The C-Numerical Range in Infinite Dimensions

Frederik vom Ende

Technical University of Munich

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Joint work with Gunther Dirr (University of Würzburg)

Finite-Dimensional Results

C-numerical range of $A, C \in \mathbb{C}^{n\times n}$:

$$W_C(A) = \{\operatorname{tr}(CU^\dagger AU) \mid U \in \mathcal{U}(n)\}$$

Important properties of $W_C(A)$:

- convex if C is normal with collinear eigenvalues *(Westwick 1975, Poon 1980)*
- star-shaped with respect to $(\operatorname{tr}(C)\operatorname{tr}(A)/n)$ for any C *(Cheung, Tsing 1996)*

Contents

Trace in Infinite Dimensions

$\mathcal{X}, \mathcal{Y}$: infinite-dimensional Hilbert spaces
$\mathcal{K}(\mathcal{X}, \mathcal{Y})$: compact operators between $\mathcal{X}$ and $\mathcal{Y}$

Lemma (Schmidt decomposition / singular value decomp.)

For $C \in \mathcal{K}(\mathcal{X}, \mathcal{Y})$, there exists a decreasing null sequence $(s_n(C))_{n\in\mathbb{N}}$ in $[0, \infty)$ and orthonormal systems $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{X}$ and $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{Y}$ such that

$$C = \sum_{n=1}^{\infty} s_n(C)\langle f_n, \cdot\rangle g_n \,,$$

where the series converges in the operator norm.

Trace in Infinite Dimensions

Trace-class:

$$\mathcal{B}^1(\mathcal{X},\mathcal{Y}) := \Big\{ C \in \mathcal{K}(\mathcal{X},\mathcal{Y}) \,\Big|\, \sum\nolimits_{n=1}^\infty s_n(C) < \infty \Big\}$$

→ Banach space under the trace norm $\nu_1(C) := \sum_{n=1}^\infty s_n(C)$
→ trace-class is two-sided ideal in the bounded operators, e.g.

$$S \in \mathcal{B}(\mathcal{Y},\mathcal{Z}), C \in \mathcal{B}^1(\mathcal{X},\mathcal{Y}), T \in \mathcal{B}(\mathcal{W},\mathcal{X}) \;\Rightarrow\; SCT \in \mathcal{B}^1(\mathcal{W},\mathcal{Z})$$

→ the trace of C

$$\operatorname{tr}(C) := \sum_{i\in I} \langle f_i, Cf_i\rangle$$

is well-defined (summable & independent of the ONB $(f_i)_{i\in I}$) and "has the usual properties as long as a trace-class operator is involved somewhere"

$W_C(T)$ in Infinite Dimensions

$\mathcal{H}$ separable, complex, inf.-dim. Hilbert space
$C \in \mathcal{B}^1(\mathcal{H})$, $T \in \mathcal{B}(\mathcal{H})$. Now the C-numerical range of T

$$W_C(T) := \{\operatorname{tr}(CU^\dagger TU) \mid U \in \mathcal{U}(\mathcal{H})\}$$

is a well-defined subset of $\mathbb{C}$

Getting Intuition for the Hausdorff Metric

Metric on the set of non-empty compact subsets of a metric space:

$$\Delta(A, B) := \max\left\{\max_{z\in A}\min_{w\in B} d(z, w), \max_{z\in B}\min_{w\in A} d(z, w)\right\}$$

Example

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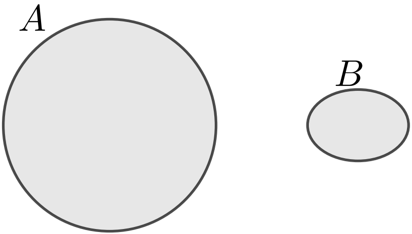

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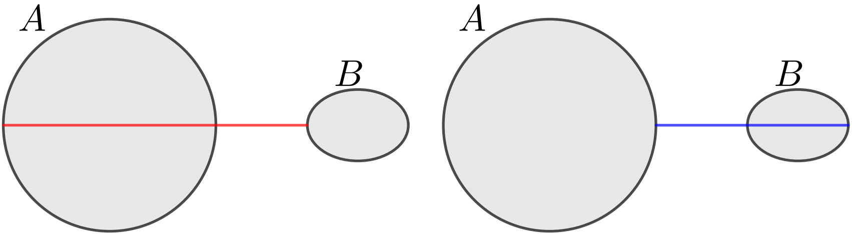

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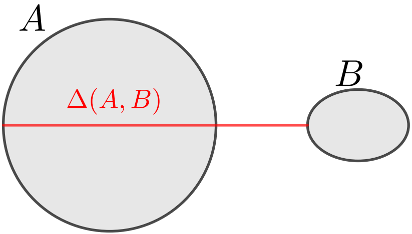

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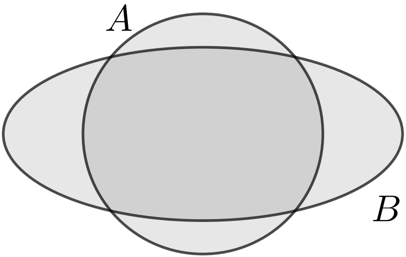

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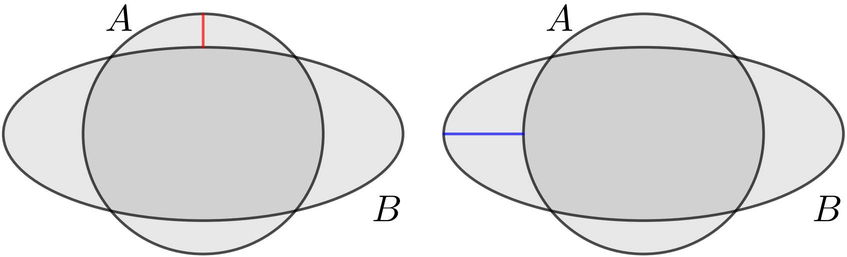

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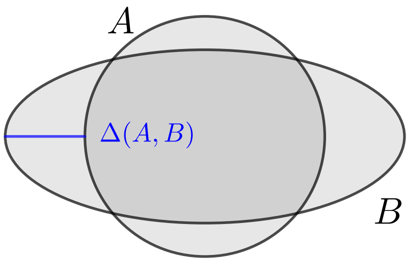

Properties of the Hausdorff Metric

Lemma

$(A_n)_{n\in\mathbb{N}}$ bounded sequence of non-empty compact subsets of $\mathbb{C}$ with $A_n \overset{n\to\infty}{\to} A$ and $(z_n)_{n\in\mathbb{N}}$ sequence of complex numbers converging to z. Then:

(a) *The sequence of convex hulls $(\operatorname{conv}(A_n))_{n\in\mathbb{N}}$ converges to $\operatorname{conv}(A)$*

(b) *If A_n is convex for all $n \in \mathbb{N}$, then A is convex*

(c) *If A_n is star-shaped with respect to z_n for all $n \in \mathbb{N}$, then A is star-shaped with respect to z*

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From Matrices to Operators (and back)

Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathcal{B}(\mathcal{H})$. Consider the maps:

- E_n : embeds the matrix into an operator

$$E_n(A) = \begin{pmatrix} A & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{pmatrix}$$

- $[\,\cdot\,]_n$: "cuts out" the upper left $n \times n$ block of an operator B and puts it into a matrix $[B]_n$
- $\cdot_n$: maps B to the operator $B_n = E_n([B]_n) = \Pi_n B \Pi_n$ where only the upper left $n \times n$-block didn't vanish ("block-approximation")

$\to$ these maps depend on the chosen orthonormal basis of $\mathcal{H}$!

Connecting Finite and Infinite Dimensions

Theorem

$C \in \mathcal{B}^1(\mathcal{H})$, $T \in \mathcal{B}(\mathcal{H})$, arbitrary ONB $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$. Then

$$\lim_{n\to\infty} W_{[C]_{2n}}([T]_{2n}) = \overline{W_C(T)}\,.$$

Proof idea.

- $W_{[C]_{2n}}([T]_{2n}) \to \overline{W_C(T)}$: Start from $\operatorname{tr}([C]_{2n}U_n^\dagger[T]_{2n}U_n)$. Embed unitary matrix U_n into unitary operator $\tilde{U}_n$ by "completing with the identity" $\to \operatorname{tr}(C\tilde{U}_n^\dagger T\tilde{U}_n) \in W_C(T)$ is ε-close
- $\overline{W_C(T)} \to W_{[C]_{2n}}([T]_{2n})$: Start from $\operatorname{tr}(CU^\dagger TU)$. For every unitary operator U exists a sequence $E_{2n}(U_n')$ of operators (given by embedded unitary matrices) such that $E_{2n}(U_n') \overset{s}{\to} U$ as $n\to\infty$. Then $\operatorname{tr}([C]_{2n}(U_n')^\dagger[T]_{2n}U_n') \in W_{[C]_{2n}}([T]_{2n})$ is ε-close □

Convexity of $\overline{W_C(T)}$

Theorem

Let $C \in \mathcal{B}^1(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H})$ be given. If C is normal with collinear eigenvalues (lie on a common line) or if T is essentially self-adjoint[a], then $\overline{W_C(T)}$ is convex.

[a] $\exists\ \theta \in \mathbb{R}$, $\xi \in \mathbb{C}$ such that $e^{-i\theta}(T - \xi\,\mathrm{id}_{\mathcal{H}})$ is self-adjoint

Proof idea.

C trace-class ($\rightarrow$ compact) and normal $\Rightarrow$ unitarily diagonalizable so due to the coll. ev., there exists $\alpha \in \mathbb{R}$ and an ONB such that $([e^{i\alpha}C]_{2n})_{n\in\mathbb{N}}$ is a sequence of hermitian matrices. Then

$$\overline{W_C(T)} = \overline{W_{e^{i\alpha}C}(e^{-i\alpha}T)} = \lim_{n\to\infty} W_{[e^{i\alpha}C]_{2n}}([e^{-i\alpha}T]_{2n})$$

(Case of T being ess. self-adj. is done similarly) □

Generalizing the Star-Center

Essential numerical range of $T \in \mathcal{B}(\mathcal{H})$

$$W_e(T) := \Big\{ \lim_{n\to\infty} \langle f_n, Tf_n \rangle \,\Big|\, (f_n)_{n\in\mathbb{N}} \text{ is ONS in } \mathcal{H} \Big\}$$

$\to$ non-empty, convex and compact subset of $\mathbb{C}$

Proposition

$T \in \mathcal{B}(\mathcal{H})$, $\mu \in \mathbb{C}$. *The following statements are equivalent.*

(a) $\mu \in W_e(T)$

(b) *There exists an orthonormal system* $(f_n)_{n\in\mathbb{N}}$ *in* $\mathcal{H}$ *such that* $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^n \langle f_j, Tf_j \rangle = \mu$.

(c) *There exists an orthonormal basis* $(e_n)_{n\in\mathbb{N}}$ *of* $\mathcal{H}$ *such that* $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^n \langle e_j, Te_j \rangle = \mu$.

Star-Shapedness of $\overline{W_C(T)}$

Theorem

Let $C \in \mathcal{B}^1(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H})$ be given. Then $\overline{W_C(T)}$ is star-shaped with respect to $\operatorname{tr}(C)W_e(T)$.

Proof.

Let any $\mu \in W_e(T)$, so there exists an ONB such that

$$\lim_{n\to\infty} \frac{\operatorname{tr}([T]_{2n})}{2n} = \lim_{n\to\infty} \frac{1}{2n} \sum_{j=1}^{2n} \langle e_j, Te_j \rangle = \mu.$$

As $\operatorname{tr}([C]_{2n}) \to \operatorname{tr}(C)$, in total $(\operatorname{tr}([C]_{2n}) \operatorname{tr}([T]_{2n})/(2n) \to \operatorname{tr}(C)\mu$ as $n \to \infty$. Thus the star-shapedness of $W_{[C]_{2n}}([T]_{2n})$ transfers, with new star-center $\operatorname{tr}(C)\mu$; but $\mu \in W_e(T)$ was chosen arbitrarily □

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Further Characterization in Finite Dimensions

Actually, if A, C are both normal, then

$$W_C(A) \subseteq \operatorname{conv}(P_C(A)) \tag{1}$$

where the C-spectrum of A

$$P_C(A) = \Big\{ \sum_{i=1}^{n} \gamma_i \alpha_{\sigma(i)} \,\big|\, \sigma \in \mathrm{Sym}_n \Big\}$$

uses the eigenvalues $(\gamma_i)_{i=1}^n$ of C and $(\alpha_i)_{i=1}^n$ of A. If additionally, the eigenvalues of C are collinear, then (1) becomes an equality.

The C-Spectrum in Infinite Dimensions

$T \in \mathcal{K}(\mathcal{H})$ normal $\Rightarrow T = \sum_{i=1}^{\infty} \tau_i \langle f_i, \cdot \rangle f_i$ with $\tau_i \to 0$ as $i \to \infty$, $(f_i)_{i\in\mathbb{N}}$ ONS in $\mathcal{H}$

$\to$ modify the eigenvalue sequence of compact (normal) operators so the diagonalization is w.r.t. an **ONB**

Definition

For $C \in \mathcal{B}^1(\mathcal{H})$ with modified eigenvalue sequence $(\gamma_n)_{n\in\mathbb{N}}$ and $T \in \mathcal{K}(\mathcal{H})$ with modified eigenvalue sequence $(\tau_n)_{n\in\mathbb{N}}$, we define the C-spectrum of T to be

$$P_C(T) = \Big\{ \sum_{n=1}^{\infty} \gamma_n \tau_{\sigma(n)} \,\Big|\, \sigma : \mathbb{N} \to \mathbb{N} \text{ is permutation} \Big\}.$$

$W_C(T)$ in the Compact Normal Case

Theorem

$C \in \mathcal{B}^1(\mathcal{H})$, $T \in \mathcal{K}(\mathcal{H})$ both normal. Then

$$P_C(T) \subseteq W_C(T) \subseteq \overline{\operatorname{conv}(P_C(T))}.$$

Proof idea.

First inclusion: choose $U = \sum_{n=1}^{\infty} \langle e_n, \cdot \rangle f_{\sigma(n)}$

Second inclusion: WLOG $w = \operatorname{tr}(CU^\dagger TU) = \sum_{i,j=1}^{\infty} \gamma_i \tau_j |\langle e_j, Ue_i\rangle|^2$. Now $S := \sum_{i,j=1}^{\infty} |\langle e_j, Ue_i\rangle|^2 \langle e_i, \cdot\rangle e_j$ is doubly-stochastic ($S \in \mathcal{D}(\mathcal{H})$); the doubly-stochastic operators coincide with the *weak closure* of the convex hull of the permutation operators $\sum_n \langle e_n, \cdot \rangle e_{\sigma(n)}$ (Kendall, 1960) so $w \in \overline{\operatorname{conv}(P_C(T))}$ □

Schur Triangulation Theorem for Operators?

dim $< \infty$: enough if one operator is normal as we get the eigenvalues of the other one by Schur

Problem: "the" diagonal of triangular (bounded) operators in general does not coincide with their spectrum (or even their point spectrum/eigenvalues); e.g. left or right shift on $\ell_2(\mathbb{N})$ (cf. Herrero, 1991)

Theorem

$C \in \mathcal{B}^1(\mathcal{H})$, $T \in \mathcal{K}(\mathcal{H})$: one of them normal and the other one upper or lower triangular. Then $\overline{P_C(T)} \subseteq \overline{W_C(T)}$.

Proof idea.

Key result (because T is compact): the non-zero diagonal elements of T give exactly the spectrum of T (without zero) & cardinality gives algebraic multiplicity □

Extension of the C-Spectrum Result

Theorem

$C \in \mathcal{B}^1(\mathcal{H})$, $T \in \mathcal{K}(\mathcal{H})$ both normal & eigenvalues of C (or T) are collinear. Then

$$\overline{W_C(T)} = \overline{\text{conv}(P_C(T))}$$

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Key result: $\lim_{n\to\infty} P_{[C]_n^e}([T]_n^g) = \overline{P_C(T)}$ where $[\cdot]_n^e$ and $[\cdot]_n^g$ are the cut-out operators w.r.t. to the ONBs $(e_n)_{n\in\mathbb{N}}$ ($(g_n)_{n\in\mathbb{N}}$) which diagonalize C (T)

Extension of the C-Spectrum Result

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$C \in \mathcal{B}^1(\mathcal{H})$, $T \in \mathcal{K}(\mathcal{H})$ both normal & eigenvalues of C (or T) are collinear. Then

$$\overline{W_C(T)} = \overline{\operatorname{conv}(P_C(T))}$$

Proof.

"$\supseteq$": First $W_{[C]_n^e}([T]_n^g) \subseteq \overline{W_{\Pi_n^e C \Pi_n^e}(\Pi_n^g T \Pi_n^g)} \overset{n\to\infty}{\to} \overline{W_C(T)}$ (limit because C, T compact). Now $W_{[C]_n^e}([T]_n^g) = \operatorname{conv}(P_{[C]_n^e}([T]_n^g))$ by assumption so

$$\begin{aligned}\overline{\operatorname{conv}(P_C(T))} &\subseteq \overline{\operatorname{conv}(\overline{P_C(T)})} = \operatorname{conv}(\overline{P_C(T)}) = \lim_{n\to\infty} \operatorname{conv}(P_{[C]_n^e}([T]_n^g)) \\ &= \lim_{n\to\infty} W_{[C]_n^e}([T]_n^g) \subseteq \lim_{n\to\infty} \overline{W_{\Pi_n^e C \Pi_n^e}(\Pi_n^g T \Pi_n^g)} = \overline{W_C(T)} \quad \square\end{aligned}$$

Open Questions (1)

Consider $T \in \mathcal{B}(\mathcal{H})$ and bounded linear functional $\gamma \in \mathcal{B}'(\mathcal{H})$. Of course $\operatorname{tr}(C\,\cdot\,) \in \mathcal{B}'(\mathcal{H})$ for any $C \in \mathcal{B}^1(\mathcal{H})$ but not every element in $\mathcal{B}'(\mathcal{H})$ is of this form. Thus: is

$$\{\gamma(U^\dagger TU) \,|\, U \in \mathcal{U}(\mathcal{H})\}$$

star-shaped?

Note: if $\gamma \in \mathcal{K}'(\mathcal{H})$, then there exists $C \in \mathcal{B}^1$ s.t. $\gamma = \operatorname{tr}(C\,\cdot\,)$ so the above set is star-shaped w.r.t. $\gamma(\mathrm{id}_{\mathcal{H}})W_e(T)$

Open Questions (2)

One can use general Schatten-p-operators ($p \in (1, \infty)$)

$$\mathcal{B}^p(\mathcal{X}, \mathcal{Y}) := \Big\{ C \in \mathcal{K}(\mathcal{X}, \mathcal{Y}) \,\Big|\, \sum_{n=1}^\infty s_n^p(C) < \infty \Big\}$$

Banach space (even Hilbert space for $p = 2$); $W_C(T)$ is well defined if $C \in \mathcal{B}^p(\mathcal{H})$, $T \in \mathcal{B}^q(\mathcal{H})$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$ (in particular contains the symmetric case).

Open Questions (2)

All results should be the same with slightly altered proofs; however, the star-center becomes

$$\frac{\operatorname{tr}([C]_{2n})\operatorname{tr}([T]_{2n})}{2n} = \frac{\operatorname{tr}[C]_{2n}}{(2n)^{1/q}}\frac{\operatorname{tr}[T]_{2n}}{(2n)^{1/p}} \overset{n\to\infty}{\to} W_e^p(C)W_e^q(T)$$

where

$$W_e^p(T) := \Big\{ \lim_{n\to\infty} \frac{1}{n^{1-\frac{1}{p}}}\sum_{j=1}^n \langle e_j, Te_j\rangle \,\Big|\, (e_n)_{n\in\mathbb{N}} \text{ is ONB of } \mathcal{H} \Big\} \subset \mathbb{C}$$

for now is called the p-essential numerical range of a Schatten-p-operator T. One can show: $W_e^p(T)$ non-empty. What else is known about it? Does it maybe already exist/other characterization?

Summary

In inf. dimensions: $C \in \mathcal{B}^1(\mathcal{H})$, $T \in \mathcal{B}(\mathcal{H})$, C-num. range of T becomes $W_C(T) = \{\operatorname{tr}(CU^\dagger TU) \,|\, U \in \mathcal{U}(\mathcal{H})\}$. Using Hausdorff metric one can show

- C normal with collinear eigenvalues or T essentially self-adjoint, then $\overline{W_C(T)}$ is convex
- Essential numerical range can be characterized via $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^n \langle e_j, Te_j\rangle = \mu$ with $(e_n)_n$ ONB of $\mathcal{H}$
- For any $C \in \mathcal{B}^1$, $T \in \mathcal{B}$, $\overline{W_C(T)}$ is star-shaped with respect to $\operatorname{tr}(C)W_e(T)$
- If one op. is normal and the other one upper/lower triangular, then $\overline{P_C(T)} \subseteq \overline{W_C(T)}$
- C and T normal & eigenvalues of C (or T) are collinear. Then $\overline{W_C(T)} = \overline{\operatorname{conv}(P_C(T))}$
- Generalization to (a) linear functionals on $\mathcal{B}(\mathcal{H})$ / (b) Schatten-p operators $\to$ p-essential numerical range?