Preservation of essential matricial range

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Joint work with

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D.R. Farenick, Matrical extensions of the numerical range: a brief survey, Linear Multilinear Algebra 34 (1993), 197-211.

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Proposition [Li,Paulsen,Poon, 2018]

Let $A_1, \ldots, A_m \in B(H)$. An m-tuple of matrices $(B_1, \ldots, B_m) \in M_q^m$ lies in $W^q(A_1, \ldots, A_m)$ if and only if for any $R_0, R_1, \ldots, R_m \in M_q$,

$$\|I_q \otimes R_0 + B_1 \otimes R_1 + \cdots + B_m \otimes R_m\|$$

$$\leq \|I_H \otimes R_0 + A_1 \otimes R_1 + \cdots + A_m \otimes R_m\|.$$

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The set $W^q(\pi(\mathbf{A}))$ is C^*-convex. That is, for any $\mathbf{B}_1, \dots, \mathbf{B}_N \in W^q(\pi(\mathbf{A}))$ and $L_1, \dots, L_N \in M_q$ satisfying $\sum_{j=1}^N L_j^* L_j = I_q$,

$$\sum_{j=1}^N L_j^* \mathbf{B}_j L_j \in W^q(\pi(\mathbf{A})).$$

Theorem [Li,Paulsen,Poon,2018]

Let $\mathbf{A} \in B(H)^m$. Then $W^q_{ess}(\mathbf{A})$ is C^*-convex. Consequently,

$$W^q_{ess}(\mathbf{A}) = W^q(\pi(\mathbf{A})).$$

Preservation problem

Problems and results [Smith and Ward, 1980]

Let $\mathbf{A} \in B(H)^m$.

- For a given positive integer N, can we find $\mathbf{K} \in K(H)^m$ such that

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Theorem [Müller, 2010]

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Theorem [Li, Paulsen, Poon, 2018]

Let $\mathbf{A} \in B(H)^m$. Suppose N is a positive integer. Then there is $\mathbf{K} \in K(H)^m$ such that for all $q \in \{1, \ldots, N\}$:

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Theorem [Li, Paulsen, Poon, 2018]

Let $\mathbf{A} \in S(H)^m$ be an m-tuple of self-adjoint operators such that $W_{ess}(1 : \mathbf{A})$ is a simplex in $\mathbb{R}^m$, i.e., a polyhedral set with $m+1$ vertices.

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The proof of the second theorem depends on a result on joint dilation.

Related results

Theorem [Binding,Farenick,Li,1995]

Let $\mathbf{A} = (A_1, \ldots, A_m) \in M_q^m$ is an m-tuple of Hermitian matrices such that $W(A_1, \ldots, A_m)$ is a simplex in $\mathbb{R}^m$.

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$$\|R_0 \otimes I + R_1 \otimes (A_1 + K_1) + \cdots + R_m \otimes (A_m + K_m)\|$$

$$\leq \max\{\|R_0 + v_{1k}R_1 + \cdots + v_{mk}R_m\| : 1 \leq k \leq m+1\}. \qquad (1)$$

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Let $\mathbf{A} = (A_1, \ldots, A_m) \in S(H)^m$. Suppose $W_{ess}^1(\mathbf{A})$ is a subset of a simplex $\mathcal{S}$ in $\mathbb{R}^m$ with vertices $v_k = (v_{1k}, \ldots, v_{m,k})$ for $k = 1, \ldots, m+1$. Then there is $\mathbf{K} = (K_1, \ldots, K_m) \in S(H)^m \cap K(H)^m$ such that for any $R_0, \ldots, R_m \in M_q$,

$$\|R_0 \otimes I + R_1 \otimes (A_1 + K_1) + \cdots + R_m \otimes (A_m + K_m)\|$$

$$\leq \max\{\|R_0 + v_{1k}R_1 + \cdots + v_{mk}R_m\| : 1 \leq k \leq m+1\}. \qquad (1)$$

In fact, $\mathbf{K}$ can be chosen such that the equality holds in (1) for any choice of $R_0, \ldots, R_m \in M_q$.

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Define the (p, q)-numerical range $\Lambda_{p,q}(\mathbf{A})$ of $\mathbf{A} = (A_1, \ldots, A_m)$ to be the set of $(B_1, \ldots, B_m) \in M_q^m$ for the existence of $X \in \mathcal{V}_{pq}$ such that

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The (p, q)-numerical range is related to the work of [Kribs, Spekkens, 2016] concerning the quantum error correctable subsystems.

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Let $\mathbf{A} \in S(H)^m$.

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The proof depends on some recent results by [Lau,Li,Poon,Sze,2018].

Further question

Suppose $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{S}(H)^m$ with $m \geq 4$. Determine whether there is $\mathbf{K} = (K_1, \ldots, K_m) \in \mathcal{K}(H)^m \cup \mathcal{S}(H)^m$ such that

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Thank you for your attention!

Thank you for supporting WONRA!

Danke!