On Restricted Numerical Range

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in collaboration with

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Numerical Range (Field of Values)

Definition

For any operator A acting on $\mathcal{H}_N$ one defines its NUMERICAL RANGE as a subset of the complex plane defined by:

$$\Lambda(A) = \{\langle x|A|x\rangle : \ |x\rangle \in \mathcal{H}^N, \ \langle x|x\rangle = 1\}.$$

In physics: Rayleigh quotient, $\mathcal{R}(A) := \langle x|A|x\rangle / \langle x|x\rangle$

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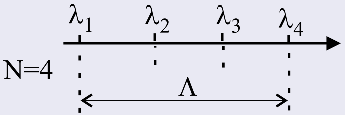
Hermitian case

For any hermitian operator $A = A^\dagger$ with spectrum $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ its **numerical range** forms an interval: the set of all possible expectation values of the observable A among arbitrary pure states, $\Lambda(A) = [\lambda_1, \lambda_N]$.



Numerical range and its properties

Compactness

$\Lambda(A)$ is a **compact** subset of $\mathbb{C}$.

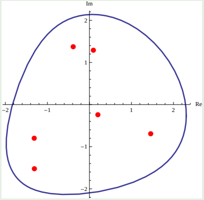
Convexity: Toeplitz (1918) – Hausdorff (1919) theorem

- $\Lambda(A)$ is a **convex** subset of $\mathbb{C}$.

100-th Aniversary !

Example

Numerical range of a random (not–hermitian) matrix of order $N = 6$
red dots represent its eigenvalues




beeline
beeline

Otton Nikodym & Stefan Banach,

talking at a bench in Planty Garden, **Cracow, summer 1916**

Numerical range for a normal matrix

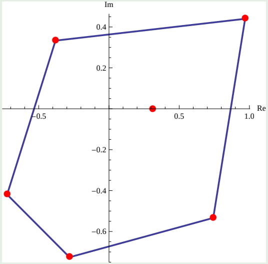
Convex hull of the spectrum & **Numerical range**

Lemma. If A is **normal** i.e. $A^\dagger A = AA^\dagger$, then $\Lambda(A) = Co(\sigma(A))$.

Proof. The eigen decomposition reads $A|\phi_1\rangle = z_i|\phi_1\rangle$ and $\langle\phi_i|\phi_j\rangle = \delta_{ij}$.
Take convex combination λ of arbitrary two eigenvalues,
$\lambda = az_1 + (1-a)z_2$ and the superposition $|\psi\rangle := \sqrt{a}|\phi_1\rangle + \sqrt{1-a}|\phi_2\rangle$.
Then $\langle\psi|A|\psi\rangle = az_1 + (1-a)z_2 = \lambda$ hence $\lambda \in \Lambda(A)$.

Example

Numerical range of a random normal matrix for $N = 6$



Generalization of NUMERICAL RANGE

Restricted Numerical Range, (Gawron *et al.* 2010)

defined by restricting the set of states to a given subset $\Omega_X \in \Omega$ of the set Ω of all normalized states,

$$\Lambda_X(A) = \{\langle\psi|A|\psi\rangle : \ |\psi\rangle \in \Omega_X\},$$

Physically motivated **Examples of restricted numerical range:**

a) **Real States**, $\Omega_R = \{|\psi\rangle \in \mathbb{R}^N\}$

leads to **Numerical range restricted to Real states**

Notions for composite Hilbert space, e.g. $\mathcal{H}_{MN} = \mathcal{H}_M \otimes \mathcal{H}_N$.

b) **Product States**, (**Schulte-Herbrueggen** *at al.* 2008)

$\Omega_P = \{|\psi_1\rangle \otimes |\psi_2\rangle\}$ leads to **Product (local) Numerical range**

c) **Maximally Entangled states**, $\Omega_E = \{U_1 \otimes U_2|\psi^+\rangle\}$

with $|\psi^+\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^N |i\rangle \otimes |i\rangle$ and local unitaries $U_1, U_2 \in U(N)$

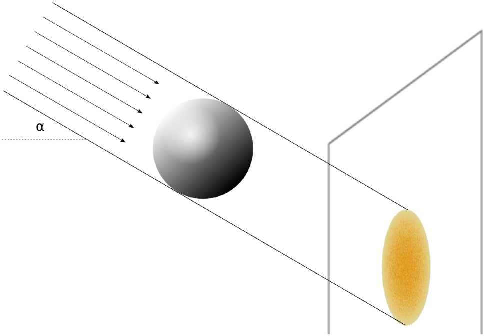
leads to **Entangled Numerical range**, (**Puchała** *et al.* 2012)

d) **Separable (mixed) states**, (in convex hull of all product states)

leads to **Separable Numerical range**

Why do we analyze **restricted numerical range** ?

Standard **numerical range** of a matrix A of order N gives us a *possible* projection of the set Ω_N of mixed states of size N on a 2-plane



Numerical range restricted to the subset Ω_X yields a *possible* projection of the set Ω_X on a 2-plane and provides information about **geometry** of the subset Ω_X.

Cognate notions

Restricted Numerical Radius

defined by restricting the set of states to a given subset $\Omega_X \in \Omega$ of the set Ω of all normalized states.

$$r_X(A) = \max\{|z| : z \in \Lambda_X(A)\}$$

Physical applications motivate examples of **restricted numerical radius**

a) **Real States**, $\Omega_R = \{|\psi\rangle \in \mathbb{R}^N\}$

leads to **Numerical radius restricted to Real states**

Notions for composite Hilbert space, e.g. $\mathcal{H}_{MN} = \mathcal{H}_M \otimes \mathcal{H}_N$.

b) **Product States**, $\Omega_P = \{|\psi_1\rangle \otimes |\psi_2\rangle\}$

leads to **Product (local) Numerical radius**

c) **Maximally Entangled states**, $\Omega_E = \{U_1 \otimes U_2|\psi^+\rangle\}$

with $|\psi^+\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle \otimes |i\rangle$ and local unitaries $U_1, U_2 \in U(N)$

leads to **Numerical Radius** with respect to **Entangled States**

d) **Separable (mixed) states**, (in convex hull of all product states)

leads to **Separable Numerical radius**.

Wawel castle in Cracow

D.& K. Ciesielscy theorem

D.& K. Ciesielscy theorem: For any $\epsilon > 0$ there exist $\eta > 0$ such that with **probability** $1 - \epsilon$ the bench **Banach** talked to **Nikodym** in **1916** was localized in η-**neighbourhood** of the **red arrow**.

Plate commemorating the discussion between

Stefan Banach and **Otton Nikodym** (**Kraków, summer 1916**)

Product (local) numerical range (PNR)

Definition

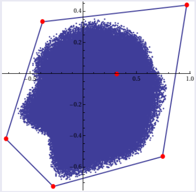
Product numerical range of an operator A acting on $\mathcal{H}_N \otimes \mathcal{H}_M$

$$\Lambda_{\otimes}(A) = \left\{ \langle \psi \otimes \phi | A | \psi \otimes \phi \rangle : \ |\psi\rangle \in \mathbb{C}^N, |\phi\rangle \in \mathbb{C}^M \right\},$$

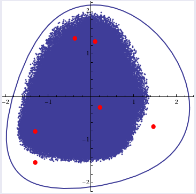
with $\langle \psi|\psi\rangle = \langle \phi|\phi\rangle = 1$.

Product numerical range forms a **subset** of numerical range

Product numerical range of a **random normal** matrix



Product numerical range of a **random** matrix



Useful lemma on dimensionality of subspaces

Lemma

Consider a composed complex Hilbert space $\mathcal{H}_n = \mathcal{H}_k \otimes \mathcal{H}_m$. *Then*

1. **Any subspace** $S_d \subset \mathcal{H}_n$ *of dimension* $d = (k-1)(m-1)+1$ *contains at least one product state,*
2. **There exists a subspace** *of dimension* $d-1 = (k-1)(m-1)$, *which* **does not** *contain any product states.*

can be proved by dimension counting ...

Wallach 2002;
Parthasarathy 2004;
Cubitt, Montanaro and Winter, 2008 .

Product numerical range for Hermitian operators

Definition

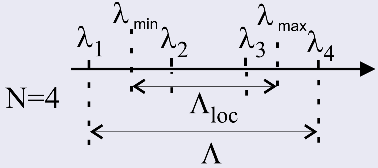
By minimal (maximal) **local values** we call: $\lambda_{\rm loc}^{\rm min}(A) = \min\left(\Lambda_{\otimes}(A)\right)$, $\lambda_{\rm loc}^{\rm max}(A) = \max\left(\Lambda_{\otimes}(A)\right)$.

Convexity for hermitian operators

For any Hermitian A its local numerical range is convex and forms an interval of the real axis, $\Lambda_{\otimes}(A) \;=\; [\lambda_{\rm loc}^{\rm min}(A), \lambda_{\rm loc}^{\rm max}(A)]$.

Proposition for 2×2 problem

Local numerical range includes the central segment of the spectrum, $\Lambda_2(A) = [\lambda_2, \lambda_3] \subset \Lambda_{\otimes}(A) \subset \Lambda(A)$.



2×2 example: **two–qubit density matrix ρ**

$$\rho(\alpha) \ := \ U_\alpha D U_\alpha^\dagger \ ,$$

with the spectrum $D = \mathrm{diag}\{0, 1/6, 2/6, 3/6\}$,
and a **non local** unitary matrix U_α of eigenvectors

$$U_\alpha \ = \ \exp(i\alpha \, \sigma_x \otimes \sigma_x) \ .$$

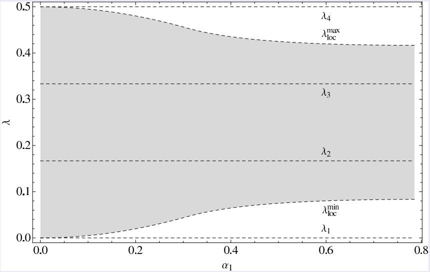


Figure: Spectrum and product numerical range (grey) for the family of states $\rho = \rho(\alpha)$

Example: **a single qubit map** Ψ

One qubit map $\Psi : \rho \to \Psi(\rho)$ can be represented by the dynamical matrix, (Choi matrix) $D_\Psi = (\Psi \otimes \mathbb{1})|\psi^+\rangle\langle\psi^+|$ where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)$,

$$D_\Psi = \begin{bmatrix} \frac{1}{2} & a & 0 & 0 \\ \bar{a} & \frac{1}{2} & b & 0 \\ 0 & \bar{b} & \frac{1}{2} & c \\ 0 & 0 & \bar{c} & \frac{1}{2} \end{bmatrix},$$

where $a, b \in \mathbb{C}$ and $c = xa$ for some $x \in R$. Checking **positivity of polynomials** (**Ł. Skowronek, K. Ż**, *J. Phys.* **A**, 2009) we find

$$\Lambda_\otimes(D_\Psi) = \left[\frac{1}{2} - M, \frac{1}{2} + M\right],$$

where

$$M = \frac{1}{4}\left(|a+c| + \sqrt{|a-c|^2 + |b|^2}\right).$$

Jamiołkowski theorem: a map Ψ is positive iff D_Ψ is **block positive**, $\langle x, y|D_\Psi|x, y\rangle \ \geq \ 0 \iff \lambda_{\min}^{loc}(D_\Psi) \ \geq \ 0$

Conclusion: The map $\Psi(a, b, c)$ is **positive** if $M(a, b, c) \ \leq \ 1/2$.

Product numerical range for Hermitian opertors II

Two-qubit **toy example**

$$H_a = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad a \in \mathbb{R}_+$$

with spectrum $(-1, -a, a, 1)$

standard numerical range: $\Lambda(H_a) = [-1, 1]$

product numerical range: $\Lambda_\otimes(H_a) = [-b, b]$, where $b = \frac{1+a}{2}$.

special case: $a = 0$ so $H_0 = \text{anti diag}(1, 0, 0, 1)$

in this case we have $\Lambda(H_0) = [-1, 1]$ and $\Lambda_\otimes(H_0) = [-1/2, 1/2]$,

so the ratio of their volumes is $\mu = \frac{\text{Vol}(\Lambda_\otimes(H_0))}{\text{Vol}(\Lambda(H_0))} = \frac{1}{2}$.

Open problem: For what **hermitian operator** $H = H^*$ **of order** 4

the ratio $\mu(H)$ **is the smallest ?**

Several parties: $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_k$

k–parties product states: $|\psi_\otimes\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_k\rangle$

k–parties Product Numerical Range: $\Lambda_\otimes(X) = \langle\psi_\otimes|X|\psi_\otimes\rangle$,

(**Schulte-Herbrueggen, Dirr, Helmke, Glaser** 2008)

k-qubit **toy example**: hermitian G_k of dimension $L = 2^k$:

anti-diagonal symmetrix matrix:

$$G_k = A(x_1, x_2, \ldots, x_{L/2},\ x_{L/2}, \ldots, x_2, x_1) = G_k^*$$

with $x_i \geq 0$ and spectrum $(\pm x_1, \pm x_2, \ldots, \pm x_{L/2})$

standard numerical range: $\Lambda(G_k) = [-x_{max}, x_{max}]$

product numerical range: $\Lambda_\otimes(G_k) = [-b, b]$, where $b = \frac{\sum_{j=1}^{L/2} x_j}{2^{k-1}}$.

special case: antidiagonal matrix $G_k = A(1, 0, 0 \ldots, 0, 0, 1)$

so that $\Lambda(G_k) = [-1, 1]$ and $\Lambda_\otimes(G_k) = [-1/2^{k-1}, 1/2^{k-1}]$,

Then the ratio of their volumes $\mu = \frac{\mathrm{Vol}(\Lambda_\otimes())}{\mathrm{Vol}(\Lambda())} = \frac{1}{2^{k-1}} \to 0$ for $k \to \infty$.

Open problem: For what **hermitian operator** $H = H^*$ **of order** 2^k

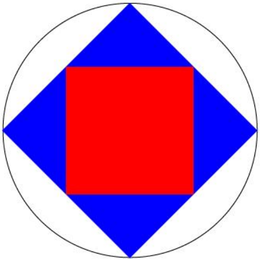
the ratio $\mu(H)$ is the smallest ?

Product numerical range for **non-hermitian** operators

Two-qubit **unitary** example

$$U = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = H_1 + iH_2$$

with spectrum $(1, i, -1, -i)$



standard numerical range : $\Lambda(U) \;=\; \text{conv hull}(1, i, -1, -i) = \diamond$

product numerical range forms a square:
$\Lambda_{\otimes}(U) \;=\; \text{conv hull}(z, \bar{z}, -z, -\bar{z})$, where $z = \frac{1+i}{2}$,

so the ratio of their volumes is $\quad \mu = \frac{\text{Vol}(\Lambda_{\otimes}(B))}{\text{Vol}(\Lambda(B))} = \frac{1}{2}$.

Open problem: For what nonhermitian **operator** B **of order** 4

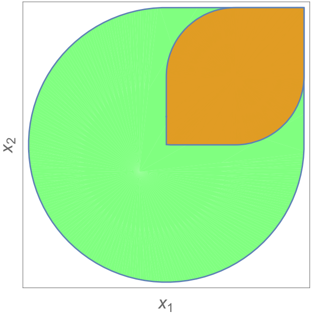
the ratio $\mu(B)$ **is the smallest ?**

Product numerical range for **non-normal** operators

Two-qubit **non-normal** example

$$X = \begin{bmatrix} 1+i & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+i \end{bmatrix}$$

with spectrum $(0, 0, 1+i, 1+i)$



standard numerical range : $\Lambda(X) = \text{conv hull}[C(0,1), \{1+i\}]$

product numerical range $\Lambda_\otimes(X)$ forms an 'eye' .

and the ratio of their volumes is $\mu = \frac{\text{Vol}(\Lambda_\otimes(X))}{\text{Vol}(\Lambda(X))} = \frac{4+\pi}{8+6\pi} \approx 0.25699.$

Joint Numerical Range (JNR) of a set of m operators

$$\Lambda(A_1, A_2, \ldots, A_m) = (\langle\psi|A_1|\psi\rangle, \langle\psi|A_2|\psi\rangle, \ldots, \langle\psi|A_m|\psi\rangle) \subset \mathbb{R}^m.$$

For $m \geq 3$ JNR is (in general) **not** a **convex set**!

Set $m = 2$, decompose $A = A_H + iA_A$ into its **Hermitian** and **anti–Hermitian** part. Then $\Lambda(A) = \Lambda(A_H, A_A)$

Proposition 3. *Take a set $\{A_1, \ldots, A_{N^2-1}\}$ of matrices of size N forming an* **orthonormal basis** *in the space of Hermitian, traceless matrices. Then*

- *$\Lambda(A_1, A_2, \ldots, A_{N^2-1})$ is affine isomorphic to the set $\Omega_N = \mathbb{C}P^{N-1}$ of* **pure quantum states** *of size N (embedded in $\mathbb{R}^{N^2-1}$),*
- *The* **convex hull** *of $\Lambda(A_1, A_2, \ldots, A_{N^2-1})$ is isomorphic to the set $\mathcal{M}_N$ of* **mixed quantum states** *of size N.*
- *$\Lambda(A_1, A_2, \ldots, A_m)$ with $m \leq N^2 - 1$ forms a projection of Ω_N into $\mathbb{R}^m$.*

Joint Numerical Range: some examples

$N = 2$: one qubit states

Let $\sigma_1, \sigma_2, \sigma_3$ denote three trace-less **Pauli matrices** of size $N = 2$. Then

- $\Lambda(\sigma_1, \sigma_2, \sigma_3) = \Omega_2 = \mathbb{C}P^1$ forms the **Bloch sphere** S^2 of all one–qubit pure states.
- The **convex hull** of $\Lambda(\sigma_1, \sigma_2, \sigma_3)$ forms the **Bloch ball**, $\mathcal{M}_2 = B_3 \subset \mathbb{R}^3$ of all one–qubit mixed states.

$N = 3$: one qutrit states

Let $\lambda_1, \dots \lambda_8$ denote eight traceless **Gell–Man matrices** of size 3: the generators of $SU(3)$.
Then

- $\Lambda(\lambda_1, \dots \lambda_8) = \Omega_3 = \mathbb{C}P^2$ forms the set of all one–qutrit pure states.
- The **convex hull** of $\Lambda(\lambda_1, \dots, \lambda_8)$ forms the set of $N = 3$ mixed states – a convex body $\mathcal{M}_3$ embedded in $\mathbb{R}^8$

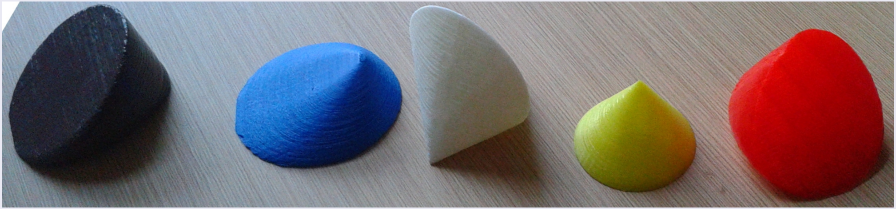
Joint Numerical Range: 3D examples for $m = 3$

$N = 3$: one qutrit

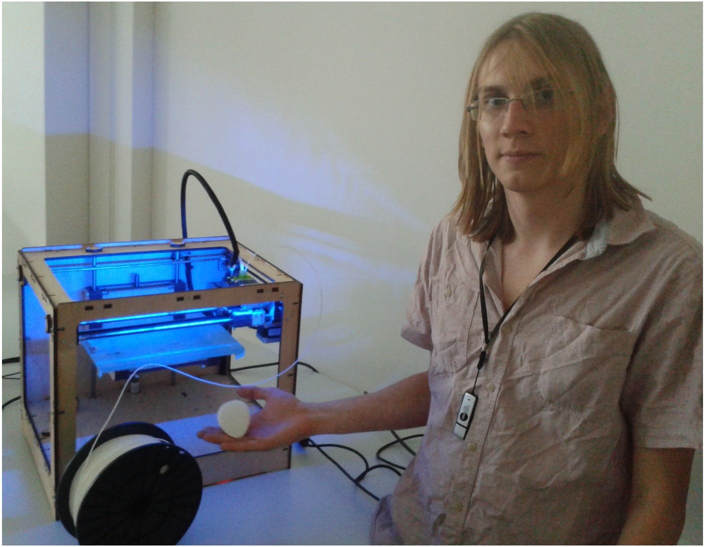
Take any triple of hermitian operators $\{A_1, A_2, A_3\}$ of size $N = 3$.

Then **joint numerical range** $\Lambda(A_1, A_2, A_3) \subset \mathbb{R}^3$ gives
a **projection** of the 8D set $\mathcal{M}_3$ of mixed states of a qutrit into **3D**.

Examples:



Different classes of $3D$ JNR: their further projections into $2D$ belong to one of **four** classes of **Keeler, Rodman, Spitkovsky** (1997).
the possible shapes of the standard numerical range for $N = 3$.



Konrad Szymański producing a 3D joint numerical range

Recall the shadows on the wall of the cave of **Plato**:

we do not understand all details of the $8D$ set $\mathcal{M}_3$ of quantum states of size three, but at least we can study its 2D and 3D **projections**

How to classify possible shapes of JNR of three Hermitian matrices A_1, A_2, A_3 of size $N = 3$?

earlier results by **Chien and Nakazato** 2010;

Chen, Ji, Li, Poon, Shen, Yu, Zeng, D. Zhou 2015,

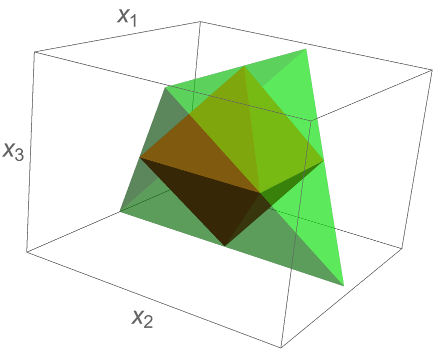
Ten classes of $3D$ numerical ranges labeled by

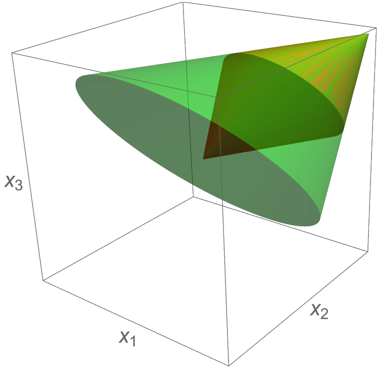
the number **of segments** s and **of ellipses** e in boundary

K. Szymański, S. Weis, K.Ż, (2018)

Separable joint numerical range (JNR) $\Lambda_{\otimes}(A_1, A_2, A_3)$

JNR of three matrices A_1, A_2, A_3 of order $2 \times 2 = 4$ gives a projection of the 15 D set of mixed states of size $N = 4$ into 3D !



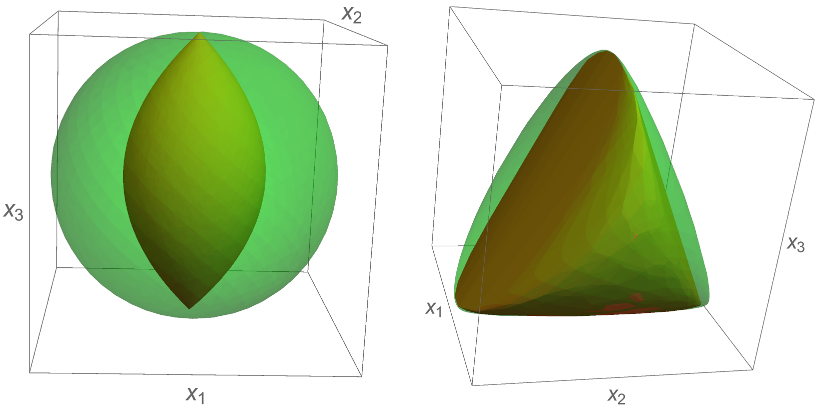


Comparison of **standard** JNR, $\Lambda(A_1, A_2, A_3)$
with **separable** JNR, $\Lambda_{\otimes}(A_1, A_2, A_3)$.

left: **octahedron** of separable states inside tetrahedron of 4 **Bell** states

Separable joint numerical range (JNR) II

Consider three matrices A_1, A_2, A_3 of order $2 \times 2 = 4$



Comparison of **standard** JNR, with **separable** JNR,

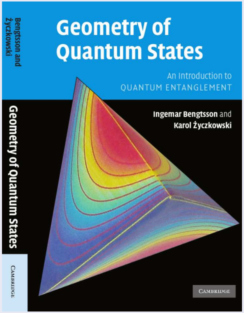
Questions: 1) What is the minmal **volume ratio** μ for 3D sets?

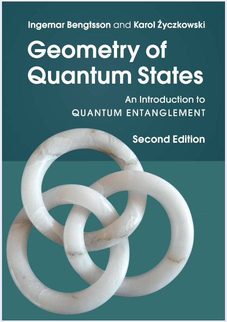
2) Classify possible shapes of JNR of 3 matrices of order $N = 4$

3) Classify possible shapes of **separable** JNR of 3 matrices of $N = 4$

more about geometry set of quantum states for $N \geq 3$:

Geometry of Quantum States
an Introduction to Quantum Entanglement





I. Bengtsson and K. Życzkowski

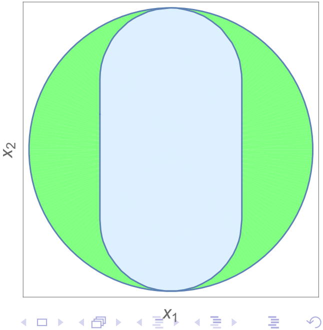
Cambridge University Press, **2006** II extended Edition **2017**

Numerical range and quantum maps

A *completely positive, trace preserving* map $\Phi : \rho \to \rho'$ is represented by **Jamiołkowski–Choi** matrix $D(\Phi) = (\Phi \otimes \mathbb{I})|\psi_+\rangle\langle\psi_+|$,
where $|\psi_+\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{n} |j\rangle \otimes |j\rangle$ is maximally entangled state.

D satisfies partial trace condition $\mathrm{Tr}_A D = \mathbb{I}$ (*).
For bistochastic maps, $\Phi(\mathbb{I}) = \mathbb{I}$, additionally $\mathrm{Tr}_B D = \mathbb{I}$ (**).
How the set of stochastic / bistochastic maps looks like?

Comparison of **standard numerical range**, with NR restricted to states satisfying (*) and (**) and corresponding to one-qubit **bistochastic maps**.



Concluding Remarks I

- Standard **Numerical Range (NR)** is a useful algebraic tool ...
- We advocate to study its generalisation - the **Restricted NR**:
  a) **product numerical range (PNR)** useful for composite systems.
  b) **separable numerical range (SNR)** and other restricted numerical ranges.
- **PNR** needs not to be **convex** or **simply connected**. Some bounds for **product numerical range** are obtained,

  but we do not know how to find it for an arbitrary operator!
- **PNR** is a versatile tool: it can be used to analyze **positivity of quantum maps**, quantum entanglement, local fidelity, **local distinguishability** and other problems in quantum theory of composite systems.
- **Open problem**: for which set of m hermitian matrices of size 2^k the ratio of the volumes: Vol(Product NR)/Vol (NR) is **minimal** ?

Bench commemorating the discussion between
Otton Nikodym and **Stefan Banach** (**Kraków, summer 1916**)

Sculpture: Stefan Dousa Fot. Andrzej Kobos

opened in Planty Garden, Cracow, Oct. 14, 2016

Key message:

Key message:

Restricted **Numerical range**

is a a) **nice algebraic tool**

b) and **usefull** also in theoretical physics !

Consider your research on
generalized numerical ranges and their properties !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Banach tells his side of the story