Spectral sets : numerical range and beyond

Michel Crouzeix

Université de Rennes 1

WONRA

Munich, June 2018

Collaboration with Anne Greenbaum

Notations

$u, v \in H$, complex Hilbert space, $\quad A \in B(H)$, linear operator,

$\langle u, v \rangle$, $\|u\| = \langle u, u \rangle^{\frac{1}{2}}$, inner product and corresponding norm,

$\|A\| = \max\{\|Au\| \,;\, u \in H, \|u\| = 1\}$, induced norm,

$W(A) = \{\langle Av, v \rangle \,;\, v \in H, \|v\| = 1\}$, $\quad$ numerical range of A.

$W(A)$ is a convex subset of $\mathbb{C}$ (Toeplitz-Hausdorff),

$\overline{W(A)} \supset \mathrm{Sp}(A)$, spectrum of A.

Notations If you prefer, you can stay in finite dimension, $H = \mathbb{C}^d$,

$u, v \in \mathbb{C}^d$, $\quad A \in \mathbb{C}^{d,d}$, $d \times d$ matrix,

$\langle u, v\rangle = v^* u$, $\|u\| = \langle u, u\rangle^{\frac{1}{2}}$, inner product and corresponding norm,

$\|A\| = \max\{\|Au\|\,; u \in H, \|u\| = 1\}$, spectral norm,

$W(A) = \{\langle Av, v\rangle\,; v \in H, \|v\| = 1\}$, numerical range of A.

$W(A)$ is a convex subset of $\mathbb{C}$ (Toeplitz-Hausdorff),

$\overline{W(A)} \supset \mathrm{Sp}(A)$, spectrum of A.

The origins : von Neumann (1951)

A closed subset $X \subset \mathbb{C}$ is a spectral set for an operator A if $\mathrm{Sp}(A) \subset X$ and if, for all rational functions f bounded in X,

$$\|f(A)\| \leq \sup_{z \in X} |f(z)|.$$

The origins : von Neumann (1951)

A closed subset $X \subset \mathbb{C}$ is a K-spectral set for an operator A if $\mathrm{Sp}(A) \subset X$ and if, for all rational functions f bounded in X,

$$\|f(A)\| \leq K \sup_{z \in X} |f(z)|.$$

The origins : von Neumann (1951)

A closed subset $X \subset \mathbb{C}$ is a K-spectral set for an operator A if $\mathrm{Sp}(A) \subset X$ and if, for all rational functions f bounded in X,

$$\|f(A)\| \leq K \sup_{z \in X} |f(z)|.$$

The interest : let us consider the commutative algebra $\mathcal{A}(X) = \{f$: uniform limits in X of bounded rational functions $r_n\}$. If $f \in \mathcal{A}(X)$, we can define $f(A) = \lim_{n\to\infty} r_n(A)$. Then, the map $f \mapsto f(A)$ is a bounded homomorphism from the algebra $\mathcal{A}(X)$ into the algebra $\mathcal{L}(H)$; in other words, we have a functional calculus.

The origins : von Neumann (1951)

A closed subset $X \subset \mathbb{C}$ is a K-spectral set for an operator A if $\mathrm{Sp}(A) \subset X$ and if, for all rational functions f bounded in X,

$$\|f(A)\| \leq K \sup_{z \in X} |f(z)|. \qquad (*)$$

The interest : If f is a uniform limit in X of rational functions r_n, we can define $f(A) = \lim_{n \to \infty} r_n(A)$.

We have

$(f+g)(A) = f(A)+g(A)$, $(fg)(A) = f(A)g(A)$, $(\lambda f)(A) = \lambda f(A)\ldots$

We can calculate with these functions of one operator almost as with ordinary functions. For instance $(\cos A)^2 + (\sin A)^2 = I$.

The estimate $(*)$ may be very useful for justifying some limit or some derivation.

The origins : von Neumann (1951)

Examples : If $A = A^*$, the spectrum of A is a spectral set for A.

If $\|A\| \leq 1$, the unit disk $\{|z| \leq 1\}$ is a spectral set for A.

If $\mathrm{Re}\langle Av, v\rangle \geq 0\ \forall v$, the half-plane $\{\mathrm{Re}\, z \geq 0\}$ is a spectral set for A.

If a bounded open domain Ω contains the spectrum $\mathrm{Sp}(A)$, then Ω is a K-spectral set for A, with

$$K = \frac{1}{2\pi}\int_{\partial\Omega} \|(\sigma I - A)^{-1}\|\, |d\sigma|.$$

The origins : Bernard and François Delyon (1999)

Let Ω be a bounded convex subset of $\mathbb{C}$ containing $W(A)$. Then Ω is a K-spectral set for A, with

$$K \leq 3 + \left(\frac{2\pi D^2}{|\Omega|}\right)^3, \quad D \text{ diameter and } |\Omega| \text{ area of } \Omega.$$

The origins : Bernard and François Delyon (1999)

Let Ω be a bounded convex subset of $\mathbb{C}$ containing $W(A)$. Then Ω is a K-spectral set for A, with

$$K \le 3 + \left(\frac{2\pi D^2}{|\Omega|}\right)^3, \quad D \text{ diameter and } |\Omega| \text{ area of } \Omega.$$

A priori, it was not clear that K would be finite. This result has allowed them to prove a conjecture of Burkholder.

The origins : Bernard and François Delyon (1999)

Let Ω be a bounded convex subset of $\mathbb{C}$ containing $W(A)$. Then Ω is a K-spectral set for A, with

$$K \leq 3 + \left(\frac{2\pi D^2}{|\Omega|}\right)^3, \quad D \text{ diameter and } |\Omega| \text{ area of } \Omega.$$

A priori, it was not clear that K would be finite. This result has allowed them to prove a conjecture of Burkholder.

I have been very interested by their result. I have worked with Bernard, Catalin Badea, Bernd Beckermann ; we have improved the estimate in many situations. In the case of 2×2 matrices, I have succeeded to show that $K \leq 2$, and that $K = 2$ implies Ω is a disk.

The origins : Bernard and François Delyon (1999)

Let Ω be a bounded convex subset of $\mathbb{C}$ containing $W(A)$. Then Ω is a K-spectral set for A, with

$$K \leq 3 + \left(\frac{2\pi D^2}{|\Omega|}\right)^3, \quad D \text{ diameter and } |\Omega| \text{ area of } \Omega.$$

These results, and some numerical simulations, plus some symmetry reasons, have lead me to the conjecture (2004) that, in a Hilbert space setting,

The numerical range is a 2-spectral set for all operators.

In other words, Delyon's result holds with $K \leq 2$.

In 2007, I have succeeded to show their result with $K \leq 11.08$

In September 2016, a new approach due to César Palencia, has improved this bound to $K \leq 1+\sqrt{2}$.

Let Ω be a bounded open convex domain of $\mathbb{C}$. Clearly the best constant K for Ω and A is given by

$$C(\Omega, A) := \sup\{\|f(A)\| \ : f \text{ rational function}, |f| \leq 1 \text{ in } \Omega\}.$$

A first bound. If Ω contains the spectrum $\mathrm{Sp}(A)$, then

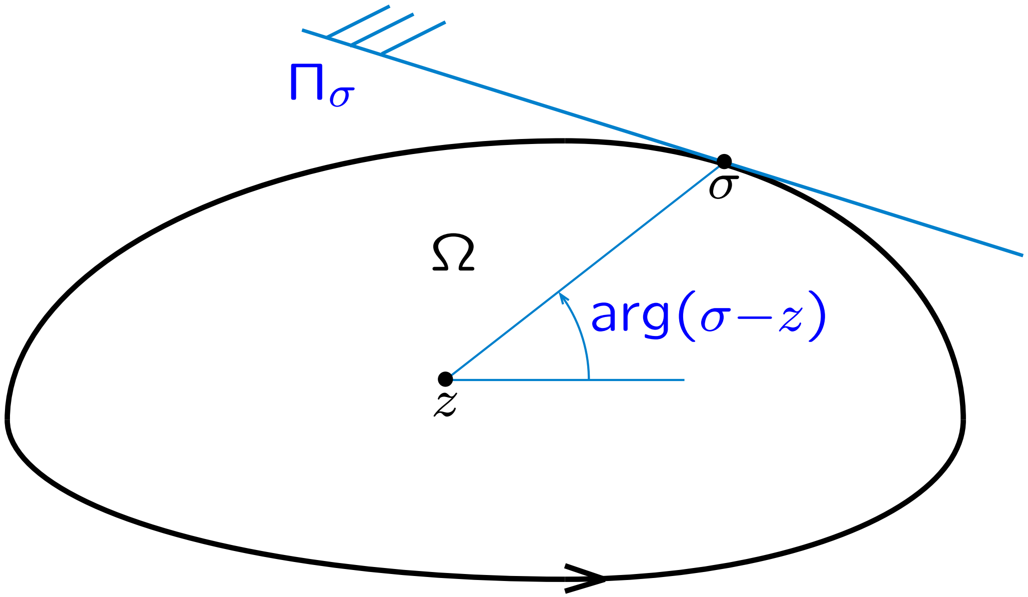
$$C(\Omega, A) \leq \frac{1}{2\pi} \int_{\partial\Omega} \|(\sigma I - A)^{-1}\| \, |d\sigma| < \infty.$$

The Palencia bound : an important tool.

The double layer potential kernel $\mu(\sigma, z)$.

s arclength of $\sigma = \sigma(s) \in \partial\Omega$, counterclockwise orientation of $\partial\Omega$

$\mu(\sigma, z) := \frac{1}{\pi}\frac{d}{ds}\arg(\sigma - z) = \frac{1}{2\pi i}\left(\frac{d\sigma/ds}{\sigma - z} - \frac{d\bar{\sigma}/ds}{\bar{\sigma} - \bar{z}}\right)$, double layer kernel.



$z \in \overline{\Omega}$ is equivalent to $\mu(\sigma, z) \geq 0, \forall \sigma \in \partial\Omega$.

We have $\mu(\sigma, z) = \frac{1}{\pi}\frac{d}{ds}\,\mathrm{arg}(\sigma - z) = \frac{1}{2\pi i}\Big(\frac{d\sigma/ds}{\sigma - z} - \frac{d\bar{\sigma}/ds}{\bar{\sigma} - \bar{z}}\Big)$ and

$$z \in \overline{\Omega} \quad \Longleftrightarrow \quad \mu(\sigma, z) \geq 0,\ \forall \sigma \in \partial\Omega \quad \Longleftrightarrow \quad \mathrm{Im}\,\Big(\sigma'(\bar{\sigma} - \bar{z})\Big) \geq 0,\ \forall \sigma \in \partial\Omega.$$

Similarly, we consider the self-adjoint operator

$$\mu(\sigma, A) := \frac{1}{2\pi i}\Big(\sigma'(\sigma I - A)^{-1} - \overline{\sigma'}(\bar{\sigma} I - A^*)^{-1}\Big).$$

We have $\mu(\sigma,z) = \frac{1}{\pi}\frac{d}{ds}\arg(\sigma-z) = \frac{1}{2\pi i}\Big(\frac{d\sigma/ds}{\sigma-z} - \frac{d\bar{\sigma}/ds}{\bar{\sigma}-\bar{z}}\Big)$ and

$$z \in \overline{\Omega} \quad \Longleftrightarrow \quad \mu(\sigma,z) \geq 0,\ \forall\sigma\in\partial\Omega \quad \Longleftrightarrow \quad \text{Im}\,\Big(\sigma'(\bar{\sigma}-\bar{z})\Big) \geq 0,\ \forall\sigma\in\partial\Omega.$$

Similarly, we consider the self-adjoint operator

$$\mu(\sigma,A) := \frac{1}{2\pi i}\Big(\sigma'(\sigma I-A)^{-1} - \overline{\sigma'}(\bar{\sigma}I-A^*)^{-1}\Big).$$

For $u \in H$ with $\|u\| = 1$, we set $z = \langle Au,u\rangle$ and $v = (\sigma I-A)u$. Then

$$\pi\langle\mu(\sigma,A)v,v\rangle = \text{Im}\,\Big(\sigma'\langle u,(\sigma I-A)u\rangle\Big) = \text{Im}\,\Big(\sigma'(\bar{\sigma}-\bar{z})\Big).$$

From this relation, we deduce

$$W(A)\subset\Omega \quad \Longrightarrow \quad \langle\mu(\sigma,A)v,v\rangle \geq 0, \forall v\in H \ \text{ and } \forall\sigma\in\partial\Omega.$$

This relation, will be also very important in our proofs.

$$W(A) \subset \Omega \quad \Longrightarrow \quad \langle \mu(\sigma, A)v, v\rangle \geq 0, \forall v \in H \text{ and } \forall \sigma \in \partial\Omega.$$

The other tools.

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} f(\sigma)\,(\sigma - z)^{-1}\,d\sigma, \quad \text{Cauchy formula},$$

$$g(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \overline{f(\sigma)}\,(\sigma - z)^{-1}\,d\sigma, \quad \text{Cauchy transform of } \overline{f},$$

$$s(z) = s(f, z) = \int_{\partial\Omega} f(\sigma)\,\mu(\sigma, z)\,ds, \quad \text{double layer transform}.$$

The other tools.

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} f(\sigma)\,(\sigma - z)^{-1}\,d\sigma, \qquad \text{Cauchy formula},$$

$$g(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \overline{f(\sigma)}\,(\sigma - z)^{-1}\,d\sigma, \qquad \text{Cauchy transform of } \overline{f},$$

$$s(z) = s(f, z) = \int_{\partial\Omega} f(\sigma)\,\mu(\sigma, z)\,ds, \qquad \text{double layer transform.}$$

Remark.

$$s(z) = f(z) + \overline{g(z)}.$$

Similarly, we define.

$$f(A) = \frac{1}{2\pi i} \int_{\partial\Omega} f(\sigma)\,(\sigma I - A)^{-1}\,d\sigma,$$

$$g(A) = \frac{1}{2\pi i} \int_{\partial\Omega} \overline{f(\sigma)}\,(\sigma I - A)^{-1}\,d\sigma,$$

$$S(A) = S(f, A) = \int_{\partial\Omega} f(\sigma)\,\mu(\sigma, A)\,ds.$$

It holds.

$$S(A) = f(A) + g(A)^*.$$

Remark. If $f = 1$ is constant, then $g = 1$ and $s = f + \overline{g} = 2$, $f(A) = g(A) = I$, $S(1, A) = \int_{\partial\Omega} \mu(\sigma, A)\,ds = 2\,I$.

The mathematical keys.

Theorem 1. If the rational function f satisfies $|f| \leq 1$ in Ω

$$g(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \overline{f(\sigma)}\,(\sigma - z)^{-1}\, d\sigma,$$

then g is holomorphic in Ω, admits a continuous extension to $\overline{\Omega}$, and satisfies $|g| \leq 1$ in Ω.

The mathematical keys.

Theorem 1. If the rational function f satisfies $|f| \leq 1$ in Ω

$$g(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \overline{f(\sigma)}\,(\sigma - z)^{-1}\,d\sigma,$$

then g is holomorphic in Ω, admits a continuous extension to $\overline{\Omega}$, and satisfies $|g| \leq 1$ in Ω.

Theorem 2. If furthermore $W(A) \subset \Omega$, then $\|S(A)\| \leq 2$.

Proof of $C(\Omega, A) \leq 1+\sqrt{2}$, (simplified by Felix Schwenninger)

Recall that $f(A) + g(A)^* = S(A)$; thus

$$\begin{aligned}(f(A)^* f(A))^2 &= f(A)^* f(A) S(A)^* f(A) - f(A)^* f(A) g(A) f(A) \\ &= f(A)^* f(A) S(A)^* f(A) - f(A)^* (fgf)(A).\end{aligned}$$

Proof of $C(\Omega, A) \leq 1+\sqrt{2}$.

Recall that $f(A) + g(A)^* = S(A)$; thus

$$\begin{aligned}(f(A)^*f(A))^2 &= f(A)^*f(A)S(A)^*f(A) - f(A)^*f(A)g(A)f(A)\\ &= f(A)^*f(A)S(A)^*f(A) - f(A)^*(fgf)(A).\end{aligned}$$

It holds $\|f(A)\| \leq C(\Omega)$; furthermore fgf is holomorphic in Ω and continuous up to the boundary, thus is a uniform limit of rational functions (Mergelyan theorem) ; also it holds $|fgf| \leq 1$ whence $\|(fgf)(A)\| \leq C(\Omega)$.

Proof of $C(\Omega, A) \leq 1+\sqrt{2}$.

Recall that $f(A) + g(A)^* = S(A)$; thus

$$\begin{aligned}(f(A)^* f(A))^2 &= f(A)^* f(A) S(A)^* f(A) - f(A)^* f(A) g(A) f(A) \\ &= f(A)^* f(A) S(A)^* f(A) - f(A)^* (fgf)(A).\end{aligned}$$

It holds $\|f(A)\| \leq C(\Omega)$; furthermore fgf is holomorphic in Ω and continuous up to the boundary, thus is a uniform limit of rational functions (Mergelyan theorem) ; also it holds $|fgf| \leq 1$ whence $\|(fgf)(A)\| \leq C(\Omega)$. With $\|S(A)\| \leq 2$ this shows that

$$\|f(A)\|^4 \leq 2\, C(\Omega)^3 + C(\Omega)^2.$$

Proof of $C(\Omega, A) \leq 1+\sqrt{2}$.

Recall that $f(A) + g(A)^* = S(A)$; thus

$$(f(A)^*f(A))^2 = f(A)^*f(A)S(A)^*f(A) - f(A)^*f(A)g(A)f(A)$$
$$= f(A)^*f(A)S(A)^*f(A) - f(A)^*(fgf)(A).$$

It holds $\|f(A)\| \leq C(\Omega)$; furthermore fgf is holomorphic in Ω and continuous up to the boundary, thus is a uniform limit of rational functions (Mergelyan theorem) ; also it holds $|fgf| \leq 1$ whence $\|(fgf)(A)\| \leq C(\Omega)$. With $\|S(A)\| \leq 2$ this shows that

$$\|f(A)\|^4 \leq 2\,C(\Omega)^3 + C(\Omega)^2.$$

Using that $C(\Omega) = \sup\{\|f(A)\| : |f| \leq 1\}$ we obtain

$$C(\Omega)^4 \leq 2\,C(\Omega)^3 + C(\Omega)^2,$$

whence $C(\Omega) \leq 1+\sqrt{2}$.

Theorem 1 bis. Assume $\partial\Omega$ of class C^2 and $|f| \leq 1$ in Ω, then g is holomorphic in Ω, admits a continuous extension to $\overline{\Omega}$, and satisfies $|g| \leq c_1(\Omega) \leq \max_{\sigma_0 \in \partial\Omega} \int_{\partial\Omega} |\mu(\sigma(s), \sigma_0)| \, ds$ in Ω.

Theorem 1 bis. Assume $\partial\Omega$ of class C^2 and $|f| \le 1$ in Ω, then g is holomorphic in Ω, admits a continuous extension to $\overline{\Omega}$, and satisfies $|g| \le c_1(\Omega) \le \max_{\sigma_0 \in \partial\Omega} \int_{\partial\Omega} |\mu(\sigma(s), \sigma_0)| \, ds$ in Ω.

Proof. Recall that $s(z) = f(z) + \overline{g(z)}$, thus

$$g(z) = \int_{\partial\Omega} \overline{f(\sigma(s))} \mu(\sigma(s), z) \, ds - \overline{f(z)}, \quad \text{if } z \in \Omega.$$

For $\sigma_0 \in \partial\Omega$, we extend g using the jump formula

$$g(\sigma_0) = \int_{\partial\Omega} \overline{f(\sigma(s))} \mu(\sigma(s), \sigma_0) \, ds,$$

The continuity has been proved by Carl Neumann ; the bound is clear on the boundary, and holds in the interior by the maximum principle.

Theorem 1 bis. Assume $\partial\Omega$ of class C^2 and $|f| \leq 1$ in Ω, then g is holomorphic in Ω, admits a continuous extension to $\overline{\Omega}$, and satisfies $|g| \leq c_1(\Omega) \leq \max_{\sigma_0 \in \partial\Omega} \int_{\partial\Omega} |\mu(\sigma(s), \sigma_0)| \, ds$ in Ω.

Proof. Recall that $s(z) = f(z) + \overline{g(z)}$, thus

$$g(z) = \int_{\partial\Omega} \overline{f(\sigma(s))} \mu(\sigma(s), z) \, ds - \overline{f(z)}, \quad \text{if } z \in \Omega.$$

For $\sigma_0 \in \partial\Omega$, we extend g using the jump formula

$$g(\sigma_0) = \int_{\partial\Omega} \overline{f(\sigma(s))} \mu(\sigma(s), \sigma_0) \, ds,$$

The continuity has been proved by Carl Neumann ; the bound is clear on the boundary, and holds in the interior by the maximum principle.

If Ω is convex, then $\mu(\sigma, \sigma_0) \geq 0$, thus $|g(\sigma_0)| \leq \int_{\partial\Omega} \mu(\sigma, \sigma_0) \, ds = 1$.

Remark. The constant $c_1(\Omega)$ may be less than $\max_{\sigma_0 \in \partial\Omega} \int_{\partial\Omega} |\mu(\sigma(s), \sigma_0)| \, ds$. For instance, if $\Omega = \{z : r < |z| < R\}$ is an annulus, it can be proved that $c_1(\Omega) = 1$ while $\max_{\sigma_0 \in \partial\Omega} \int_{\partial\Omega} |\mu(\sigma(s), \sigma_0)| \, ds = 3$.

We consider $\lambda_m(s) = \min\{\lambda : \lambda \text{ eigenvalue of } \mu(\sigma(s), A)\}$.

Theorem 2 bis. Assume that $\mathrm{Sp}(A) \subset \Omega$ and $|f| \leq 1$ in Ω then, with $\alpha(s) = \min(0, -\lambda_{min}(s))$,

$$\|S(f, A)\| \leq 2\, c_2(A, \Omega) \leq 2\Big(1 + \int_{\partial\Omega} \alpha(s))\, ds\Big).$$

We consider $\lambda_m(s) = \min\{\lambda : \lambda \text{ eigenvalue of } \mu(\sigma(s), A)\}$.

Theorem 2 bis. Assume that $\mathrm{Sp}(A) \subset \Omega$ and $|f| \leq 1$ in Ω then, with $\alpha(s) = \min(0, -\lambda_{min}(s))$,

$$\|S(f, A)\| \leq 2\, c_2(A, \Omega) \leq 2\Big(1 + \int_{\partial\Omega} \alpha(s)\, ds\Big).$$

Proof. It holds

$$S(A) = \int_{\partial\Omega} f(\sigma(s))\Big(\mu(\sigma(s), A) + \alpha(s)\Big)\, ds - \int_{\partial\Omega} f(\sigma(s))\alpha(s)\, ds$$

Note that $\mu(\sigma(s), A) + \alpha(s) \geq 0$, thus

$$\|S(A)\| \leq \Big\| \int_{\partial\Omega} (\mu(\sigma(s), A) + \alpha(s)\Big)\, ds\Big\| + \int_{\partial\Omega} \alpha(s)\, ds.$$

Which gives the result since $\int_{\partial\Omega} \mu(\sigma(s), A)\, ds = 2I$.

We consider $\lambda_m(s) = \min\{\lambda : \lambda \text{ eigenvalue of } \mu(\sigma(s), A)\}$.

Theorem 2 bis. Assume that $\mathrm{Sp}(A) \subset \Omega$ and $|f| \leq 1$ in Ω then, with $\alpha(s) = \min(0, -\lambda_{min}(s))$,

$$\|S(f, A)\| \leq 2\, c_2(A, \Omega) \leq 2\Big(1 + \int_{\partial\Omega} \alpha(s)\, ds\Big).$$

Proof. It holds

$$S(A) = \int_{\partial\Omega} f(\sigma(s))\Big(\mu(\sigma(s), A) + \alpha(s)\Big)\, ds - \int_{\partial\Omega} f(\sigma(s))\alpha(s)\, ds$$

Note that $\mu(\sigma(s), A) + \alpha(s) \geq 0$, thus

$$\|S(A)\| \leq \Big\| \int_{\partial\Omega} (\mu(\sigma(s), A) + \alpha(s)\Big)\, ds \Big\| + \int_{\partial\Omega} \alpha(s)\, ds.$$

If $W(A) \subset \Omega$ and if Ω is convex, then $\mu(\sigma, A) \geq 0$, thus $\alpha(s) = 0$, whence $\|S(A)\| \leq 2$.

Remark. The best constant $c_2(\Omega)$ may be less than $1+\int_{\partial\Omega} \alpha(s)\, ds$. For instance, if $\Omega = \{z : r < |z| < R\}$, if $\|A\| \leq R$, and if $\|A^{-1}\| \leq 1/r$, then $c_2 = 1$ while $1 + \int_{\partial\Omega} \alpha(s)\, ds = 2$.

Theorem. If, for all rational functions f bounded by 1 in Ω it holds $|g| \leq c_1$ and $\|S(f, A)\| \leq 2c_2$, then

$$C(\Omega) \leq c_2 + \sqrt{c_2^2 + c_1}$$

Theorem. If, for all rational functions f bounded by 1 in Ω it holds $|g| \leq c_1$ and $\|S(f, A)\| \leq 2c_2$, then

$$C(\Omega) \leq c_2 + \sqrt{c_2^2 + c_1}$$

Proof.

Recall that

$$(f(A)^* f(A))^2 = f(A)^* f(A) S(A)^* f(A) - f(A)^* (fgf)(A).$$

Thus

$$\|f(A)\|^4 \leq 2\, c_2 C(\Omega)^3 + c_1 C(\Omega)^2.$$

Using that $C(\Omega) = \sup\{\|f(A)\| : |f| \leq 1\}$ we obtain

$$C(\Omega)^4 \leq 2\, c_2 C(\Omega)^3 + c_1 C(\Omega)^2,$$

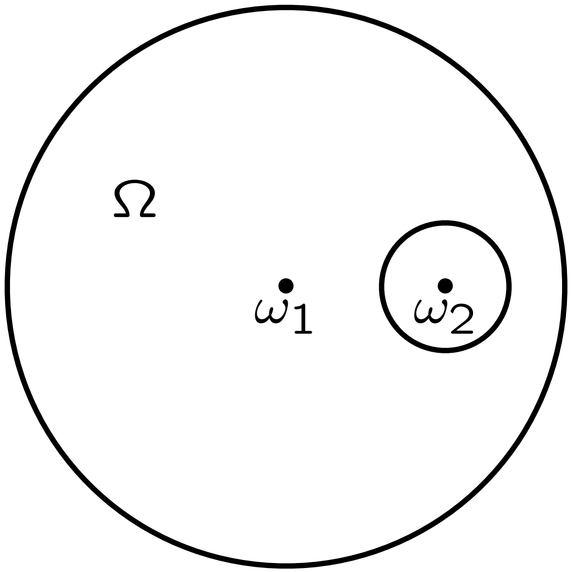
whence $C(\Omega) \leq c_2 + \sqrt{c_2^2 + c_1}$.

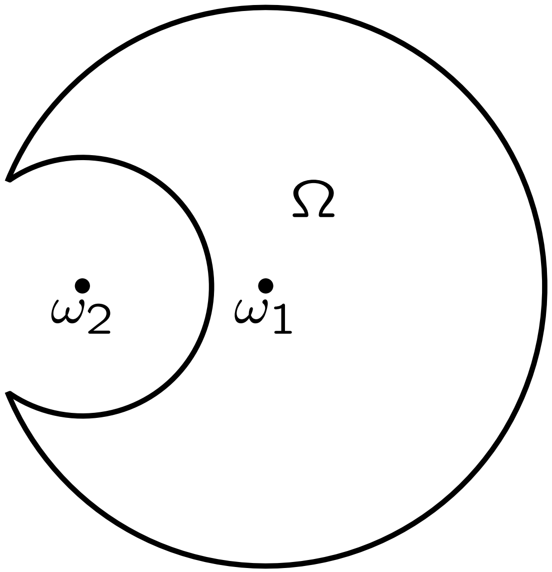
Example 1. With the assumptions

$$\Omega = \{z \in \mathbb{C} : |z - \omega_1| < R,\ |z - \omega_2| > r\},$$

$$\|A - \omega_1 I\| \le R, \quad \|(A - \omega_2 I)^{-1}\| \le 1/r,$$

We can show $C(\Omega, A) \le 1 + \sqrt{2}$.



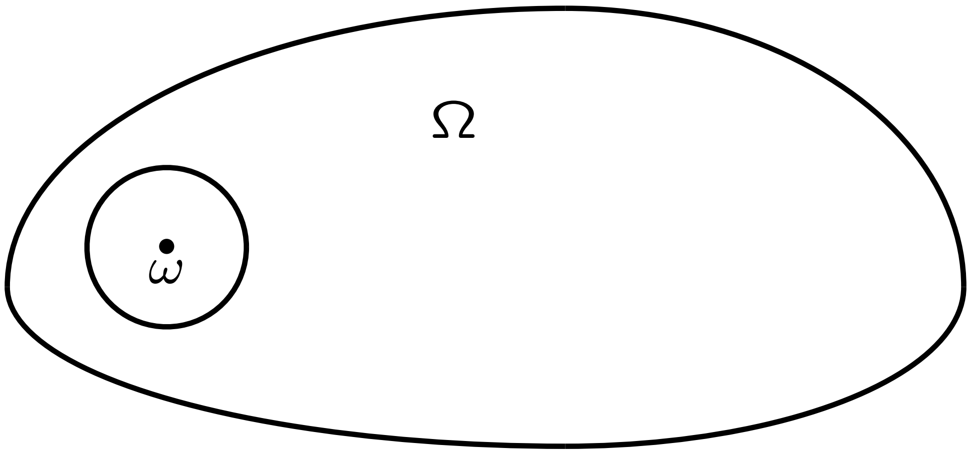


Example 2. With the assumptions Ω_1 convex

$$\Omega = \{z \in \Omega_1 \,:\; |z-\omega| > r\}, \qquad \{z \,:\; |z-\omega| \le r\} \subset \Omega_1,$$

$$W(A) \subset \Omega_1, \quad w((A-\omega_2 I)^{-1}) \le 1/r,$$

We can show $C(\Omega, A) \le 3 + 2\sqrt{3}$.



Thank you for your attention!