\text{anti-symmetric}]_{\text{'fermionic'}}$

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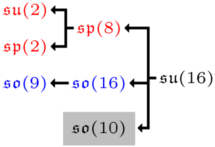

Corollary (*J. Math. Phys.* **52**, 113510 (2011))

Let Σ_A, Σ_B be control systems with system algebras $\mathfrak{k}_A, \mathfrak{k}_B$ over a given Hilbert space $\mathcal{H}$.
Then

- Σ_A *can simulate* $\Sigma_B \Leftrightarrow \mathfrak{k}_B$ *is a subalgebra of* $\mathfrak{k}_A$.

system type		'fermionic'	'bosonic'	system alg.
n-spins-$\frac{1}{2}$	no. of levels	order of coupling		
(A) —XX— ○ —XX— ○ ········ ○	n	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+1)$

NB: no. of spins maps into no. of levels (as in Jordan-Wigner transformation).

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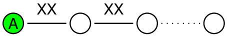
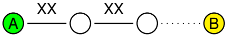
	n	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+1)$
	$n+1$	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+2)$

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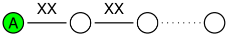
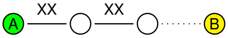
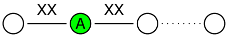
system type n-spins-$\frac{1}{2}$	no. of levels	'fermionic' —— order of coupling	'bosonic' ——	system alg.
A $\overset{XX}{—}$ ○ $\overset{XX}{—}$ ○ ········ ○	n	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+1)$
A $\overset{XX}{—}$ ○ $\overset{XX}{—}$ ○ ········ B	$n+1$	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+2)$
○ $\overset{XX}{—}$ A $\overset{XX}{—}$ ○ ········ ○ for $n \bmod 4 \in \{0,1\}$	n	up to n	–	$\mathfrak{so}(2^n)$

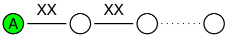
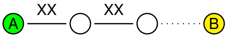
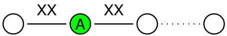
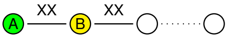
NB: no. of spins maps into no. of levels (as in Jordan-Wigner transformation).

system type n-spins-$\frac{1}{2}$	no. of levels	'fermionic' ——— order of coupling	'bosonic' ———	system alg.
	n	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+1)$
	$n+1$	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+2)$
				
for $n \bmod 4 \in \{0,1\}$	n	up to n	–	$\mathfrak{so}(2^n)$
for $n \bmod 4 \in \{2,3\}$	n	–	up to n	$\mathfrak{sp}(2^{n-1})$

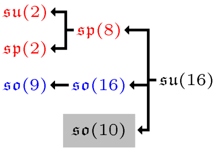
NB: no. of spins maps into no. of levels (as in Jordan-Wigner transformation).

system type n-spins-$\frac{1}{2}$	no. of levels	'fermionic' order of coupling	'bosonic' order of coupling	system alg.
(A) —XX— ( ) —XX— ( ) ······· ( )	n	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+1)$
(A) —XX— ( ) —XX— ( ) ······· (B)	$n+1$	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+2)$
( ) —XX— (A) —XX— ( ) ······· ( )				
for $n \bmod 4 \in \{0,1\}$	n	up to n	–	$\mathfrak{so}(2^n)$
for $n \bmod 4 \in \{2,3\}$	n	–	up to n	$\mathfrak{sp}(2^{n-1})$
(A) —XX— (B) —XX— ( ) ······· ( )	n	up to n	up to n	$\mathfrak{su}(2^n)$

NB: no. of spins maps into no. of levels (as in Jordan-Wigner transformation).

system type n-spins-$\frac{1}{2}$	no. of levels	'fermionic'	'bosonic'	system alg.
		order of coupling		
	n	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+1)$
	$n+1$	quadratic (i.e. 2)	–	$\mathfrak{so}(2n+2)$
				
for n mod 4 $\in \{0,1\}$	n	up to n	–	$\mathfrak{so}(2^n)$
for n mod 4 $\in \{2,3\}$	n	–	up to n	$\mathfrak{sp}(2^{n-1})$
	n	up to n	up to n	$\mathfrak{su}(2^n)$

NB: no. of spins maps into no. of levels (as in Jordan-Wigner transformation).



Classical features of $W(A)$ and $W(C, A)$:

- $W(A)$ and $W(C, A)$ are *compact* and *connected*. GOLDBERG & STRAUSS 1977
- $W(A)$ is *convex*. HAUSDORFF 1919, TOEPLITZ 1918
- $W(C, A)$ is *star-shaped*. CHEUNG & TSING '96
- $W(C, A)$ is *convex* if C or A Hermitian. WESTWICK '75
- $W(C, A)$ is a *circular disk centered at the origin* if C or A are unitarily similar to block-shift form LI & TSING '91

Definition (Lin. Multin. Alg. 56 (2008) 3–26 and 27–51)

The *relative C-numerical range* is the set

$$W_{\mathbf{K}}(C,A) := \{\operatorname{tr}(C^\dagger K A K^\dagger) \mid K \in \mathbf{K} \subsetneq SU(N)\} \subseteq W_C(A),$$

where the unitary orbit is restricted to a subgroup **K**.
Ex.:*local operations* $K \in SU(2) \otimes SU(2) \otimes \cdots \otimes SU(2)$

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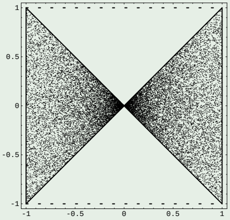
Example (I non convex)

$$A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$$

$$C := \operatorname{diag}(1,0,0,0)$$



Definition (Lin. Multin. Alg. **56** (2008) 3–26 and 27–51)

The *relative C-numerical range* is the set

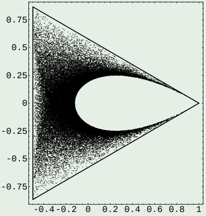
$$W_{\mathbf{K}}(C, A) := \{\mathrm{tr}\,(C^\dagger K A K^\dagger) \mid K \in \mathbf{K} \subsetneq SU(N)\} \subseteq W_C(A),$$

where the unitary orbit is restricted to a subgroup **K**.
Ex.: *local operations* $K \in SU(2) \otimes SU(2) \otimes \cdots \otimes SU(2)$

Example (II neither star-shaped nor simply connected)

$$A := \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\pi/3} \end{pmatrix}^{\otimes 3}$$

$$C := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{\otimes 3}$$



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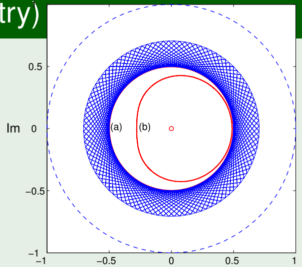
Ex.: *local operations* $K \in SU(2) \otimes SU(2) \otimes \cdots \otimes SU(2)$

Example (III distinct circ. symmetry)

$$A := \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

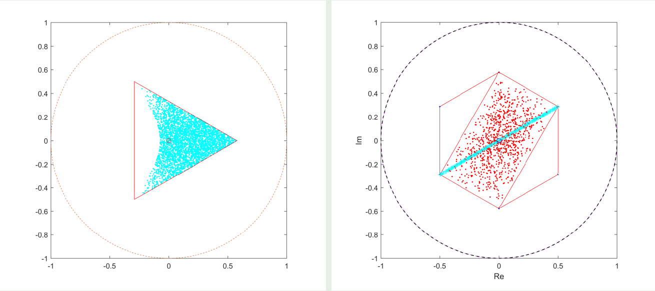
$$C := \operatorname{diag}(1\ 0\ 0\ 0)$$



Example (IV $\mathbf{K}$ = $USp(4/2)$ vs $SO(4)$ vs $SU(2)^{\otimes 2}$)

$$A=\frac{1}{\sqrt{3}}\mathrm{diag}(0, e^{i2\pi/3}, e^{i4\pi/3}, 1)$$

$$C=\mathrm{diag}(1,0,0,0) \qquad C=\frac{1}{2}\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

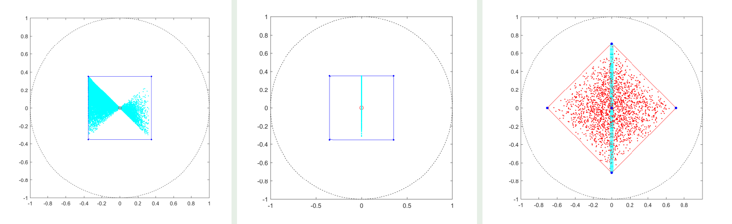


The Relative C-Numerical Range

Example (V $\mathbf{K}$ = $USp(4/2)$ vs $SO(4)$ vs $SU(2)^{\otimes 2}$)

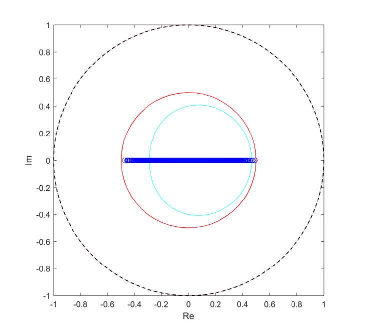
$$A = \frac{1}{\sqrt{8}} \mathrm{diag}(1+i, 1-i, -1-i, -1+i)$$

$$C = \mathrm{diag}(1,0,0,0) \qquad C = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad C = \frac{1}{2}\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} =: \frac{1}{2} J$$



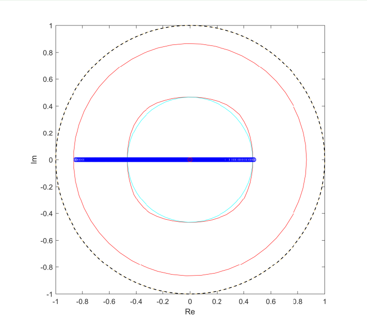
Example (VI $\mathbf{K} = USp(4/2)$ vs $SO(4)$ vs $SU(2)^{\otimes 2}$)

$$A = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad C = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =: \frac{1}{2} K$$



Example (VII $\mathbf{K}$ = $USp(4/2)$ vs $SO(4)$ vs $SU(2)^{\otimes 2}$)

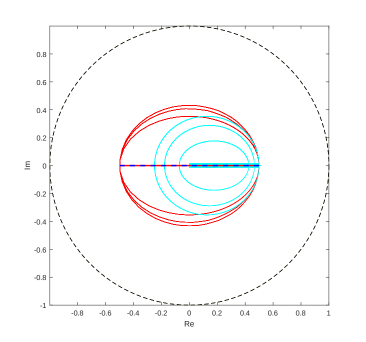
$$A = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad C = \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \frac{1}{2} KJ$$



Motivation
I. Systems Theory
II. Relative $W_C(A)$
III. Gradient Flows
Conclusions & Outlook

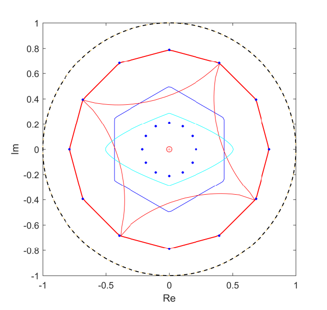
Example (VIII $\mathbf{K}$ = $USp(4/2)$ vs $SO(4)$ vs $SU(2)^{\otimes 2}$)

$$A \in \left\{ \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1&0&0&0\\1&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1&0&0&0\\1&0&0&0\\1&0&0&0\\1&0&0&0 \end{pmatrix} \right\} \qquad C = \frac{1}{2}K$$



Example (IX $\mathbf{K}$ = $USp(4/2)$ vs $SO(4)$ vs $SU(2)^{\otimes 2}$)

$$A = \frac{1}{\sqrt{3}} \text{diag}(0, e^{i*2\pi/3}, e^{i*4\pi/3}, 1) \qquad C = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \frac{1}{2} KJ$$



Lin.Multilin.Alg. **56** (2008), 27

Theorem

Let $\mathbf{K}$ *be a compact connected subgroup of* $U(N)$ *with Lie algebra* $\mathfrak{k}$*, and let* $\mathfrak{t}$ *be a torus algebra of* $\mathfrak{k}$*. Then the relative* C*-numerical range* $W_{\mathbf{K}}(C, A_+)$ *of a matrix* $A_+ \in \mathrm{Mat}_N(\mathbb{C})$ *is a circular disc centered at the origin of the complex plane for all* $C \in \mathrm{Mat}_N(\mathbb{C})$ *if and only if there exists a* $K \in \mathbf{K}$ *and a* $\Delta \in \mathfrak{t}$ *such that* $KA_+K^\dagger$ *is an eigenoperator to* ad_Δ *with a non-zero eigenvalue*

$$\mathrm{ad}_\Delta(KA_+K^\dagger) \equiv [\Delta, KA_+K^\dagger] = ip\,(KA_+K^\dagger) \quad \textit{and} \quad p \neq 0 \quad .$$

Theorem

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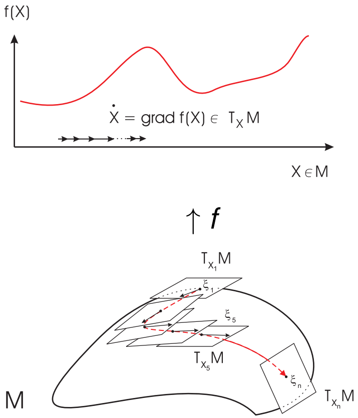
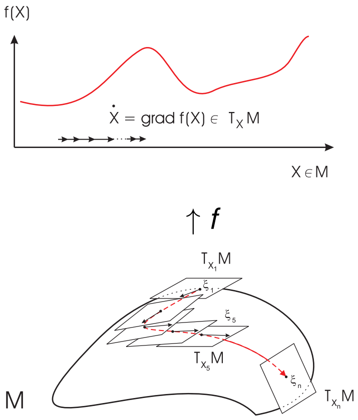
$$\mathrm{ad}_\Delta(KA_+K^\dagger) \equiv [\Delta, KA_+K^\dagger] = ip\,(KA_+K^\dagger) \quad \text{and} \quad p \neq 0 \quad .$$

If $KA_+K^\dagger$ *is an eigenoperator of* ad_Δ *to eigenvalue* $+ip$ *and* $A_- := A_+^\dagger$*, then* $KA_-K^\dagger$ *has eigenvalue* $-ip$*.*
A_+ *and* A_- *share the same relative* C*-numerical range of circular symmetry,* $W_{\mathbf{K}}(C, A_+) = W_{\mathbf{K}}(C, A_-)$*.*


f(X)
$\dot{X}$ = grad f(X) ∈ T_XM
X ∈ M
↑ f
T_{X_1}M
ξ_1
ξ_5
T_{X_5}M
ξ_n
T_{X_n}M
M



- quality function $f : M \to \mathbb{R}\,, X \mapsto f(X)$
  drives into (local) maximum by gradient flow
  to $\dot{X} = \operatorname{grad} f(X)$ on M

Example: $f(U) := \operatorname{Re}\operatorname{tr}\{C^\dagger UAU^\dagger\}$

write $[\cdot,\cdot]_S$ as skew-herm. part

- calculate Lie derivative
  $Df(U)(\Omega U) = \langle [UAU^\dagger, C^\dagger]_S^\dagger U | \Omega U\rangle$

  identifying $Df(U)(\Omega U) = \langle \operatorname{grad} f(U) | \Omega U\rangle$, where $\xi \in T_U SU(N)$ reads $\xi = \Omega U$ and $\Omega \in \mathfrak{su}(N)$;

- obtain gradient vector field
  $\operatorname{grad} f(U) = [UAU^\dagger, C^\dagger]_S^\dagger\, U$
- integrate gradient system $\dot{U} = \operatorname{grad} f(U)$ by
- discretisation scheme $U_{k+1} = e^{-\alpha_k [U_k A U_k^\dagger, C^\dagger]_S}\, U_k$

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Generalised task:
approximate A_0 by elements on the sum of orbits

- unitary similarity:

$$\min_{U_j} \Big|\Big| \sum_{j=1}^{N} U_j A_j U_j^\dagger - A_0 \Big|\Big|_2^2$$

- N coupled gradient flows:

$$U_{k+1}^{(j)} = \exp\{-\alpha_k^{(j)} [A_k^{(j)}, A_{0jk}^\dagger]_S\}\, U_k^{(j)}$$

notations:

$$A_k^{(j)} := U_k^{(j)} A_j U_k^{(j)\dagger}; \quad A_{0jk} := A_0 - \sum_{\substack{\nu=1\\ \nu\neq j}}^{N} A_k^{(\nu)}; \quad [\cdot,\cdot]_S \text{ skew-herm. part}$$

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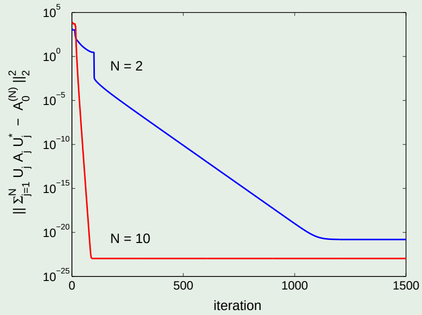
Example (unitary similarity)

Define in $\mathbb{C}^{10\times 10}$

$$A_j := \operatorname{diag}(1,3,5,\ldots,19) + \tfrac{j-1}{10}\mathbb{1}_{10}$$
$$A_0^{(N)} := \operatorname{diag}(a_1,\ldots,a_{10}) \quad ,$$

where $a_1,\ldots,a_{10}$ are eigenvalues of $\sum_{j=1}^N U_j^{(r)} A_j U_j^{(r)\dagger}$ with random unitaries (distributed by Haar measure). For $N=2$ and $N=10$ the above flows give:



Conclusion for Reachable Sets

by Group Orbits

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no vs constant vs switchable noise Γ_L

- closed coherently controllable (cc) systems:
  Reach $\rho_0 = \mathcal{O}_{\mathbf{K}}(\rho_0) := \{K\rho_0 K^\dagger \,|\, K \in \mathbf{K} \subseteq SU(N)\}$,
  where $\mathbf{K} = \exp \mathfrak{k}$ generated by system algebra $\mathfrak{k}$
- open systems, cc with constant Markovian noise Γ:
  Reach $\rho_0 = \mathbf{S}\,\mathrm{vec}\,\rho_0$, where
  $\mathbf{S} \simeq e^{A_\ell} e^{A_{\ell-1}} \cdots e^{A_1}$ with $A_1, A_2, \dots, A_\ell \in \mathfrak{w}$
- open systems, cc with switchable Markovian noise:
  - unital: $\overline{\text{Reach}\,\rho_0} = \{\rho \in \mathfrak{pos}_1 \,|\, \rho \prec \rho_0\}$
  - non-unital: $\overline{\text{Reach}\,\rho_0} = \mathfrak{pos}_1$

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  $\mathbf{S} \simeq e^{A_\ell} e^{A_{\ell-1}} \cdots e^{A_1}$ with $A_1, A_2, \ldots, A_\ell \in \mathfrak{w}$
- open systems, cc with switchable Markovian noise:
  - unital: $\overline{\text{Reach}\,\rho_0} = \{\rho \in \mathfrak{pos}_1 \,|\, \rho \prec \rho_0\}$
  - non-unital: $\overline{\text{Reach}\,\rho_0} = \mathfrak{pos}_1$

no vs constant vs switchable noise Γ_L

- closed coherently controllable (cc) systems:
  Reach $\rho_0 = \mathcal{O}_{\mathbf{K}}(\rho_0) := \{K\rho_0 K^\dagger \,|\, K \in \mathbf{K} \subseteq SU(N)\}$,
  where $\mathbf{K} = \exp \mathfrak{k}$ generated by system algebra $\mathfrak{k}$

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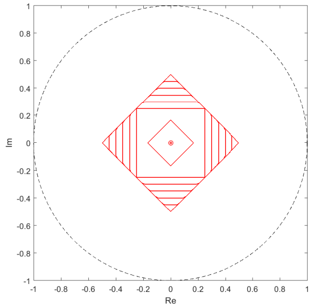
Example (Semigroup Orbit $\mathbf{S}_{LK}(\operatorname{diag}(1,0,0,0))$)



$C = \frac{1}{2}\operatorname{diag}(1, i, -i, 1)$

$A = \operatorname{diag}(1, 0, 0, 0)$
$\operatorname{diag}(0.9, 0.1, 0, 0)$
$\operatorname{diag}(0.8, 0.2, 0, 0)$
$\operatorname{diag}(0.7, 0.3, 0, 0)$
$\operatorname{diag}(0.6, 0.4, 0, 0)$
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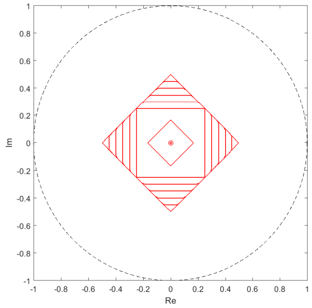

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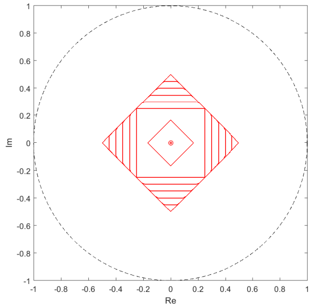

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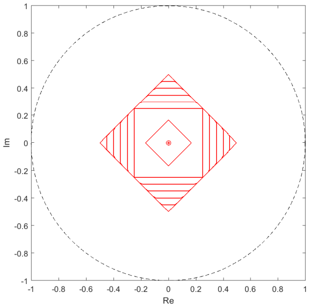

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Conclusions & Outlook

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   - reflect specific quantum dynamics (via $\mathbf{K} = \exp(\mathfrak{k})$)
   - come with a geometry not fully explored yet
   - can naturally be generalised to projections of semigroup orbits in the sense $W_{\mathbf{S}}(C, A) := \langle C | \mathbf{S} A \rangle$

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   - powerful tool for geometric optimisation and control
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Acknowledgements

Thanks go to:

Gunther Dirr, Uwe Helmke†, Robert Zeier
Chi-Kwong Li, Yiu-Tung Poon, Frederik vom Ende

integrated EU programme; excellence network; DFG research group NV centres

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