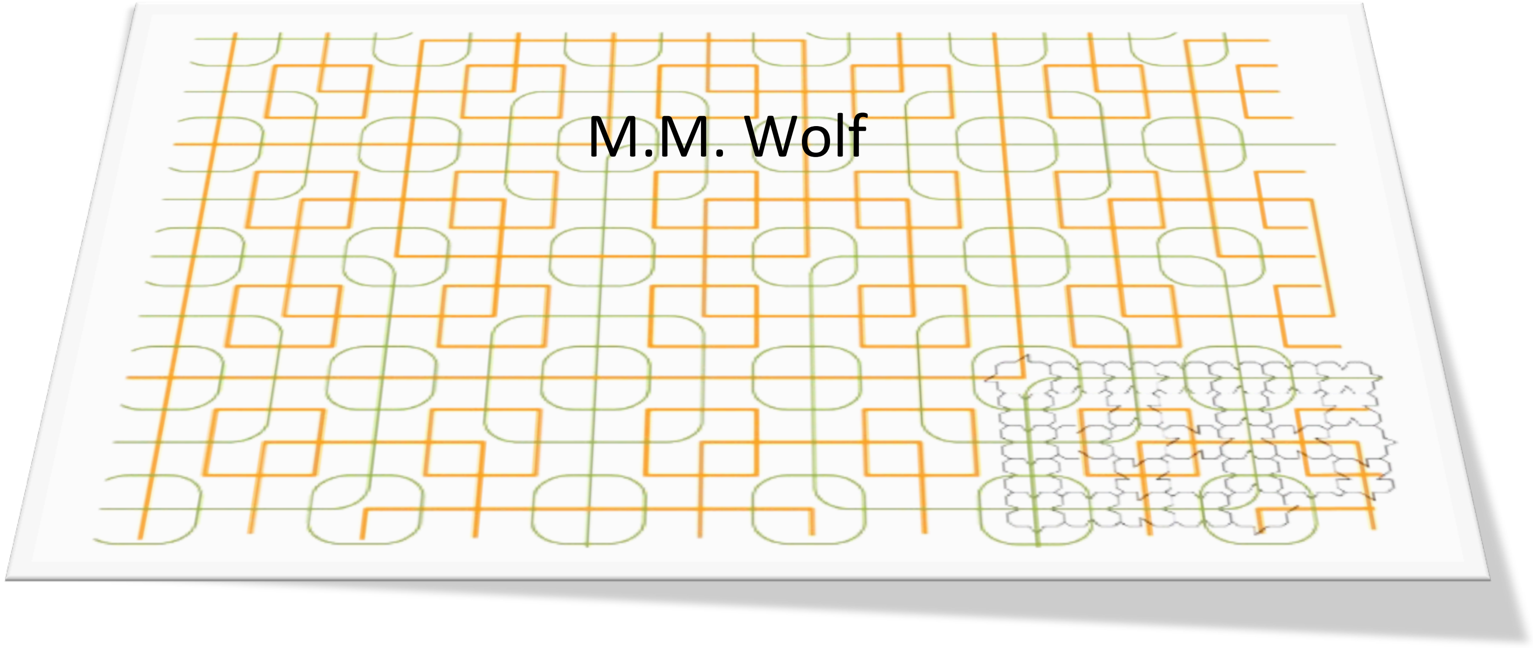
Undecidable problems in Quantum Theory

M.M. Wolf



TECHNISCHE
UNIVERSITÄT
MÜNCHEN

Timeline

1868

Timeline

768

1868

Timeline

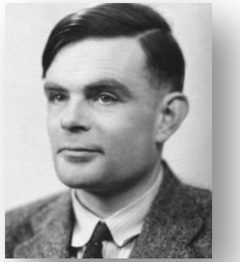
768

1868

193X

Timeline





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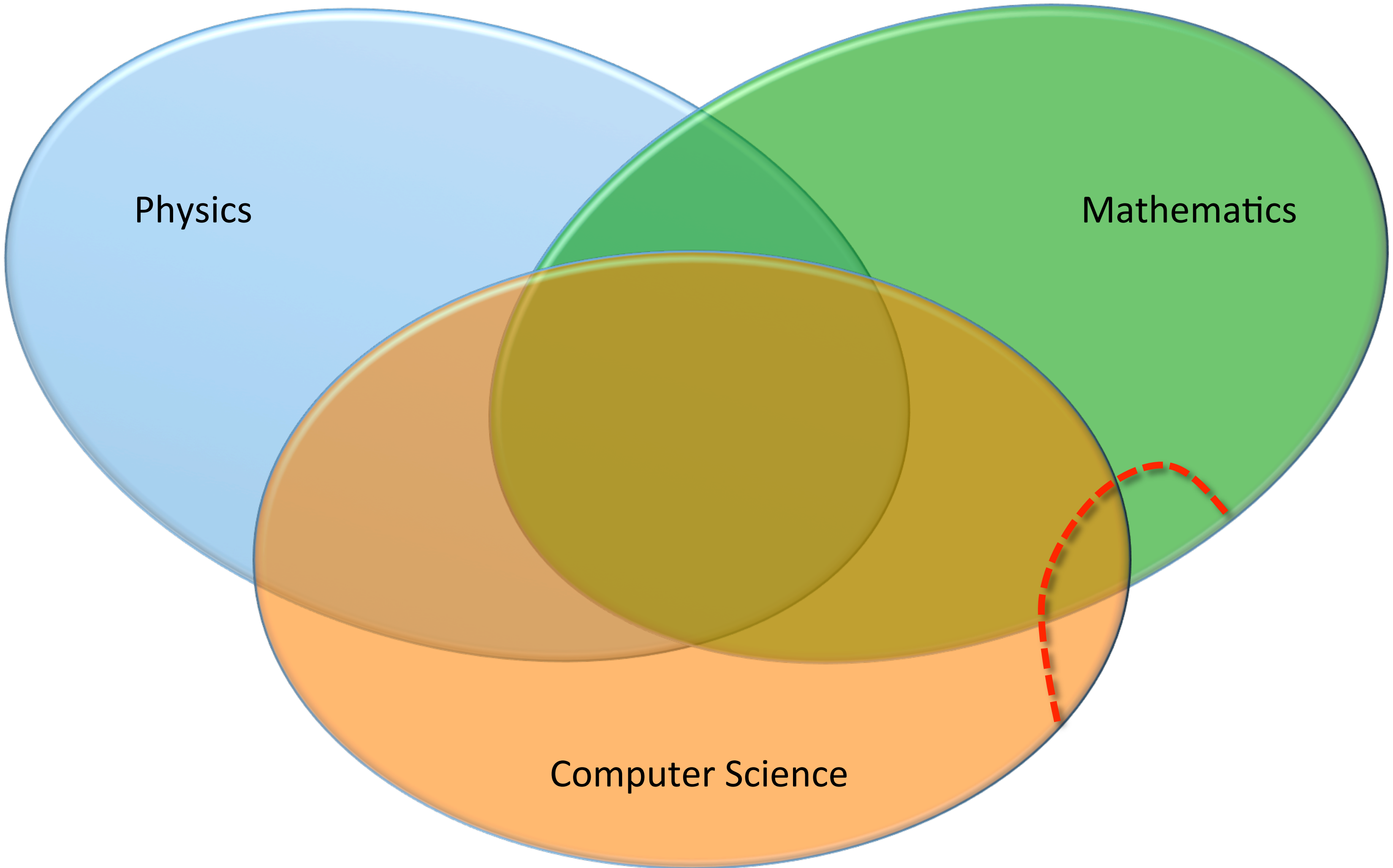
Where is the border between provable & unprovable ?



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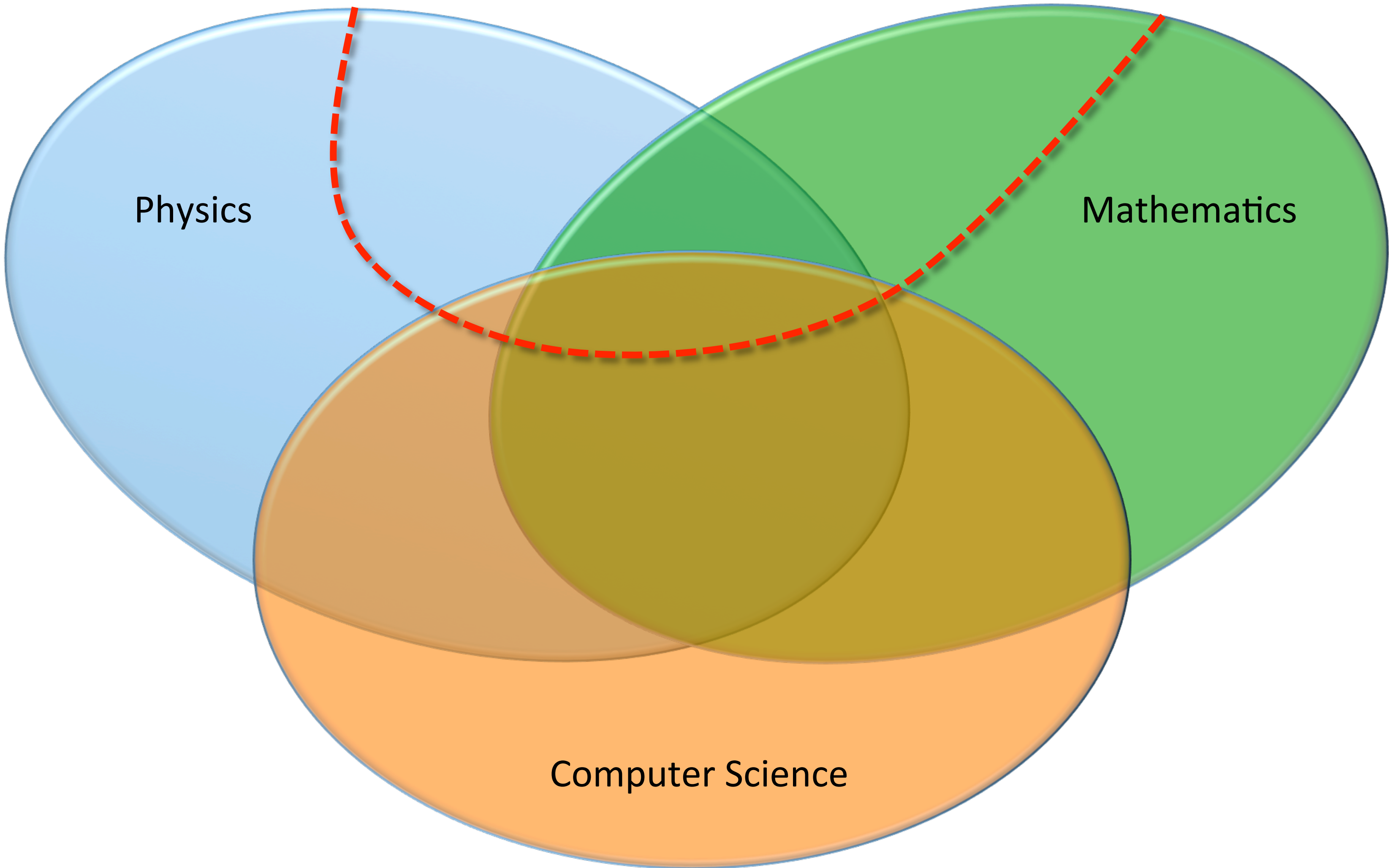


Decision problems in QT

- Is a given state entangled?
- Does a given channel have quantum capacity?
- Is a given set of controls sufficient to reach a certain target?
- Is a given Hamiltonian gapped?

When expressed in formal language these contain quantification over infinite sets

Undecidable

Algorithmic

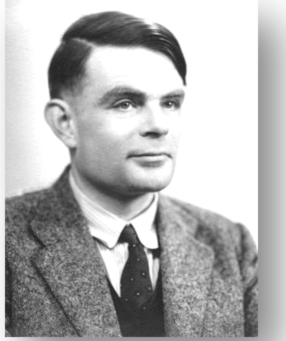
Undecidable = uncomputable by any Turing machine

Requires infinite set of instances

Axiomatic

Undecidable = independent of the axioms of a formal system

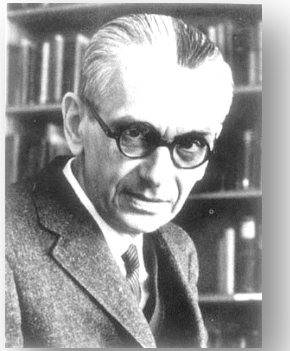
Single instances may be concerned, but are hard to identify



Turing

$\Rightarrow$

for consistent effectively presented axiomatic systems



Gödel

Proving algorithmic decidability

I. Finite domain

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II. Tarski-Seidenberg quantifier elimination

Proving algorithmic decidability

I. Finite domain

II. Tarski-Seidenberg quantifier elimination

$$\exists x \in \mathbb{R} : \; a_1 x^2 > a_2$$

$$\Leftrightarrow a_2 < 0 \vee a_1 > 0$$

Is decidable within a computable ordered field (e.g. for algebraic numbers)

Proving algorithmic decidability

I. Finite domain

II. Tarski-Seidenberg quantifier elimination

$\phi(x,a)$ Boolean combination of polynomial (in-)equalities with integer coefficients and variables

$(x,a) \in \mathbb{R}^n \times \mathbb{R}^m_{alg}$

$\psi(a) := \Big(Q_1 x_1 \ldots Q_n x_n : \phi(x,a)\Big),\ Q_i \in \{\forall, \exists\}$

There exists a quantifier-free formula $\theta(a)$ that is

i. Boolean combination of polynomial inequalities,

ii. effectively computable,

iii. equivalent in the sense that

$$\forall a : \ \big(\psi(a) \ \Leftrightarrow \ \theta(a)\big)$$

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Examples: Is a given state entangled?
Is x in numerical range of A?

Proving undecidability

Halting Problem:

Does a universal TM halt upon a given input?

A given problem is undecidable if the Halting Problem can be reduced to it.

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In formal language this involved quantification over $\mathbb{N}$ (instead of $\mathbb{R}$), which makes quantifier elimination impossible. Expressions like $\exists n \in \mathbb{N}$ or $\lim\limits_{\mathbb{N} \ni n \to \infty}$ appear as:

- Large block size limit
- Number of steps/rounds in a protocol
- Thermodynamic limit

Quantum control problems

Given a set of quantum channels $\mathcal{T} = \left\{T_i : \mathbb{C}^{d\times d} \to \mathbb{C}^{d\times d}\right\}_{i=1}^{k}$

Is there a finite sequence $T := T_{i_n} \cdots T_{i_1}$ with $T_{i_l} \in \mathcal{T}$ s.t. …

1) … for given initial state ρ , target state ψ and fidelity threshold $\lambda \in (0,1)$
   $\langle\psi|T(\rho)|\psi\rangle > \lambda$?

2) … for given target unitary U and fidelity threshold $\lambda \in (0,1)$
   $\int \langle\phi|U^* T(|\phi\rangle\langle\phi|)U|\phi\rangle d\phi > \lambda$?

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   For $(k,d) > (3,3)$ this is undecidable by reduction from the Halting Problem via PCP

[Wolf, Cubitt, Perez-Garcia, 2011]
[Buchinger, Wolf, 2015]

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$$\int \langle\phi|U^* T(|\phi\rangle\langle\phi|)U|\phi\rangle d\phi > \lambda \text{ ?}$$

Is decidable by realizing that $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_0$ with

$\mathcal{T}_1 := \{T_i \in \mathcal{T} | \det(T_i) = 1\}$ generates algebraic group, enabling quantifier elimination

$\mathcal{T}_0 := \{T_i \in \mathcal{T} \big| \, |\det(T_i)| < 1\}$ leads to upper bound of used elements

[Buchinger, Wolf, 2015]

MIND THE GAP

Spectral gap

for quantum spin lattices closely related to

- quantum phase transitions
- criticality
- decay of correlations

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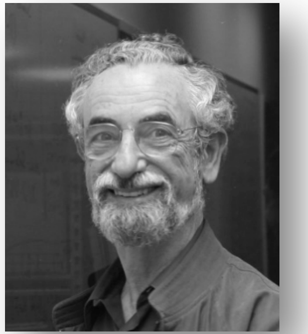
Seminal results

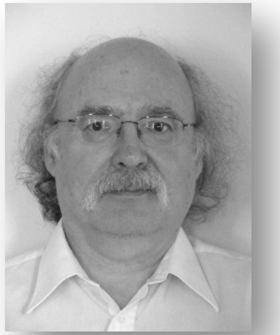
- Lieb-Schultz-Mattis
- Affleck-Kennedy-Lieb-Tasaki
- Hastings

Notorious open problems

- Haldane conjecture
- 2D AKLT
- Yang-Mills existence and mass gap



Lieb

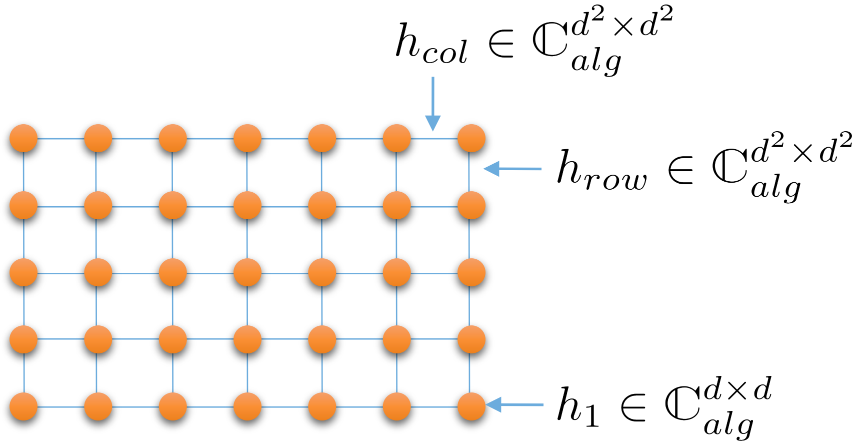


Haldane

Yang & Mills

Spectral gap

$H_L :=$ transl. inv. Hamiltonian on $L \times L$ lattice with nearest neighbour interactions $h_{col},\ h_{row}$ and on-site term h_1



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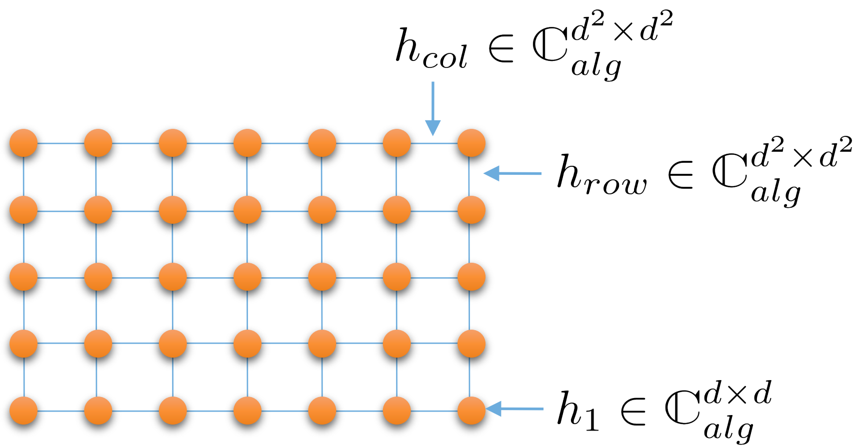
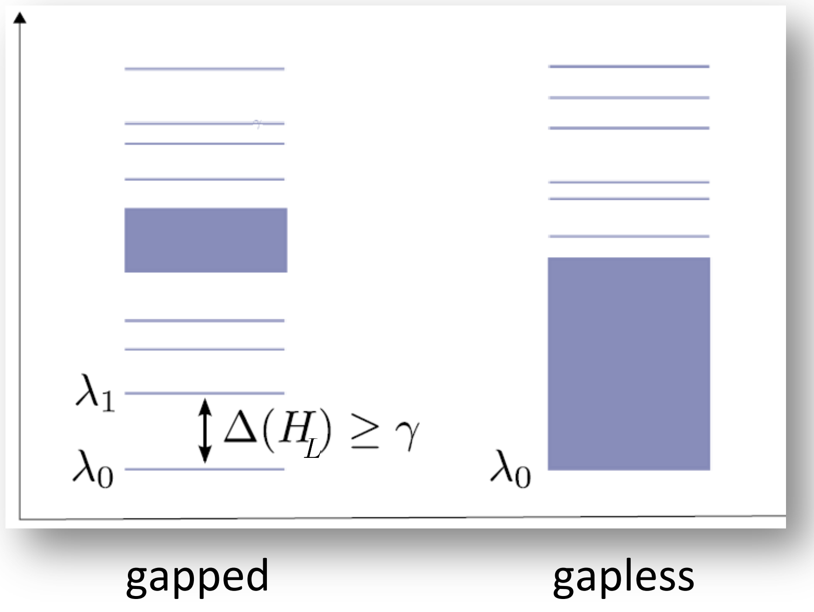


Gapped: unique ground state and uniformly bounded gap for sufficiently large L

Gapless: asymptotically continuous spectrum above the ground state



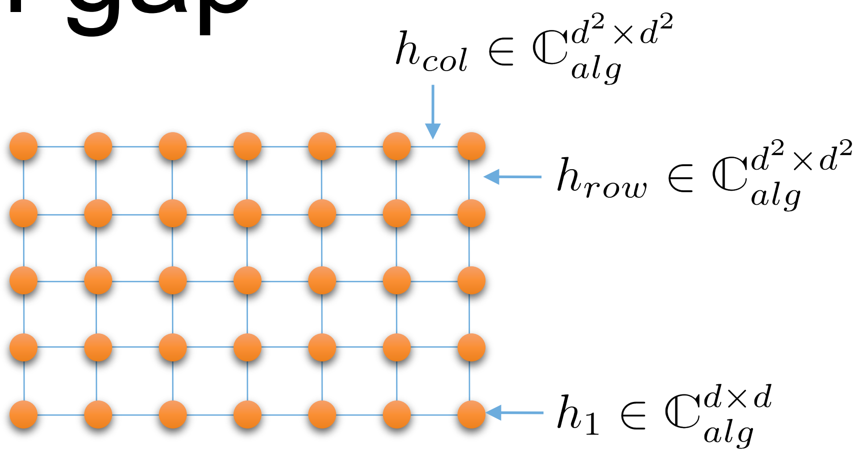
Undecidability of spectral gap

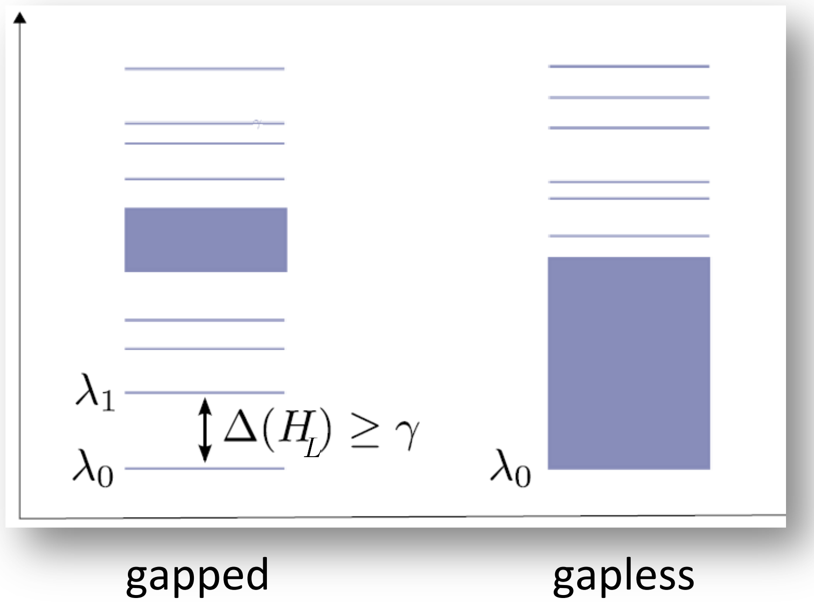
Thm. [Cubitt, Perez-Garcia, Wolf, *Nature '15*]:

There is a $d \in \mathbb{N}$ so that the spectral gap problem for algebraic transl.-inv. nearest-neighbour Hamiltonians on the 2D square lattice is undecidable. This holds even under the promise that

i. the system is gapped or gapless in the strong sense
ii. in the gapped case

$$\gamma \geq \max\{||h_{row}||, ||h_{col}||, ||h_1||\}$$





Reduction I

$\varrho(H) :=$ ground state energy density of H

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Given

i. Hamiltonians H_u for which it is undecidable whether $\varrho(H_u) \nearrow 0$ or $\varrho(H_u) > 0$,
ii. a gapples Hamiltonian H_d with $\lambda_0(H_d) = 0$,

we can construct a Hamiltonian H with

$$\text{spec } H = \{0\} \cup \left\{\text{spec } H_u + \text{spec } H_d\right\} \cup S$$

where $S \geq 1$.

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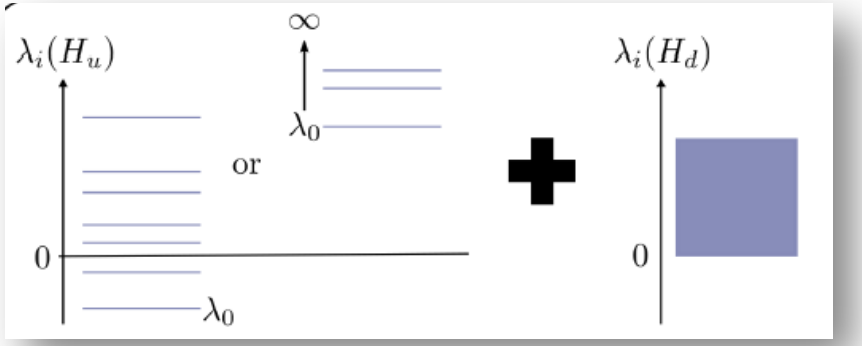

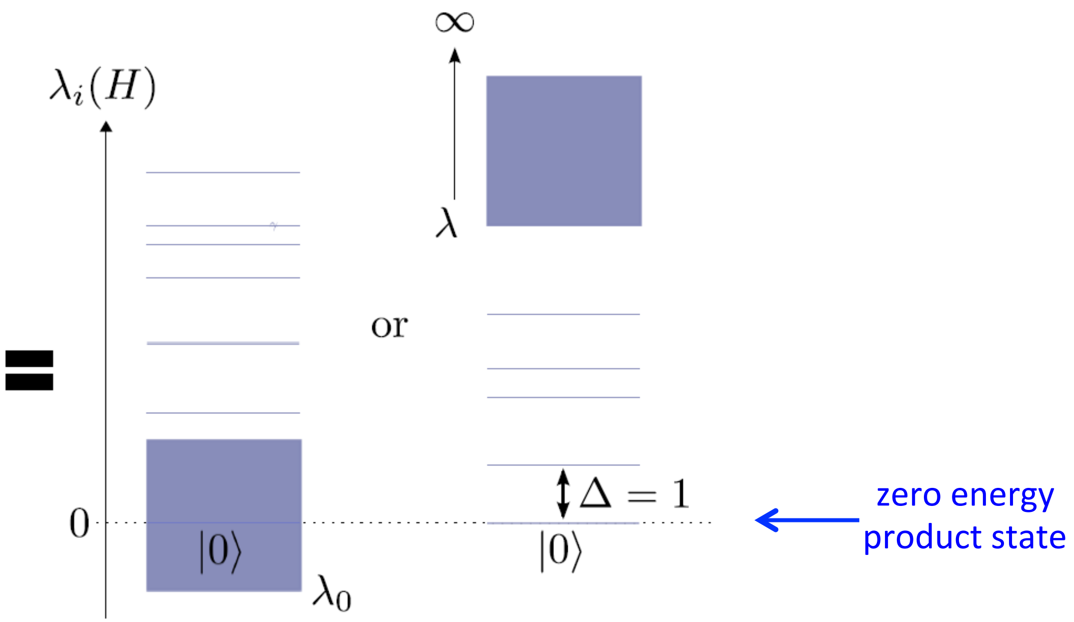

Reduction II

Feynman, Kitaev, …, Gottesmann, Irani: "computational history state" $\frac{1}{\sqrt{\tau}}\sum_{t=1}^{\tau}|t\rangle|\psi_t\rangle$
as ground state of trans.-inv. nearest-neighbour Hamiltonian in 1D.

Idea: encode the evolution of a universal Turing machine and penalize the halting state

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Problem I: where is the input?
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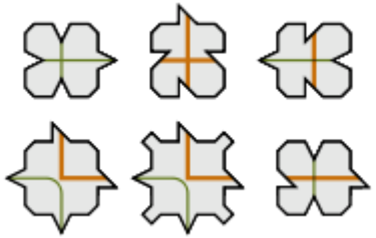
Solution I: quantum phase estimation
Solution II: run Turing machines on edges of aperiodic tilings

Quasi-periodic tilings

Robinson tiles ['71]:



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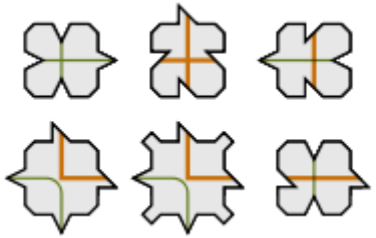
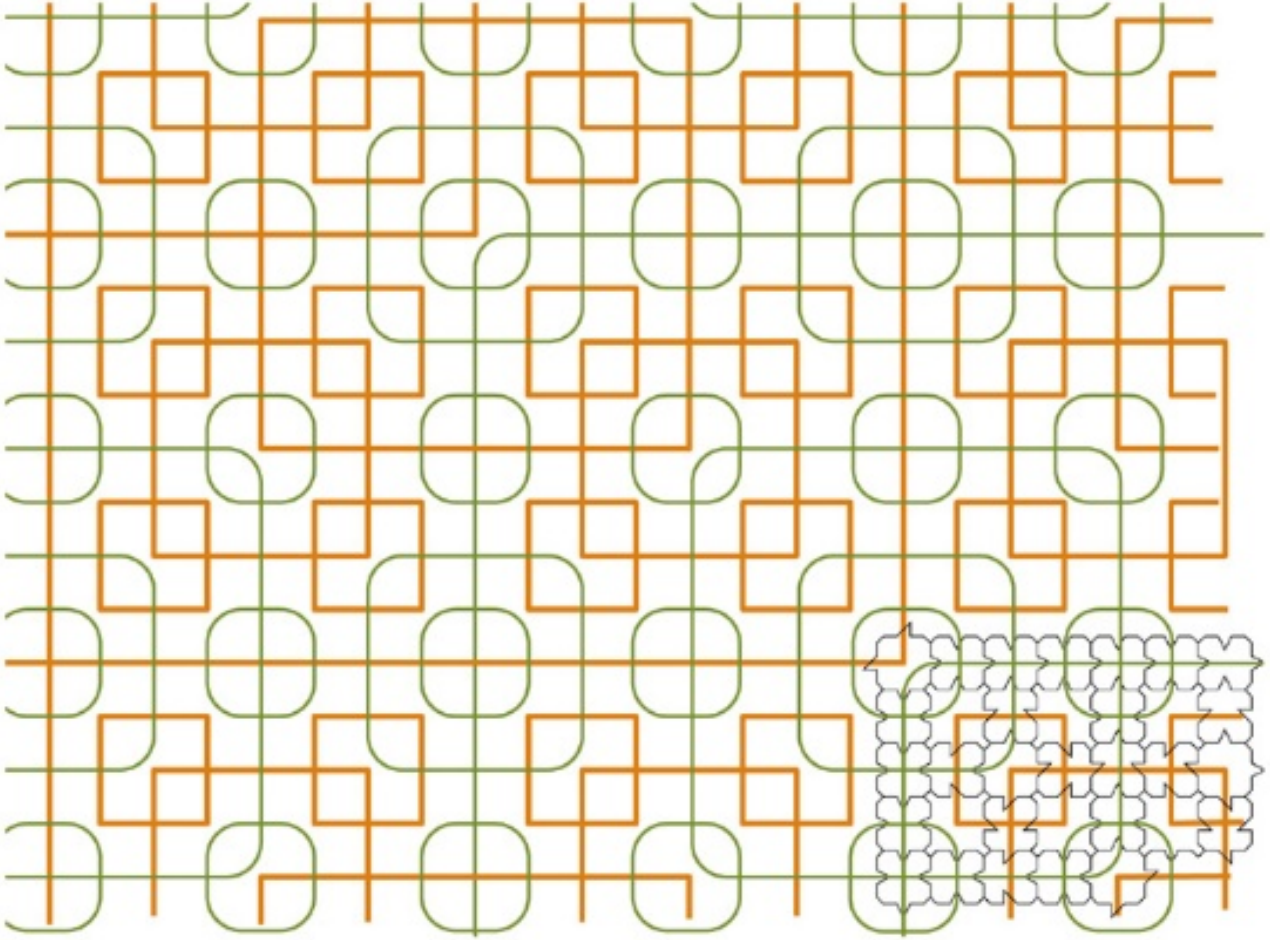
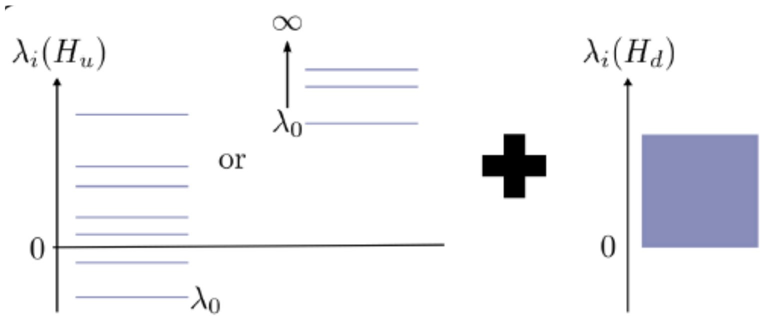
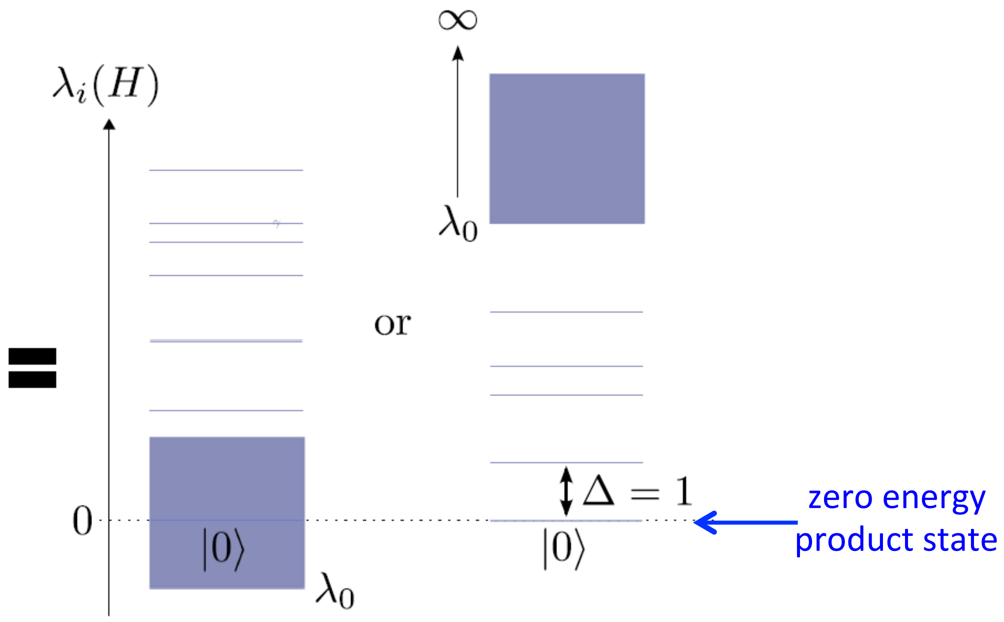




Extensions

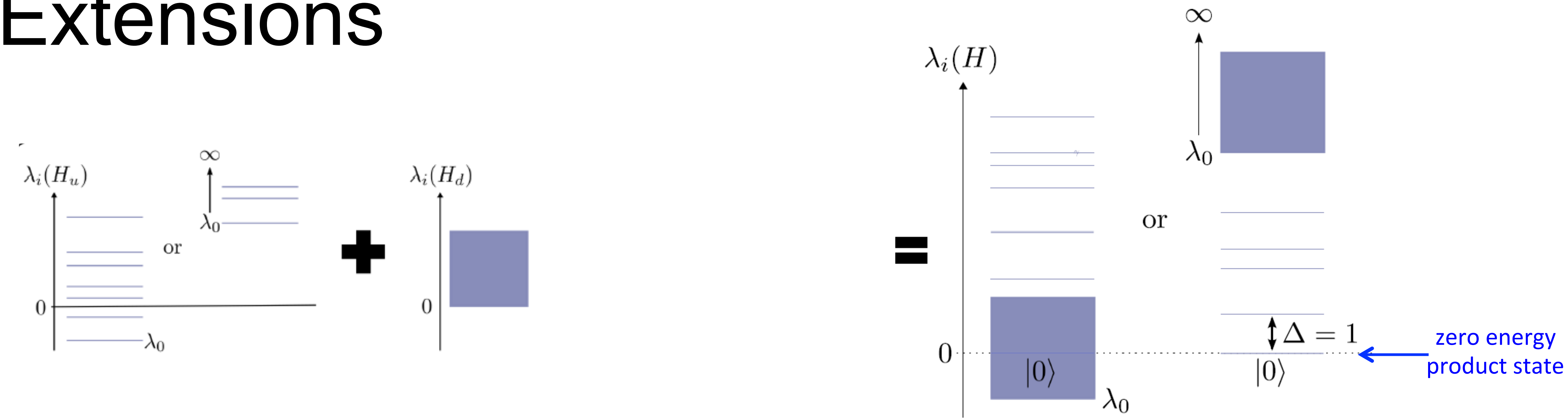


$\lambda_i(H_u)$
0
λ_0
or
∞
λ_0
+
$\lambda_i(H_d)$
0



=
$\lambda_i(H)$
0
$|0\rangle$
λ_0
or
∞
λ_0
$\Delta = 1$
$|0\rangle$
zero energy
product state

Extensions

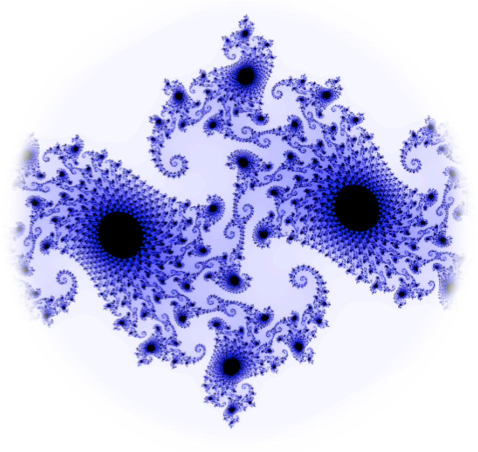


Replacing H_d with other Hamiltonians, one obtains undecidability of essentially all properties defined in the thermodynamic limit that differ from the low energy behaviour of a gapped system with product ground state. E.g.:

- critical correlations
- area law violation
- specific excitations
- topological order

Discussion

- There cannot be a generally valid algorithm/criterion
- There are unprovable instances, but we cannot pinpoint them
- Thermodynamic limit & extrapolations have to be treated with care
- Result also shows extreme form of instability
- Decidability for small *d* or 1D is open
- New physical phenomena? "Size-driven phase transitions"



Size-driven phase transitions

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[Bausch, Lucia, Cubitt, Perez-Garcia, Wolf, PNAS 2018]

Lower bounds on threshold system size after which a size-driven quantum phase transition from classical to topological ground state occurs (d = local Hilbert space dimension):

d	5	6	7	8	9	10
L	15	84	420	2310	$3\ 10^7$	10^{36534}

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What can we learn from undecidability?

- Stop looking for a general algorithm
- Be careful with limits and extrapolations
- Hint at new phenomena?
- Proof tool (purification limitations for FCS, single letter formulas, etc.)

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