Classification of joint numerical ranges of three hermitian matrices of size three

talk at

The 14th Workshop on Numerical Ranges and Numerical Radii

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speaker

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Overview

1. Introduction
2. Problems with 3D Joint Numerical Ranges of 3-by-3 Matrices
3. Solution: Graph Embedding (definition of classes)
4. Finding Examples (all classes are populated)
5. Conclusion

Introduction

Joint Numerical Ranges

let $F_1, \ldots, F_k \in M_d$ denote hermitian d-by-d matrices, the state space (mixed states) of a *-subalgebra $\mathcal{A} \subset M_d$ is

$$\mathcal{M}(\mathcal{A}) = \{\rho \in \mathcal{A} \mid \rho \geq 0, \mathrm{tr}(\rho) = 1\},$$

the joint algebraic numerical range of $F = (F_1, \ldots, F_k)$ is

$$L_F = \{(\mathrm{tr}(\rho F_1), \ldots, \mathrm{tr}(\rho F_k)) : \rho \in \mathcal{M}(M_d)\} \subset \mathbb{R}^k,$$

the joint numerical range (JNR) of F is

$$W_F = \{(\langle\psi|F_1\psi\rangle, \ldots, \langle\psi|F_k\psi\rangle) : |\psi\rangle \in \mathbb{C}^d, \langle\psi|\psi\rangle = 1\}$$

with $\langle\varphi|\psi\rangle = \overline{\varphi_1}\psi_1 + \cdots + \overline{\varphi_d}\psi_d$

Lemma

$$\mathrm{conv}(W_F) = L_F.$$

Convexity of Numerical Ranges

Theorem (Toeplitz and Hausdorff)

W_{F_1,F_2} is convex. Hence $W_{F_1,F_2} = L_{F_1,F_2}$.

Math. Z. 2 (1918), 187 and Math. Z. 3 (1919), 314

Theorem (Au-Yeung and Poon)

If $d \geq 3$ then W_{F_1,F_2,F_3} is convex. Hence $W_F = L_F$.

Southeast Asian Bull. Math. 3 (1979), 85

there is no easy rule to decide convexity of $W_{F_1,\ldots,F_k}$ if $k \geq 4$

Li and Poon, SIAM J. Matrix Anal. Appl. 21 (2000), 668

Boundary Generating Curve ($k = 2$)

consider the hypersurface

$$V_{F_1,F_2} = \{(u_0 : u_1 : u_2) \in \mathbb{P}^2_{\mathbb{C}} \mid \det(u_0 \mathbb{1} + u_1 F_1 + u_2 F_2) = 0\}$$

with d-by-d identity matrix $\mathbb{1}$

and its dual curve

$$V^*_{F_1,F_2} \subset \mathbb{P}^{2\,*}_{\mathbb{C}}$$

closure of the set of tangent lines at smooth points of V

the boundary generating curve of F_1, F_2 is

$$V^*_{F_1,F_2}(\mathbb{R}) = \{(x_1, x_2) \in \mathbb{R}^2 \mid (1 : x_1 : x_2) \in V^*_{F_1,F_2}\} \subset \mathbb{R}^2$$

Theorem (Kippenhahn)

W_{F_1,F_2} is the convex hull of $V^*_{F_1,F_2}(\mathbb{R})$.

Mathematische Nachr. 6 (1951), 193

Classification of Numerical Ranges ($k = 2$)

$d = 2$, the numerical range W_{F_1,F_2} is an ellipse (possibly degenerate)

$d = 3$, Kippenhahn (1951) derived a classification of W_{F_1,F_2} from the boundary generating curve $V^*_{F_1,F_2}(\mathbb{R})$,
see also Keeler et al. LAA 252 (1997), 115

$d = 4$, Chien and Nakazato derived a classification of W_{F_1,F_2} from $V^*_{F_1,F_2}(\mathbb{R})$, Electronic J. Lin. Alg. 23 (2012), 755

Definition. 3-by-3 matrices $F_1, \ldots, F_k$ are unitarily reducible (otherwise unitarily irreducible) if there is a unitary matrix U such that $U^* F_1 U, \ldots, U^* F_k U$ are of direct sum form

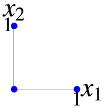
$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}.$$

Numerical Ranges, $d = 3$, Unitarily Reducible

Drawings: boundary generating curves $V^*_{F_1,F_2}(\mathbb{R})$ (blue)

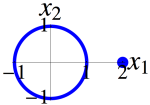
1) $V^*_{F_1,F_2}(\mathbb{R})$ consists of three points

e.g. $F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$



2) $V^*_{F_1,F_2}(\mathbb{R})$ is the union of an ellipse and a point

e.g. $F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

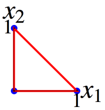


Numerical Ranges, $d = 3$, Unitarily Reducible

Drawings: boundaries of the numerical ranges W_{F_1,F_2} (red)

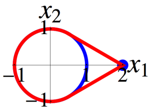
1) W_{F_1,F_2} is a triangle

e.g. $F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$



2) W_{F_1,F_2} is the convex hull of an ellipse and a point

e.g. $F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

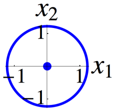


Numerical Ranges, $d = 3$, Unitarily Irreducible

Drawings: boundary generating curves $V^*_{F_1,F_2}(\mathbb{R})$ (blue)

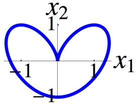
1) $V^*_{F_1,F_2}(\mathbb{R})$ is the union of an ellipse and a point inside

e.g. $F_1 = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & -i & -\frac{i}{2} \\ i & 0 & \frac{i}{2} \\ \frac{i}{2} & -\frac{i}{2} & 0 \end{pmatrix}$



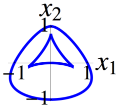
2) $V^*_{F_1,F_2}(\mathbb{R})$ is a quartic curve

e.g. $F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$



3) $V^*_{F_1,F_2}(\mathbb{R})$ is a sextic curve

e.g. $F_1 = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & 1 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

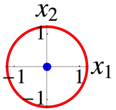


Numerical Ranges, $d = 3$, Unitarily Irreducible

Drawings: boundaries of the numerical ranges W_{F_1,F_2} (red)

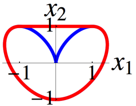
1) W_{F_1,F_2} is an ellipse

e.g. $F_1 = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & -i & -\frac{i}{2} \\ i & 0 & \frac{i}{2} \\ \frac{i}{2} & -\frac{i}{2} & 0 \end{pmatrix}$



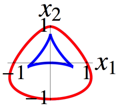
2) W_{F_1,F_2} is the convex hull of a quartic curve

e.g. $F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$



3) W_{F_1,F_2} is the convex hull of a sextic curve

e.g. $F_1 = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & 1 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$



Problems with Three-Dimensional Joint Numerical Ranges

Boundary generating surface ($k = 3$)

consider the hypersurface

$$V_{F_1,F_2,F_3} = \{u \in \mathbb{P}^3_{\mathbb{C}} : \det(u_0 \mathbb{1} + u_1 F_1 + \cdots + u_3 F_3) = 0\}$$

and its dual variety

$$V^*_{F_1,F_2,F_3} \subset \mathbb{P}^{3\,*}_{\mathbb{C}}$$

closure of the set of tangent planes at smooth points of V

the boundary generating surface of F_1, F_2, F_3 is

$$V^*_{F_1,F_2,F_3}(\mathbb{R}) = \{x \in \mathbb{R}^3 \mid (1 : x_1 : x_2 : x_3) \in V^*_{F_1,F_2,F_3}\} \subset \mathbb{R}^2$$

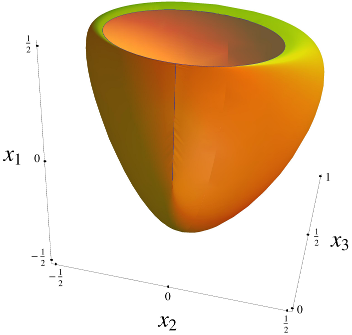
Observation (Chien and Nakazato, LAA 432 (2010), 173)
$V^*_{F_1,F_2,F_3}(\mathbb{R})$ can contain lines, hence $V^*_F(\mathbb{R}) \subset W_F$ is impossible and $\operatorname{conv}(V^*_F(\mathbb{R})) = W_F$ fails.

Example 1

$$F_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ F_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

boundary generating surface

$$V^*_{F_1,F_2,F_3}(\mathbb{R})$$
$$= \{x \in \mathbb{R}^3 \mid -4x_1^2x_3^2 - 4x_2^2x_3^2 + 4x_3^3 - 4x_3^4 + 4x_1x_2^2x_3 - x_2^4 = 0\}$$



Depicted surface:

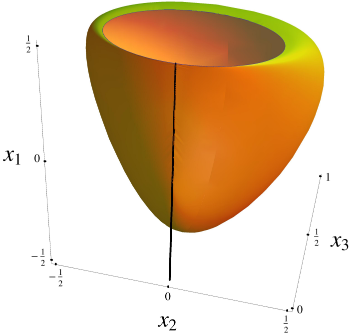
Intersection of $V^*_{F_1,F_2,F_3}(\mathbb{R})$ with the boundary of W_{F_1,F_2,F_3}

Example 1

$$F_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ F_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

boundary generating surface

$$V^*_{F_1,F_2,F_3}(\mathbb{R})$$
$$= \{x \in \mathbb{R}^3 \mid -4x_1^2x_3^2 - 4x_2^2x_3^2 + 4x_3^3 - 4x_3^4 + 4x_1x_2^2x_3 - x_2^4 = 0\}$$



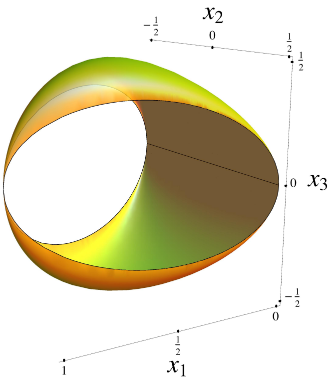
the x_1-axis lies in $V^*_{F_1,F_2,F_3}(\mathbb{R})$

Example 2

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ F_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ F_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

boundary generating surface

$$V^*_{F_1,F_2,F_3}(\mathbb{R}) = \{x \in \mathbb{R}^3 \mid -x_1^2 x_2^2 + x_1 x_3^2 - x_1^2 x_3^2 - x_3^4 = 0\}$$



Depicted surface:

Intersection of $V^*_{F_1,F_2,F_3}(\mathbb{R})$

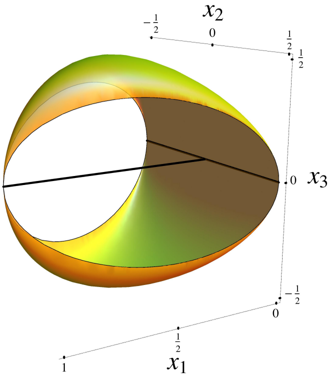
with the boundary of W_{F_1,F_2,F_3}

Example 2

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; F_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; F_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

boundary generating surface

$$V^*_{F_1,F_2,F_3}(\mathbb{R}) = \{x \in \mathbb{R}^3 \mid -x_1^2 x_2^2 + x_1 x_3^2 - x_1^2 x_3^2 - x_3^4 = 0\}$$



the x_1- and x_2-axes

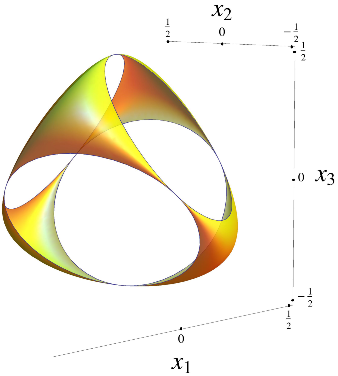
lie in $V^*_{F_1,F_2,F_3}(\mathbb{R})$

Example 3

$$F_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ F_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ F_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

boundary generating surface = *Roman surface*

$$V^*_{F_1,F_2,F_3}(\mathbb{R}) = \{x \in \mathbb{R}^3 \mid x_1x_2x_3 - x_1^2x_2^2 - x_1^2x_3^2 - x_2^2x_3^2 = 0\}$$



Depicted surface:

Intersection of $V^*_{F_1,F_2,F_3}(\mathbb{R})$

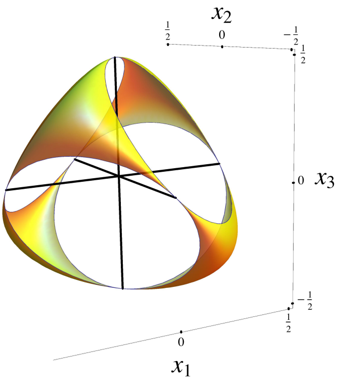
with the boundary of W_{F_1,F_2,F_3}

Example 3

$$F_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ F_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ F_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

boundary generating surface = *Roman surface*

$$V^*_{F_1,F_2,F_3}(\mathbb{R}) = \{x \in \mathbb{R}^3 \mid x_1 x_2 x_3 - x_1^2 x_2^2 - x_1^2 x_3^2 - x_2^2 x_3^2 = 0\}$$



all three coordinate axes

lie in $V^*_{F_1,F_2,F_3}(\mathbb{R})$

Classification of JNRs: State of the Art

Kippenhahn's assertion does not generalize from $k = 2$ to $k = 3$, an algebraic geometry approach seems unavailable !

very little is known about $W_F = W_{F_1,\ldots,F_k}$, $k \geq 3$, except for

- corner points (conical points) imply F unitarily reducible

  Binding and Li, LAA 151 (1991), 157

- ovals and reconstruction of F from W_F

  Krupnik and Spitkovsky, LAA 419 (2006), 569

- a maximum of 4 ellipses on the boundary of W_F if $k = d = 3$

  Chien and Nakazato, LAA 430 (2009), 204

Our Approach: Study configurations of exposed faces on the boundary of W_F.

Solution: Graph Embedding

Exposed Faces

an exposed face of a convex set $C \subset \mathbb{R}^n$ is the set of maximizers of a linear functional,

$$\mathbb{F}_C(u) = \operatorname{argmax}\{\langle x, u\rangle : x \in C\}, \qquad u \in \mathbb{R}^n,$$

or the empty set; let $F(u) = u_1 F_1 + \cdots + u_k F_k$, $u \in \mathbb{R}^k$, and

$$\mathbb{E} : \mathcal{M}(M_d) \to W_F,\ \rho \mapsto (\mathrm{tr}(\rho F_1), \ldots, \mathrm{tr}(\rho F_k));$$

then

$$\mathbb{E}^{-1}\left(\mathbb{F}_{W_F}(u)\right) = \mathbb{F}_{\mathcal{M}(M_d)}(F(u))$$

and

$$\mathbb{F}_{\mathcal{M}(M_d)}(F(u)) = \mathcal{M}(pM_dp)$$

where p is the spectral projection of $F(u)$ corresponding to the maximal eigenvalue

Large Faces

we assume $k = d = 3$ and call large face an exposed face of W_F which is neither $\varnothing$, nor a singleton, nor equal to all of W_F

Lemma (Szymański, SW, Życzkowski)

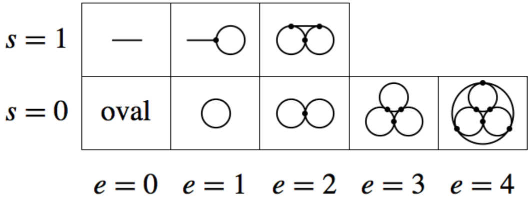
1) Every large face is a segment or a filled ellipse.
2) Each two distinct large faces intersect in a singleton.
3) If G_1, G_2, G_3 are mutually distinct large faces and $G_1 \cap G_2 \cap G_3 = \varnothing$, then W_F has a corner point.
4) If there are two distinct large faces which are segments, then W_F has a corner point.

Graph Embedding

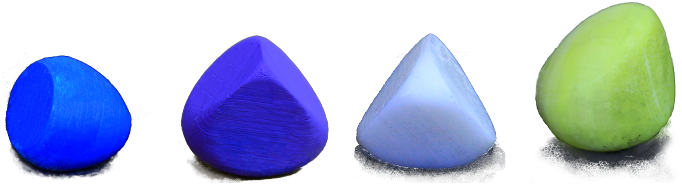
2) and 3) of the lemma show that a complete graph K_n embeds into the union of large faces with one vertex on each large face

the boundary of W_F is homeomorphic to the sphere S^2 so $n \leq 4$

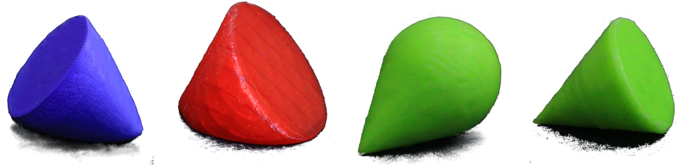
Theorem (Szymański, SW, Życzkowski)
Let $k = d = 3$. If W_F has no corner point, then the set of large faces has one of the following configurations.



Finding Examples



(a) $s = 0, e = 1$ (b) $s = 0, e = 3$ (c) $s = 0, e = 4$ (d) $s = 1, e = 0$



(e) $s = 1, e = 1$ (f) $s = 1, e = 2$ (g) $s = \infty, e = 0$ (h) $s = \infty, e = 1$

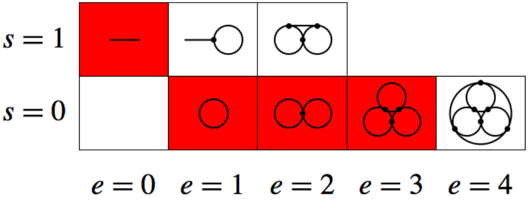
Figure : 3D printouts of exemplary joint numerical ranges of three 3-by-3 hermitian matrices from random density matrices: s denotes the number of segments, e the number of ellipses in the boundary

Finding Candidates

searching for candidates belonging to each class, we used

- random matrices
- guessing

and found some new examples (red)



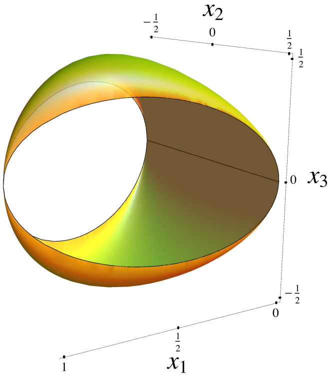
Question: How to determine the class of an example?

Large Faces of Example 2

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad F_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad F_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

the exposed faces with normal vectors $u = (-1, 0, 0), (1, \pm 2, 0)$ are large faces as the maximal eigenvalues are degenerate:

$$F(-1,0,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad F(1,\pm 2,0) = \begin{pmatrix} 0 & \pm 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



how can we be sure there are no further large faces?

by finding all degenerate eigenvalues in the hermitian pencil spanned by F_1, F_2, F_3

Discriminant as a Sum of Squares

the discriminant of the polynomial $p(\lambda) = -\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ is

$$\begin{aligned} \mathrm{Disc}_\lambda(p) &= -(27a_3^2 + 18a_1a_2a_3 - 4a_1^3a_3 + 4a_2^3 - a_1^2a_2^2) \\ &= (\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2 \end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the roots of p

the discriminant of $A \in M_d$ is $\mathrm{Disc}(A) = \mathrm{Disc}_\lambda(\det(A - \lambda\mathbb{1}))$

Theorem (Ilyushechkin)
Let $A \in M_d$ be a normal matrix. Let A_* be the $d^2 \times d$-matrix with columns the coefficients of $\mathbb{1}, A, A^2, \ldots, A^{d-1}$, all in the same order. Let M_ν denote the ν-minor of A_*. Then

$$|\mathrm{Disc}(A)| = \sum_{\nu\subset\{1,\ldots,d\}^2, |\nu|=d} |M_\nu|^2.$$

Mathematical Notes 51 (1992), 230

Squared Minors of $F(u_1, u_2, u_3)$ from Example 2

$$\Big\{\frac{1}{16}\,u1^2\,u2^2\,u3^2,\ 0,\ \frac{u2^2\,u3^4}{64},\ \frac{u2^4\,u3^2}{64},\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ \frac{u2^4\,u3^2}{64},\ \frac{1}{64}\left(-4\,u1^2\,u2+u2^3\right)^2,$$

$$\frac{1}{16}\,u1^2\,u2^2\,u3^2,\ \frac{u3^6}{64},\ \frac{u2^2\,u3^4}{64},\ 0,\ \frac{u2^2\,u3^4}{64},\ \frac{u2^4\,u3^2}{64},\ \frac{u2^2\,u3^4}{64},\ \frac{u2^4\,u3^2}{64},\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,$$

$$\frac{u2^4\,u3^2}{64},\ \frac{1}{64}\left(-4\,u1^2\,u2+u2^3\right)^2,\ 0,\ \frac{u3^6}{64},\ 0,\ \frac{u1^2\,u3^4}{16},\ \frac{u2^2\,u3^4}{64},\ 0,\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,$$

$$\frac{u2^2\,u3^4}{64},\ \frac{u2^4\,u3^2}{64},\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ 0,\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ 0,\ 0,\ 0,\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ 0,\ 0,\ 0,$$

$$0,\ 0,\ \frac{u2^4\,u3^2}{64},\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ \frac{u2^4\,u3^2}{64},\ \frac{1}{64}\,u2^2\left(4\,u1^2-u2^2+u3^2\right)^2,\ 0,\ 0,\ \frac{u2^4\,u3^2}{64},\ 0,$$

$$\frac{1}{16}\,u1^2\,u2^2\,u3^2,\ \frac{u2^4\,u3^2}{64},\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ 0,\ 0,\ 0,\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ \frac{u2^2\,u3^4}{64},\ 0,\ \frac{u2^2\,u3^4}{64},$$

$$\frac{1}{64}\left(-u2^2\,u3+u3^3\right)^2,\ 0,\ 0,\ \frac{u2^2\,u3^4}{64},\ 0,\ 0,\ \frac{u2^2\,u3^4}{64},\ \frac{u2^4\,u3^2}{64},\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ \frac{u2^4\,u3^2}{64},$$

$$\frac{1}{64}\,u2^2\left(4\,u1^2-u2^2+u3^2\right)^2,\ 0,\ 0,\ \frac{u2^4\,u3^2}{64},\ 0,\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ \frac{u2^4\,u3^2}{64},\ \frac{u2^2\,u3^4}{64},\ 0,$$

$$\frac{1}{16}\,u1^2\,u2^2\,u3^2,\ \frac{u2^2\,u3^4}{64},\ \frac{1}{64}\left(-u2^2\,u3+u3^3\right)^2,\ \frac{1}{16}\,u1^2\,u2^2\,u3^2,\ 0,\ \frac{u2^2\,u3^4}{64},\ 0,\ \frac{u2^2\,u3^4}{64}\Big\}$$

Squared Minors of $F(u_1, u_2, u_3)$ from Example 2

$$
\Big\{ \frac{1}{16} u1^2\, u2^2\, u3^2,\ 0,\ \frac{u2^2\, u3^4}{64},\ \frac{u2^4\, u3^2}{64},\ \frac{1}{16} u1^2\, u2^2\, u3^2,\ \frac{u2^4\, u3^2}{64},\ \frac{1}{64}\left(-4\, u1^2\, u2 + u2^3\right)^2,
$$

$$
\frac{1}{16} u1^2\, u2^2\, u3^2,\ \frac{u3^6}{64},\ \frac{u2^2\, u3^4}{64},\ 0,\ \frac{u2^2\, u3^4}{64},\ \frac{u2^4\, u3^2}{64},\ \frac{u2^2\, u3^4}{64},\ \frac{u2^4\, u3^2}{64},\ \frac{1}{16} u1^2\, u2^2\, u3^2,
$$

$$
\frac{u2^4\, u3^2}{64},\ \frac{1}{64}\left(-4\, u1^2\, u2 + u2^3\right)^2,\ 0,\ \frac{u3^6}{64},\ 0,\ \frac{u1^2\, u3^4}{16},\ \frac{u2^2\, u3^4}{64},\ 0,\ \frac{1}{16} u1^2\, u2^2\, u3^2,
$$

$$
\frac{u2^2\, u3^4}{64},\ \frac{u2^4\, u3^2}{64},\ \frac{1}{16} u1^2\, u2^2\, u3^2,\ 0,\ \frac{1}{16} u1^2\, u2^2\, u3^2,\ 0,\ 0,\ 0,\ \frac{1}{16} u1^2\, u2^2\, u3^2,\ 0,\ 0,\ 0,
$$

$$
0,\ 0,\ \frac{u2^4\, u3^2}{64},\ \frac{1}{16} u1^2\, u2^2\, u3^2,\ \frac{u2^4\, u3^2}{64},\ \frac{1}{64} u2^2 \left(4\, u1^2 - u2^2 + u3^2\right)^2,\ 0,\ 0,\ \frac{u2^4\, u3^2}{64},\ 0,
$$

$$
\frac{1}{16} u1^2\, u2^2\, u3^2,\ \frac{u2^4\, u3^2}{64},\ \frac{1}{16} u1^2\, u2^2\, u3^2,\ 0,\ 0,\ 0,\ \frac{1}{16} u1^2\, u2^2\, u3^2,\ \frac{u2^2\, u3^4}{64},\ 0,\ \frac{u2^2\, u3^4}{64},
$$

$$
\frac{1}{64}\left(-u2^2\, u3 + u3^3\right)^2,\ 0,\ 0,\ \frac{u2^2\, u3^4}{64},\ 0,\ 0,\ \frac{u2^2\, u3^4}{64},\ \frac{u2^4\, u3^2}{64},\ \frac{1}{16} u1^2\, u2^2\, u3^2,\ \frac{u2^4\, u3^2}{64},
$$

$$
\frac{1}{64} u2^2 \left(4\, u1^2 - u2^2 + u3^2\right)^2,\ 0,\ 0,\ \frac{u2^4\, u3^2}{64},\ 0,\ \frac{1}{16} u1^2\, u2^2\, u3^2,\ \frac{u2^4\, u3^2}{64},\ \frac{u2^2\, u3^4}{64},\ 0,
$$

$$
\frac{1}{16} u1^2\, u2^2\, u3^2,\ \frac{u2^2\, u3^4}{64},\ \frac{1}{64}\left(-u2^2\, u3 + u3^3\right)^2,\ \frac{1}{16} u1^2\, u2^2\, u3^2,\ 0,\ \frac{u2^2\, u3^4}{64},\ 0,\ \frac{u2^2\, u3^4}{64} \Big\}
$$

Evaluation of Squared Minors from Example 2

- if the exposed face $\mathbb{F}_{W_F}(u)$ is a large face then $u_3 = 0$ because

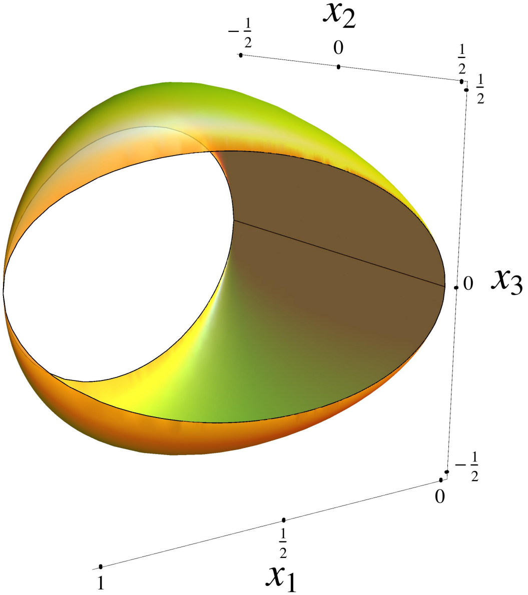
$$u_3 \neq 0 \quad \Rightarrow \quad |\mathrm{Disc}(F(u))| \geq |M_{\{(1,1),(1,3),(2,2)\}}|^2 = \frac{u_3^6}{64} > 0$$

- if the exposed face $\mathbb{F}_{W_F}(u_1, u_2, 0)$ is a large face then $u_2 = 0$ or $u_2 = \pm 2u_1$ since

$$|\mathrm{Disc}(F(u))| \geq |M_{\{(1,1),(1,2),(3,3)\}}|^2 = \frac{u_2^2(u_2^2 - 4u_1^2)^2}{64}$$

Result: Example 2 has exactly three large faces, the segment $\mathbb{F}_{W_F}(-1,0,0)$ and the two ellipses $\mathbb{F}_{W_F}(1,\pm 2,0)$.

All Large Faces of Example 2



$\mathbb{F}_{W_F}(-1, 0, 0)$

$\mathbb{F}_{W_F}(1, \pm 2, 0)$

Conclusion

Summary:

We have a classification of joint numerical ranges in the simplest three-dimensional case of $k = d = 3$.

Questions:

- Can we find classifications of L_F for $k > 3$, $d = 3$?
  probably yes, but graph embedding into S^{k-1} is no constraint any more
- Can we find a classification of W_F for $k = 3$, $d = 4$?
  unclear, even determining large faces is very hard, as $\mathrm{Disc}(F(u))$ is a sum of $\binom{64}{4} = 1820$ squares

Reference:

Konrad Szymański, SW, Karol Życzkowski, *Classification of joint numerical ranges of three hermitian matrices of size three*, LAA 545 (2018), 148

Thank you