Faces of sets of operators with numerical range in a prescribed polyhedron

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$\mathscr{H}$ complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$

$\mathscr{S}_{\mathscr{H}} = \{x \in \mathscr{H}; \quad \|x\| = 1\}$ the unit sphere

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The numerical range of $A \in \mathcal{B}(\mathscr{H})$ is

$$W(A) = \{\langle Ax, x\rangle; \quad x \in \mathscr{S}_{\mathscr{H}}\}.$$

Definition and properties

$K \subseteq \mathbb{C}$ non-empty set

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- $K_1 \subseteq K_2 \implies \mathscr{W}^{K_1} \subseteq \mathscr{W}^{K_2}$.

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- Let $K \subseteq \mathbb{C}$ be a non-empty closed convex set. Then

$$rb(\mathscr{W}^K) = \{A \in \mathscr{W}^K : \quad \overline{W(A)} \cap rb(K) \neq \emptyset\}.$$

Affine transformation

- $\theta: \mathbb{C} \to \mathbb{C}$ invertible affine transformation

$$\theta(u+iv) = au + bv + e + i(cu + dv + f)$$

$$a, b, c, d, e, f \in \mathbb{R}, \quad ad - bc \neq 0.$$

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It follows: $\mathscr{W}^{\theta(K)} = \Theta(\mathscr{W}^K)$.

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Faces

A convex subset $\mathscr{F} \subseteq \mathscr{W}^K$ is a face if

$$A \in \mathscr{F}, \quad B, C \in \mathscr{W}^K, \quad 0 < t < 1 : \quad A = tB + (1-t)C$$

$$\Rightarrow \quad B, C \in \mathscr{F}.$$

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$$rb(\mathscr{W}^K) = \{A \in \mathscr{W}^K; \quad \overline{W(A)} \cap rb(K) \neq \emptyset\};$$

- $A \in \mathscr{W}^K$ is extreme point if $\{A\}$ is a face.

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- A extreme point of $\mathscr{W}^K \Leftrightarrow \Theta(A)$ extreme point of $\mathscr{W}^{\theta(K)}$.

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$\Longrightarrow \mathscr{W}^K$ is sot-closed and convex.

What are faces of $\mathscr{W}^K$?

- $\dim(\mathscr{H}) < \infty$

  $K = \bigcap_{j=1}^{n} K_j$, where K_j are closed convex sets on $\mathbb{C}$

$\mathscr{F}$ face of $\bigcap_{j=1}^{n} \mathscr{W}^{K_j} \Longleftrightarrow \exists \mathscr{F}_j$ faces of $\mathscr{W}^{K_j}$, $\mathscr{F} = \bigcap_{j=1}^{n} \mathscr{F}_j$.

Faces of $\mathscr{W}^K$ for a polyhedron set K

- $K \subseteq \mathbb{C}$ closed convex set
- $F \subseteq K$ face
- $\mathscr{R} \subseteq \mathscr{S}_{\mathscr{H}}$.

Define $\mathscr{G}_F^K(\mathscr{R}) = \{A \in \mathscr{W}^K; \quad \langle Ax, x\rangle \in F \quad \forall\, x \in \mathscr{R}\}$.

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Theorem

- $\mathscr{G}_F^K(\mathscr{R})$ *is a face of* $\mathscr{W}^K$.

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- *$\mathscr{G}_F^K(\mathscr{R})$ is a face of $\mathscr{W}^K$.*
- *For every index set J and $F_j \subseteq K$ a face, $\mathscr{R}_j \subseteq \mathscr{S}_{\mathscr{H}}$, the set $\bigcap_{j\in J} \mathscr{G}_{F_j}^K(\mathscr{R}_j)$ is a face of $\mathscr{W}^K$.*

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- $\dim(\mathscr{H}) < \infty$.
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If $\mathscr{F}$ *is a face of* $\mathscr{W}^K$*, then* $\exists\, \mathscr{R}_1, \ldots, \mathscr{R}_n \subseteq \mathscr{S}_{\mathscr{H}}$:

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- $K = \mathbb{H} = \{z \in \mathbb{C}; \quad \mathrm{Re}(z) \geq 0,\ \mathrm{Im}(z) \geq 0\}$ sector.

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- $\mathscr{F} \subseteq \mathscr{W}^{\mathbb{C}^+}$ is a face $\Longleftrightarrow$ $\exists\ P_{\mathscr{F}}$ orth. projection:

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  $\exists \mathscr{F}_1 \in \mathscr{F}(\mathscr{W}^{\mathbb{C}^+}), \quad \mathscr{F}_2 \in \mathscr{F}(\mathscr{W}^{\mathbb{R}}) : \quad \mathscr{F} = \mathscr{F}_1 \cap \mathscr{F}_2.$

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$\mathbb{R}_+ = \{r \in \mathbb{R}; \quad r \geq 0\}$ half line

- $\{0\}$ the only proper face of $\mathbb{R}^+$.
- $A = R + iS \in \mathscr{W}^{\mathbb{R}^+} \iff R \geq 0 \quad \text{and} \quad S = 0.$

What are faces of $\mathscr{W}^{\mathbb{R}^+}$?

- $\mathscr{F} \subseteq \mathscr{W}^{\mathbb{R}^+}$ is a face $\iff \exists\, P_{\mathscr{F}}$ orth. projection:

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W. N. Anderson Jr., G. E. Trapp, *The extreme points of a set of positive semidefinite operators*, Lin. Alg. App. **106** (1988), 209-217;

S.-L. Eriksson, H. Leutwiler *A potential-theoretic approach to parallel addition*, Math. Ann. **274** (1986), 301-317.

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  $\mathscr{F}_2 = \{R \in \mathscr{W}^{\mathbb{R}} : \ 0 \leq I - R \leq \lambda P_{\mathscr{F}_2}, \text{ for some } \lambda \geq 0\}.$

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- Extreme points of $\mathscr{W}^{\mathbb{I}}$ are orthogonal projections.

Faces of $\mathscr{W}^K$ for a polyhedron set K

$K = \mathbb{S} = \{z \in \mathbb{C}; \quad 0 \leq \mathrm{Re}(z) \leq 1\}$ strip

- $\mathscr{F} \subseteq \mathscr{W}^{\mathbb{S}}$ is a face $\iff$ $\exists P_{\mathscr{F}}, Q_{\mathscr{F}}$ orthogonal projections:

$$\mathscr{F} = \{R + iS \in \mathscr{W}^{\mathbb{S}} : \ Q_{\mathscr{F}} \leq R \leq P_{\mathscr{F}}\};$$

- $\mathscr{W}^{\mathbb{S}}$ has no extreme points.

$K = \mathbb{P} = \{z \in \mathbb{C};\quad 0 \leq \mathrm{Re}(z) \leq 1,\ \mathrm{Im}(z) \geq 0\}$ half strip

- $\mathscr{F} \subseteq \mathscr{W}^{\mathbb{P}}$ is a face $\iff \exists P_{\mathscr{F}_1}, Q_{\mathscr{F}_1}, P_{\mathscr{F}_2}$ orth. projections:

$$\mathscr{F} = \{R + iS \in \mathscr{W}^{\mathbb{P}} :\ Q_{\mathscr{F}_1} \leq R \leq P_{\mathscr{F}_1} \text{ and } 0 \leq S \leq \lambda P_{\mathscr{F}_2},\ \text{for some } \lambda \geq 0\};$$

- Extreme points of $\mathscr{W}^{\mathbb{P}}$ are orthogonal projections.

Faces of $\mathscr{W}^K$ for a polyhedron set K

$$K = \mathbb{H} = \{z \in \mathbb{C}; \quad \text{Re}(z) \geq 0, \ \text{Im}(z) \geq 0\} \ \text{sector}$$

- $\mathscr{F} \subseteq \mathscr{W}^{\mathbb{H}}$ is a face $\iff \exists P_{\mathscr{F}}, Q_{\mathscr{F}}$ orthogonal projections:

$$\mathscr{F} = \{R + iS \in \mathscr{W}^{\mathbb{H}} : \ 0 \leq R \leq \lambda P_{\mathscr{F}} \text{ and } 0 \leq S \leq \mu Q_{\mathscr{F}}, \text{ for some } \lambda, \mu \geq 0\};$$

- $\{0\}$ is the only extreme point of $\mathscr{W}^{\mathbb{H}}$.

Example

$K = \overline{\mathbb{D}}$ *closed unit disc*

$\mathscr{W}^{\overline{\mathbb{D}}} = \{A \in \mathcal{B}(\mathscr{H}); w(A) \leq 1\}$ *closed unit ball in* $(\mathcal{B}(\mathscr{H}), w)$.

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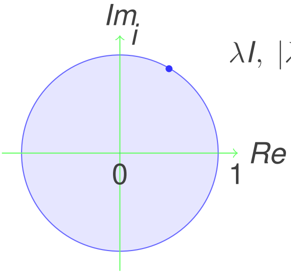

$\lambda I, \ |\lambda| = 1$

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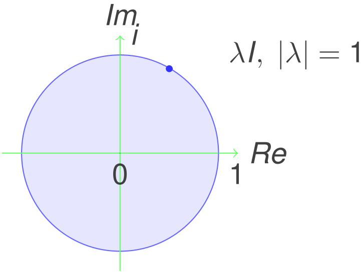
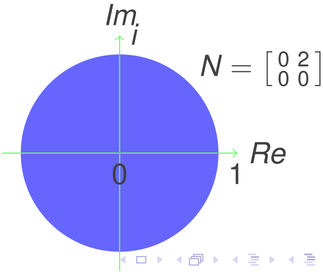
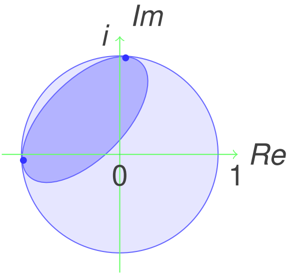
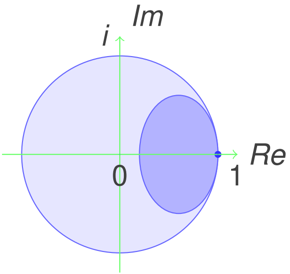




Faces of $\mathscr{W}^{\overline{\mathbb{D}}}$





A. Hopenwasser, R. L. Morre, V. I. Paulsen *C^*-extreme points*, Transactions AMS **266** (1981), 291-307;

M. A. Dritschel, H. J. Woerdeman *Model Theory and Linear Extreme points in the Numerical Radius Unit Ball*, Memoirs AMS **615** (1997).