Preservation of essential matricial range

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Joint work with

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The joint $q$-matricial range

- Let $B(H)$ be the algebra of bounded linear operators acting on the Hilbert space $H$ equipped with the inner product $\langle x, y \rangle$. 

D.R. Farenick, Matrical extensions of the numerical range: a brief survey, Linear Multilinear Algebra 34 (1993), 197-211.

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- If $\dim H = n$, we identify $H = \mathbb{C}^n$ with $\langle x, y \rangle = y^* x$, and identify $B(H)$ with $M_n$, the algebra of $n \times n$ complex matrices.
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If $\dim H = n$, we identify $H = \mathbb{C}^n$ with $\langle x, y \rangle = y^* x$, and identify $B(H)$ with $M_n$, the algebra of $n \times n$ complex matrices.

Let $V_q$ be the set of partial isometries $X : \mathbb{C}^q \to H$ such that $X^* X = I_q$.
Let $B(H)$ be the algebra of bounded linear operators acting on the Hilbert space $H$ equipped with the inner product $\langle x, y \rangle$.

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The joint spatial $q$-matricial range of $A = (A_1, \ldots, A_m) \in B(H)^m$ is defined as

$$W_s^q(A) = \left\{ (X^*A_1X, \ldots, X^*A_mX) : X \in \mathcal{V}_q \right\}.$$
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  $$W_s^q(A) = \{(X^*A_1X, \ldots, X^*A_mX) : X \in \mathcal{V}_q\}.$$ 
- So, $(B_1, \ldots, B_m) \in W_s^q(A)$ if and only if there is a unitary $U = [X|\tilde{X}]$ such that 
  $$U^*A_jU = \begin{pmatrix} B_j & \ast \\ \ast & \ast \end{pmatrix}, \quad j = 1, \ldots, m.$$
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- The $q$-matricial range is useful in the study of operators.

- For example, two compact operators $A, B \in B(H)$ are unitarily similar if and only if $W^q(A) = W^q(B)$ for all positive integer $q$.
Let $B(H)$ be the algebra of bounded linear operators acting on the Hilbert space $H$ equipped with the inner product $\langle x, y \rangle$.

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Let $V_q$ be the set of partial isometries $X : \mathbb{C}^q \to H$ such that $X^*X = I_q$.

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$$W_q^s(A) = \{(X^*A_1X, \ldots, X^*A_mX) : X \in V_q\}.$$

So, $(B_1, \ldots, B_m) \in W_q^s(A)$ if and only if there is a unitary $U = [X | \tilde{X}]$ such that

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The $q$-matricial range is useful in the study of operators.

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Let $\mathcal{A}$ be a $C^*$-algebra.

The joint (algebra) $q$-matricial range of $A = (A_1, \ldots, A_m) \in \mathcal{A}^m$ is defined by

\[ W_q(A) = \{ \phi(A) = (\phi(A_1), \ldots, \phi(A_m)) : \phi \text{ is a unital complete positive linear map from } \mathcal{A} \text{ to } M_q \}. \]
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For \( A \in B(H)^m \) it is hard to check whether \( (B_1, \ldots, B_m) \in M_q^m \) lies in \( W^q_s(A) \).
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For $A \in B(H)^m$ it is hard to check whether $(B_1, \ldots, B_m) \in M_q^m$ lies in $W^q_s(A)$. For $W^q(A)$, we have the following.

**Proposition [Li, Paulsen, Poon, 2018]**

Let $A_1, \ldots, A_m \in B(H)$. An $m$-tuple of matrices $(B_1, \ldots, B_m) \in M_q^m$ lies in $W^q(A_1, \ldots, A_m)$ if and only if for any $R_0, R_1, \ldots, R_m \in M_q$,

$$\|I_q \otimes R_0 + B_1 \otimes R_1 + \cdots + B_m \otimes R_m\| \leq \|I_H \otimes R_0 + A_1 \otimes R_1 + \cdots + A_m \otimes R_m\|.$$
Let $K(H)$ be the set of compact operators in $B(H)$, $B(H)/K(H)$ be the Calkin algebra, and $\pi : B(H) \to B(H)/K(H)$ is the canonical surjection.

Then the joint essential $q$-matricial range of $A \in B(H)$ is defined by
\[ \cap \{ W_q(A + K) : K \in K(H) \} , \]
which equals $W_q(\pi(A))$ with $\pi(A) = (\pi(A_1), \ldots, \pi(A_m))$.

Define the essential spatial $q$-matricial range by
\[ W_{q,\text{ess}}(A) = \cap \{ \overline{W}_q(A + K) : K \in K(H) \} . \]
The set $W_q(\pi(A))$ is $C^*$-convex.

That is, for any $B_1, \ldots, B_N \in W_q(\pi(A))$ and $L_1, \ldots, L_N \in M_q$ satisfying
\[ \sum_{j=1}^N L_j^* L_j = I_q, \]
\[ \sum_{j=1}^N L_j^* B_j L_j \in W_q(\pi(A)) . \]

Theorem [Li, Paulsen, Poon, 2018]
Let $A \in B(H)$. Then $W_{q,\text{ess}}(A)$ is $C^*$-convex. Consequently, $W_{q,\text{ess}}(A) = W_q(\pi(A))$. 

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Theorem [Li, Paulsen, Poon, 2018] Let $A \in B(H)^m$. Then $W^q_{\text{ess}}(A)$ is $C^*$-convex. Consequently, $W^q_{\text{ess}}(A) = W^q(\pi(A))$. 

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Define the essential spatial $q$-matricial range by

$$W^q_{ess}A) = \cap\{\text{cl} (W^q_s(A + K)) : K \in K(H)^m\}.$$
Essential matricial range

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Define the essential spatial $q$-matricial range by

$$ W^q_{ess}A = \cap \{ \text{cl} (W^q_s(A + K)) : K \in K(H)^m \}. $$

The set $W^q(\pi(A))$ is $C^*$-convex. That is, for any $B_1, \ldots, B_N \in W^q(\pi(A))$ and $L_1, \ldots, L_N \in M_q$ satisfying $\sum_{j=1}^N L_j^*L_j = I_q$,

$$ \sum_{j=1}^N L_j^*B_jL_j \in W^q(\pi(A)). $$

**Theorem [Li,Paulsen,Poon,2018]**

Let $A \in B(H)^m$. Then $W^q_{ess}(A)$ is $C^*$-convex. Consequently,

$$ W^q_{ess}(A) = W^q(\pi(A)). $$
Preservation problem

Problems and results [Smith and Ward, 1980]

Let $A \in B(H)^m$.

- For a given positive integer $N$, can we find $K \in K(H)^m$ such that
  \[ W^q(A + K) = W^q(\pi(A)) \quad \text{for all } q \in \{1, \ldots, N\} \]
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Theorem [Müller, 2010]

Let $A \in B(H)^m$. There is $K \in K(H)^m$ such that
$$W^{1_{ess}}(A) = \text{cl}(W^{1_{s}}(A + K)) = W^{1_{s}}(A + K) = W^{1_{s}}(\pi(A)).$$
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Theorem [Müller, 2010]

Let $A \in B(H)^m$. There is $K \in K(H)^m$ such that

\[ W^1_{ess}(A) = \text{cl} \left( W^1_s(A + K) \right) = W^1(A + K) = W^1(\pi(A)). \]
Theorem [Li, Paulsen, Poon, 2018]

Let \( \mathbf{A} \in B(H)^m \). Suppose \( N \) is a positive integer. Then there is \( \mathbf{K} \in K(H)^m \) such that for all \( q \in \{1, \ldots, N\} \):

\[
W^q_{\text{ess}}(\mathbf{A}) = W^q(\pi(\mathbf{A})) = \text{cl}(W^q_{\text{ss}}(\mathbf{A} + \mathbf{K})) = W^q(\mathbf{A} + \mathbf{K}).
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Theorem [Li, Paulsen, Poon, 2018]

Let \( A \in S(H)^m \) be an \( m \)-tuple of self-adjoint operators such that \( W_{\text{ess}}(1 : A) \) is a simplex in \( \mathbb{R}^m \), i.e., a polyhedral set with \( m + 1 \) vertices.
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The proof of the first theorem depends on the connection of maps from \( B(H) \) to \( M_q \) and maps from \( B(H)/K(H) \) to \( M_q \).
Results

Theorem [Li, Paulsen, Poon, 2018]
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The proof of the second theorem depends on a result on joint dilation.
Theorem [Binding,Farenick,Li,1995]

Let \( A = (A_1, \ldots, A_m) \in M_q^m \) is an \( m \)-tuple of Hermitian matrices such that \( W(A_1, \ldots, A_m) \) is a simplex in \( \mathbb{R}^m \).

Proposition [Li,Paulsen,Poon, 2018]

Let \( A = (A_1, \ldots, A_m) \in S(H)^m \).

Suppose \( W_1 \text{ess}(A) \) is a subset of a simplex \( S \) in \( \mathbb{R}^m \) with vertices \( v_k = (v_1^k, \ldots, v_m^k) \) for \( k = 1, \ldots, m+1 \).

Then there is \( K = (K_1, \ldots, K_m) \in S(H)^m \cap K(H)^m \) such that for any \( R_0, \ldots, R_m \in M_q \),

\[
\| R_0 \otimes I + R_1 \otimes (A_1 + K_1) + \cdots + R_m \otimes (A_m + K_m) \| \leq \max \{ \| R_0 + v_1^1 R_1 + \cdots + v_m^m R_m \| : 1 \leq k \leq m+1 \}.
\]

(1)

In fact, \( K \) can be chosen such that the equality holds in (1) for any choice of \( R_0, \ldots, R_m \in M_q \).
Theorem [Binding,Farenick,Li,1995]

Let \( A = (A_1, \ldots, A_m) \in M_q^m \) is an \( m \)-tuple of Hermitian matrices such that \( W(A_1, \ldots, A_m) \) is a simplex in \( \mathbb{R}^m \). Then \( B = (B_1, \ldots, B_m) \in S(H)^m \) satisfies \( W(B) \subseteq W(A) \) if and only if there is an partial isometry \( X \) such that \( B_j = X^* (I \otimes A_j) X \) for \( j = 1, \ldots, m \).

Proposition [Li,Paulsen,Poon, 2018]

Let \( A = (A_1, \ldots, A_m) \in S(H)^m \). Suppose \( W_1 \text{ess}(A) \) is a subset of a simplex \( S \) in \( \mathbb{R}^m \) with vertices \( v_k = (v_1^k, \ldots, v_m^k) \) for \( k = 1, \ldots, m+1 \). Then there is \( K = (K_1, \ldots, K_m) \in S(H)^m \cap K(H)^m \) such that for any \( R_0, \ldots, R_m \in M_q \),

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\| R_0 \otimes I + R_1 \otimes (A_1+K_1) + \cdots + R_m \otimes (A_m+K_m) \| \leq \max \{ \| R_0 + v_1^1 R_1 + \cdots + v_i^k R_k + \cdots + v_m^m R_m \| : 1 \leq k \leq m+1 \}.
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### Theorem [Binding, Farenick, Li, 1995]

Let $A = (A_1, \ldots, A_m) \in M_q^m$ be an $m$-tuple of Hermitian matrices such that $W(A_1, \ldots, A_m)$ is a **simplex** in $\mathbb{R}^m$. Then $B = (B_1, \ldots, B_m) \in S(H)^m$ satisfies $W(B) \subseteq W(A)$ if and only if there is a partial isometry $X$ such that

$$B_j = X^*(I \otimes A_j)X \quad \text{for } j = 1, \ldots, m.$$
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\[\text{(1)}\]
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Proposition [Li, Paulsen, Poon, 2018]

Let \( A = (A_1, \ldots, A_m) \in S(H)^m \). Suppose \( W_{ess}^1(A) \) is a subset of a simplex \( S \) in \( \mathbb{R}^m \) with vertices \( v_k = (v_{1k}, \ldots, v_{mk}) \) for \( k = 1, \ldots, m + 1 \).
Related results

**Theorem [Binding,Farenick,Li,1995]**

Let $\mathbf{A} = (A_1, \ldots, A_m) \in M_q^m$ is an $m$-tuple of Hermitian matrices such that $W(A_1, \ldots, A_m)$ is a simplex in $\mathbb{R}^m$. Then $\mathbf{B} = (B_1, \ldots, B_m) \in S(H)^m$ satisfies $W(\mathbf{B}) \subseteq W(\mathbf{A})$ if and only if there is an partial isometry $X$ such that

$$B_j = X^*(I \otimes A_j)X \quad \text{for} \ j = 1, \ldots, m.$$ 

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### Related results

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\[
\| R_0 \otimes I + R_1 \otimes (A_1 + K_1) + \cdots + R_m \otimes (A_m + K_m) \| \\
\leq \max\{\| R_0 + v_{1k} R_1 + \cdots + v_{mk} R_m \| : 1 \leq k \leq m + 1 \}.
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In fact, $K$ can be chosen such that the equality holds in (1) for any choice of $R_0, \ldots, R_m \in M_q$. 

Chi-Kwong Li, College of William & Mary  
Preservation of essential matricial range
Define the \((p, q)\)-numerical range \(\Lambda_{p,q}(A)\) of \(A = (A_1, \ldots, A_m)\) to be the set of \((B_1, \ldots, B_m) \in M_q^m\) for the existence of \(X \in \mathcal{V}_{pq}\) such that

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When \(q = 1\), we get the rank \(p\)-numerical range \(\Lambda_p(A)\) introduced in [Choi, Kribs, Zyczkowski, 2006] for the study of quantum error correction.
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The \((p, q)\)-numerical range is related to the work of [Kribs, Spekkens, 2016] concerning the quantum error correctable subsystems.
Theorem [Li,Paulsen,Poon,2018]

Let $A \in S(H)^m$.

(1) For any positive integer $N$, there is $K \in S_K(H)^m$ such that for every $q \in \{1, \ldots, N\}$,
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The proof depends on some recent results by [Lau,Li,Poon,Sze,2018].
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(2) If $W_{ess}(A)$ is a simplex in $\mathbb{R}^m$, then there is $K \in S_{K}(\mathcal{H})^m$ such that for every positive integer $q$,?
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Suppose $A = (A_1, \ldots, A_m) \in S(H)^m$ with $m \geq 4$. Determine whether there is $K = (K_1, \ldots, K_m) \in \mathcal{K}(H)^m \cup S(H)^m$ such that

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for every positive integer $q$. For $m = 1$, yes. For $m \geq 4$, No. By a result of Paulsen. How about $m = 2, 3$? Any comments and suggestions are welcomed! Thank you for your attention!
Further question

Suppose $A = (A_1, \ldots, A_m) \in S(H)^m$ with $m \geq 4$. Determine whether there is $K = (K_1, \ldots, K_m) \in K(H)^m \cup S(H)^m$ such that

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Thank you for supporting WONRA!

Danke!

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