Star-shapedness of numerical range of a set of matrices

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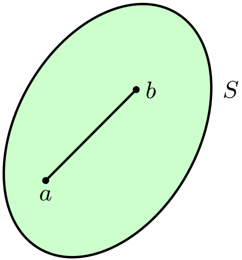
Notations

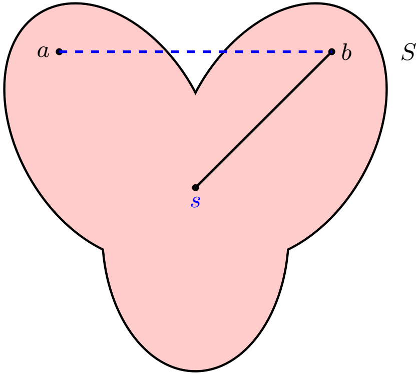
- $\mathbb{R}$: the set of real numbers
- $\mathbb{C}$: the set of complex numbers
- M_n: the set of $n \times n$ complex matrices
- H_n: the set of $n \times n$ Hermitian matrices
- $B(\mathcal{H})$: the set of bounded linear operators on a complex Hilbert space $\mathcal{H}$
- $S(\mathcal{H})$: the set of self-adjoint operators in $B(\mathcal{H})$
- $S_F(\mathcal{H})$: the set of finite rank operators in $S(\mathcal{H})$
- $S_K(\mathcal{H})$: the set of compact operators in $S(\mathcal{H})$
- $\mathbf{cl}\,(S)$: the closure of a set S

Convexity and Star-shapedness

A set S is said to be convex if for any $a, b \in S$, the line segment joining a and b also lies in S. That is,

$$ta + (1-t)b \in S \quad \text{for all } t \in [0,1].$$





A set S is said to be star-shaped if there is a star center $s \in S$ such that the line segment joining s and any other point $b \in S$ also lies in S. That is,

$$ts+(1-t)b \in S \quad \text{for all } t \in [0,1].$$

Numerical range

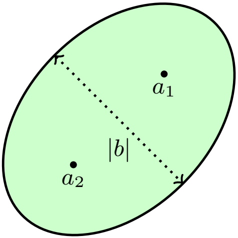
A: $n \times n$ matrix

The **numerical range** (field of values) is defined as

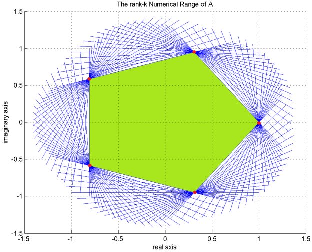
$$\begin{aligned} W(A) &= \{x^*Ax \in \mathbb{C} : \|x\| = 1,\, x \in \mathbb{C}^n\} \\ &= \{\langle x|A|x\rangle : \langle x|x\rangle = 1\} \end{aligned}$$

- The spectrum of A, $\sigma(A) \subseteq W(A)$.
- For any $a, b \in \mathbb{C}$, $W(aA + bI) = aW(A) + b$.
- For any unitary U, $W(U^*AU) = W(A)$.
- $W(A)$ is always nonempty.
- $W(A) = \{\mu\} \iff A = \mu I$.
- $W(A) \subseteq \mathbb{R} \iff A = A^*$.
- $W(A) \subseteq [0, \infty) \iff A$ is positive semidefinite.
- $W(A)$ and $W(A^{-1})$ lie in the unit disk $\iff A$ is unitary.
- If A is Hermitian with eigenvalues $a_1 \geq a_2 \geq \cdots \geq a_n$, then
$$W(A) = [a_n, a_1]\,.$$

- If $A = \begin{bmatrix} a_1 & b \\ 0 & a_2 \end{bmatrix}$, then $W(A)$ is the elliptical disk with foci a_1 and a_2 and minor axis of length $|b|$.



- If A is normal with eigenvalues $a_1, \ldots, a_n$, then
$$W(A) = \operatorname{conv}\{a_1, \ldots, a_n\},$$
which is a convex polygon.

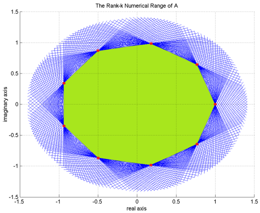


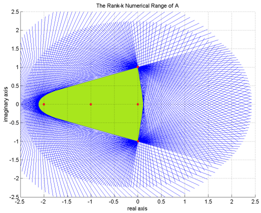
Numerical range

- In general,

$$W(A) = \bigcap_{t\in[0,2\pi)} \left\{\mu \in \mathbb{C} : e^{-it}\mu + e^{it}\bar{\mu} \le \lambda_1(e^{-it}A + e^{it}A^\dagger)\right\}$$

where $\lambda_k(H)$ denotes the k-th largest eigenvalue of Hermitian H.





- $W(A)$ is always convex. **(Toeplitz-Hausdorff Theorem [1918-1919])**

  Generalization: k-numerical range, c-numerical range, q-numerical range, C-numerical range, kth generalized numerical range, product numerical range, restricted numerical range, rank-k numerical range, indefinite numerical range, nuclear numerical range, q-matricial range...

Numerical range

Given a matrix C, the C-numerical range of A is defined as

$$W_C(A) = \{\operatorname{tr}(CU^\dagger AU) : U \text{ is unitary}\}.$$

- If $C = E_{11}$, $W_C(A) = W(A)$.
- If $C = E_{11} + \cdots + E_{kk}$, then it reduces to k-numerical range

$$W_C(A) = W_k(A) = \left\{ \sum_{j=1}^{k} x_j^* A x_j : x_1, \ldots, x_k \text{ are orthonormal in } \mathbb{C}^n \right\},$$

  which is convex. [Halmos, Berger (1963)]

- If $C = \operatorname{diag}(c_1, \ldots, c_n)$, then it reduces to c-numerical range

$$W_C(A) = W_c(A) = \left\{ \sum_{j=1}^{n} c_j x_j^* A x_j : x_1, \ldots, x_n \text{ are orthonormal in } \mathbb{C}^n \right\},$$

  which is also convex when C is Hermitian. [Westwick (1975), Poon (1980)]

- In general, $W_C(A)$ is not convex.
- $W_C(A)$ is convex if there is $\gamma \in \mathbb{C}$ such that $\hat{C} = C - \gamma I$ satisfies one of the following conditions.
  - $\hat{C}$ is rank one.
  - $\hat{C}$ is a multiple of a Hermitian matrix.
  - $\hat{C}$ is unitarily similar to a block matrix of the form $[C_{ij}]_{i,j}$ such that $C_{11}, \ldots, C_{mm}$ are square and $C_{ij} = 0$ if $j \neq i + 1$.
- $W_C(A)$ is star-shaped. [Cheung, Tsing, LAMA (1996)]

The joint numerical range of m-tuple $\mathbf{A} = (A_1, \ldots, A_m)$ is defined by

$$\begin{aligned} W(\mathbf{A}) &= \{(x^*A_1x, x^*A_2x, \ldots, x^*A_mx) : \|x\| = 1, x \in \mathbb{C}^n\} \\ &= \{\text{all joint measurements of pure states by the observables } A_1, \ldots, A_m\}. \end{aligned}$$

Write $A_j = H_j + iG_j$ for Hermitian matrices H_j and G_j,

$$W(A_1, \ldots, A_m) \subseteq \mathbb{C}^m \cong W(H_1, G_1, H_2, G_2, \ldots, H_m, G_m) \subseteq \mathbb{R}^{2m}.$$

Here, assume A_j are Hermitian.

- $W(A_1, A_2, A_3)$ is NOT convex when

  $$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

  Actually, $W(A_1, A_2, A_3) = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1^2 + a_2^2 + a_3^2 = 1\}$.

- For $n \geq 3$, $W(A_1, A_2, A_3)$ is always convex.

  [Au-Yeung & Poon, LAA 27:69-79 (1979)]

- For $m \geq 4$, $W(\mathbf{A})$ may not be convex in general.

- $W(\mathbf{A})$ is star-shaped if the dimension n is sufficiently large.

  [Li & Poon, Studia 194:91-104 (2009)]

Generalized joint numerical ranges

For any $\mathbf{A} = (A_1, \ldots, A_m) \in H_n^m$,

$$W(\mathbf{A}) = \{(x^*A_1x, x^*A_2x, \ldots, x^*A_mx) : \|x\| = 1,\ x \in \mathbb{C}^n\}$$

$$= \{(\mu_1, \ldots, \mu_m) \in \mathbb{R}^m : X^*A_jX = \mu_j \text{ with } X \in M_{n\times 1}, X^*X = 1\}$$

(numerical range)

$$\Lambda_k(\mathbf{A}) = \{(\mu_1, \ldots, \mu_m) \in \mathbb{R}^m : X^*A_jX = \mu_j I_k \text{ with } X \in M_{n\times k}, X^*X = I_k\}$$

(rank-k numerical range)

$$W(q : \mathbf{A}) = \left\{(B_1, \ldots, B_m) \in H_q^m : X^*A_jX = B_j \text{ with } X \in M_{n\times q}, X^*X = I_q\right\}$$

(q-matricial range)

$$\Lambda_{p,q}(\mathbf{A}) = \left\{(B_1, \ldots, B_m) \in H_q^m : X^*A_jX = I_p \otimes B_j \text{ with } X \in M_{n\times pq}, X^*X = I_{pq}\right\}$$

((p, q)-matricial range)

All these ranges are non-empty and star-sharped when the dimension is sufficiently large. [Lau, Li, Poon, S. JFA, in press]

- A quantum channel ϕ of the form

$$A \mapsto \sum_{j=1}^{r} F_j A F_j^\dagger \quad \text{with} \quad \sum_{j=1}^{r} F_j^\dagger F_j = I$$

has a k-dimensional quantum error correcting code if and only if

$$\Lambda_k(F_1^\dagger F_1, F_1^\dagger F_2, \ldots, F_m^\dagger F_m) \neq \emptyset.$$

[Knill and Laflamme, PRA 55:900-911 (1997)], [Choi, Kribs, Zyczkowski, RMP 58:77-91 (2006)]

- A quantum channel ϕ has correctable (recovery) subsystems with size $p \times q$ if and only if

$$\Lambda_{p,q}(F_1^\dagger F_1, F_1^\dagger F_2, \ldots, F_m^\dagger F_m) \neq \emptyset.$$

[Kribs, Laflamme, Poulin, and Lesosky, QIC 6:383-399 (2006)]

Numerical range of a set of matrices

$\mathcal{F}$: A nonempty set of matrices in M_n.

$$
\begin{aligned}
W(\mathcal{F}) &= \bigcup \{W(A) : A \in \mathcal{F}\} \\
&= \{\text{all possible measurements of states by some observable } A \in \mathcal{F}\}
\end{aligned}
$$

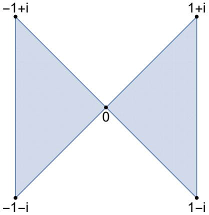
Example Let $A = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}$ and

$$\mathcal{F} = \operatorname{conv}\{A, -A\} = \{(2t-1)A : t \in [0,1]\}.$$

Then

$$W(\mathcal{F}) = \operatorname{conv}\{0, 1+i, 1-i\} \cup \operatorname{conv}\{0, -1-i, -1+i\},$$

which is NOT convex.



Basic Properties

- For any $a, b \in \mathbb{C}$, $W(a\mathcal{F} + bI) = aW(\mathcal{F}) + b$.
- For any unitary U, $W(\mathcal{F}) = W(U^*\mathcal{F}U)$, where $U^*\mathcal{F}U = \{U^\dagger AU : A \in \mathcal{F}\}$.
- $\mathcal{F}$ is bounded $\Longrightarrow W(\mathcal{F})$ is bounded.
- $\mathcal{F}$ is connected $\Longrightarrow W(\mathcal{F})$ is connected.
- $\mathcal{F}$ is compact $\Longrightarrow W(\mathcal{F})$ is compact.
- $\mathcal{F}$ is convex $\not\Longrightarrow W(\mathcal{F})$ is convex.
- $\mathcal{F}$ is closed $\not\Longrightarrow W(\mathcal{F})$ is closed.

Example $\mathcal{F} = \left\{ \begin{bmatrix} a + \frac{i}{a} & 0 \\ 0 & 0 \end{bmatrix} : a > 0 \right\} \cup \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$
and

$$W(\mathcal{F}) = \{a + ib : a, b > 0, ab \leq 1\} \cup \{0\}.$$

More Properties

- $W(\mathcal{F}) = \{\mu\} \iff \mathcal{F} = \{\mu I\}$.
- $W(\mathcal{F}) \subseteq$ a straight line $L \iff$
  - $\mathcal{F} \subseteq \{\mu I : \mu \in L\}$, or
  - $\exists \alpha \in \mathbb{C}$ s.t. $\alpha(A - \mathrm{tr}\,(A)/n)$ are Hermitian for all $A \in \mathcal{F}$ and

    $\{(\mathrm{tr}\,A)/n : A \in \mathcal{F}\}$ is collinear.
- $W(\mathcal{F})$ is a convex polygon $\iff W(\mathcal{F}) = \mathrm{conv}\,\{v_1, \ldots, v_m\}$,
  where v_j is an eigenvalue of some $A_j \in \mathcal{F}$.

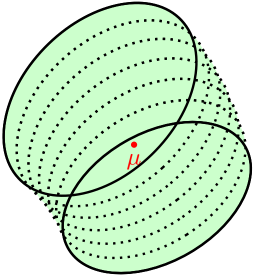
Numerical range of a set of matrices

Assume $\mathcal{F}$ is convex

- If $\mathcal{F}$ contains a scalar matrix μI, then $W(\mathcal{F})$ is star-shaped with μ as a star center.

- Suppose $\bigcap_{A\in\mathcal{F}} W(A) \neq \emptyset$. Then $W(\mathcal{F})$ is star-shaped with μ as a star center for all $\mu \in \bigcap_{A\in\mathcal{F}} W(A)$.

- If $\operatorname{tr} A = \mu$ for all $A \in \mathcal{F}$, then $W(\mathcal{F})$ is star-shaped with μ as a star center.



Assume $\mathcal{F} = \operatorname{conv} \mathcal{G}$

- Suppose $\bigcap_{A\in\mathcal{G}} W(A) \neq \emptyset$. Then $W(\mathcal{F})$ is star-shaped with μ as a star center for all $\mu \in \bigcap_{A\in\mathcal{G}} W(A)$.

Numerical range of a set of matrices

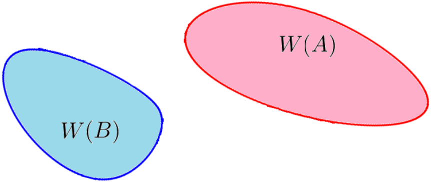
Assume $\mathcal{F} = \mathrm{conv}\,\{A, B\}$ ($\mathcal{G} = \{A, B\}$)

- If $W(A) \cap W(B) \neq \emptyset$, then $W(\mathcal{F})$ is star-shaped with μ as a star center for all $\mu \in W(A) \cap W(B)$.

- If $W(A) \cup W(B)$ lies on a line, then

  $$W(\mathcal{F}) = \mathrm{conv}\,\{W(A), W(B)\} \text{ is convex.}$$

- Otherwise, there are two non-parallel lines L_1 and L_2 intersecting at μ such that for each $j = 1, 2$, L_j is a common supporting line of $W(A)$ and $W(B)$ separating the two convex sets; the set $W(\mathcal{F})$ is star-shaped with star-center μ.



Numerical range of a set of matrices

Assume $\mathcal{F} = \operatorname{conv}\{A_1, \ldots, A_m\}$ ($\mathcal{G} = \{A_1, \ldots, A_m\}$)

- $W(\mathcal{F}) = W^{\otimes}(A_1 \oplus A_2 \oplus \cdots \oplus A_m)$, where

  $W^{\otimes}(X) = \{\langle u{\otimes}v|X|u{\otimes}v\rangle : \langle u|u\rangle = \langle v|v\rangle = 1\}$. (product numerical range)

  [Gawron, Puchala, Miszczak, Skowronek, Zyczkowski, JMP 51:102204 (2010)]

- $W(\mathcal{F}) = W(\hat{\mathcal{F}}) = W(\operatorname{conv}\hat{\mathcal{F}})$, where

  $$\hat{\mathcal{F}} = \{A_x = \operatorname{diag}(\langle x|A_1|x\rangle, \ldots, \langle x|A_m|x\rangle) : \langle x|x\rangle = 1,\ |x\rangle \in \mathbb{C}^n\}.$$

- Suppose $\bigcap_{j=1}^{m} W(A_j) \neq \emptyset$. Then $W(\mathcal{F})$ is star-shaped

  with μ as a star center for all $\mu \in \bigcap_{j=1}^{m} W(A_j)$.

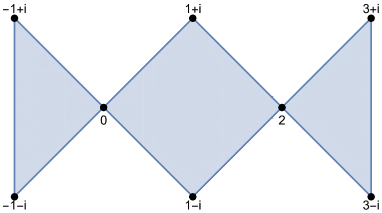
- $\exists\ \{A_1, A_2, A_3\}$ with $W(A_1) \cap W(A_2) \cap W(A_3) = \emptyset$ and $W(\mathcal{F})$ is NOT star-shaped.

Example Let $A = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}$ and $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ with

$$\begin{aligned} \mathcal{F}_1 &= \operatorname{conv}\{A, -A\} = \{(2t-1)A : t \in [0,1]\} \\ \mathcal{F}_2 &= \operatorname{conv}\{A, -A+4I_2\} = \{(2t-1)(A-2I_2)+2I_2 : t \in [0,1]\}. \end{aligned}$$

Notice that $\mathcal{F}$ is star-shaped with A as a star center. But

$$W(\mathcal{F}) = \operatorname{conv}\{0, -1+i, -i-i\} \cup \operatorname{conv}\{0, 2, 1-i, 1+i\} \cup \operatorname{conv}\{2, 3+i, 3-i\}.$$



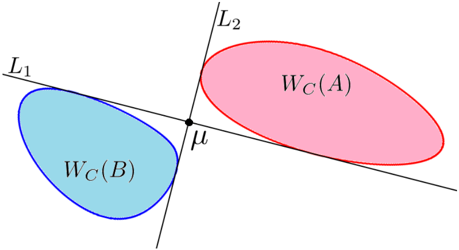
C-Numerical range of a set of matrices

Assume $\mathcal{F} = \operatorname{conv}\{A, B\}$ and **both $W_C(A)$ and $W_C(B)$ are convex.**

- If $W_C(A) \cap W_C(B) \neq \emptyset$, then $W_C(\mathcal{F})$ is star-shaped with μ as a star center for all $\mu \in W_C(A) \cap W_C(B)$.
- If $W_C(A) \cup W_C(B)$ lies on a line, then

$$W_C(\mathcal{F}) = \operatorname{conv}\{W_C(A), W_C(B)\} \text{ is convex.}$$

- Otherwise, there are two non-parallel lines L_1 and L_2 intersecting at μ such that for each $j = 1, 2$, L_j is a common supporting line of $W_C(A)$ and $W_C(B)$ separating the two convex sets; the set $W_C(\mathcal{F})$ is star-shaped with star-center μ.

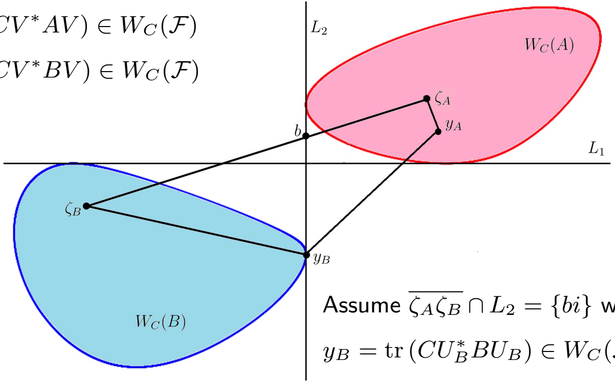


C-Numerical range of a set of matrices

Given $\zeta = \mathrm{tr}\,(CV^*(tA + (1-t)B)V) \in W_C(\mathcal{F})$

$\zeta_A = \mathrm{tr}\,(CV^*AV) \in W_C(\mathcal{F})$

$\zeta_B = \mathrm{tr}\,(CV^*BV) \in W_C(\mathcal{F})$



Assume $\overline{\zeta_A\zeta_B} \cap L_2 = \{bi\}$ with $b > 0$.

$y_B = \mathrm{tr}\,(CU_B^*BU_B) \in W_C(\mathcal{F}) \cap L_2$

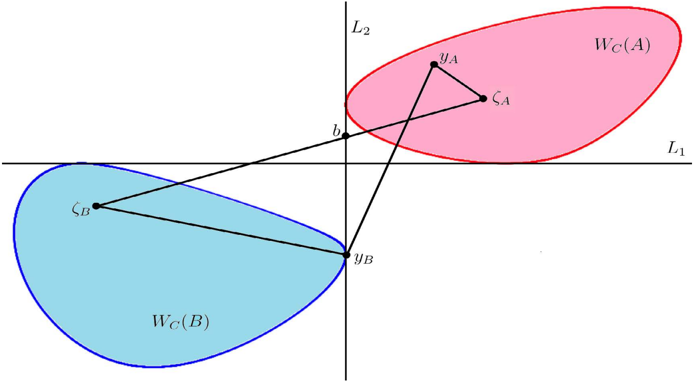
$y_A = \mathrm{tr}\,(CU_B^*AU_B) \in W_C(\mathcal{F})$

$$\overline{\zeta_A\zeta_B} \cup \overline{\zeta_B y_B} \cup \overline{y_B y_A} \cup \overline{y_A\zeta_A} \subseteq W_C(\mathcal{F})$$

$$V \to U \text{ continuously} \implies \mathrm{conv}\,\{\overline{\zeta_A\zeta_B} \cup \overline{\zeta_B y_B} \cup \overline{y_B y_A} \cup \overline{y_A\zeta_A}\} \subseteq W_C(\mathcal{F})$$

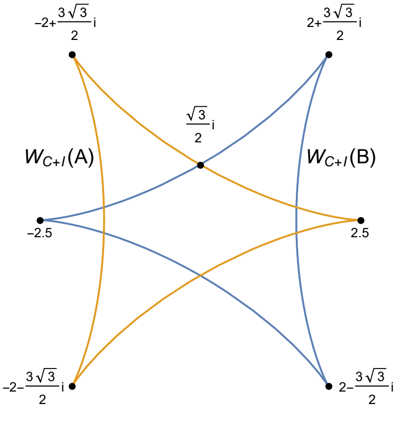
$$\implies \overline{\zeta 0} \subseteq W_C(\mathcal{F})$$

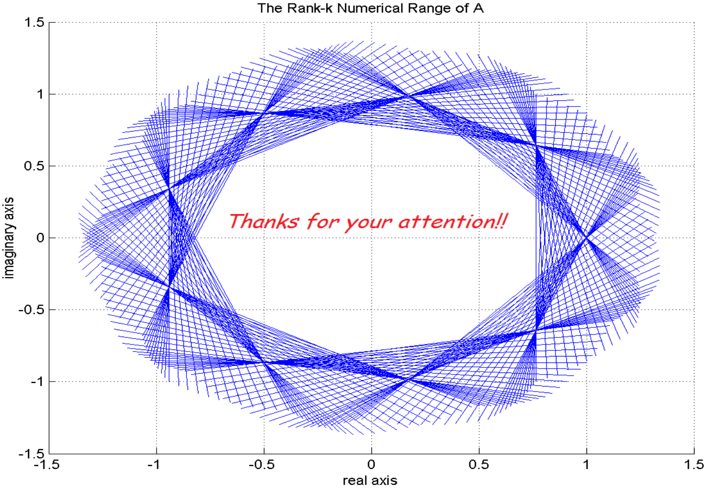
C-Numerical range of a set of matrices



C-Numerical range of a set of matrices

Example Let $w = e^{2\pi i/3}$ and $C = \operatorname{diag}(1, w, w^2)$. Suppose $A = C - \frac{1}{6}I$, $B = e^{\pi i/3}C + \frac{1}{6}I$ and $\mathcal{F} = \operatorname{conv}\{A, B\}$. Then both $W_{C+I}(A)$ and $W_{C+I}(B)$ are not star-shaped. Furthermore, $W_{C+I}(\mathcal{F})$ is not star-shaped.




The Rank-k Numerical Range of A
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5
imaginary axis
real axis
Thanks for your attention!!