An inverse numerical range problem for determinantal representations

Mao-Ting Chien

Soochow University, Taiwan

Based on joint work with Hiroshi Nakazato

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Outline

1. Introduction

Let A be an $n \times n$ matrix. The numerical range of A is defined as

$$W(A) = \{x^* A x; x \in \mathbf{C}^n, \|x\| = 1\}.$$

The determinantal ternary form associated to A:

$$F_A(t, x, y) = \det(tI_n + x\Re(A) + y\Im(A)),$$

where $\Re(A) = (A + A^*)/2$, $\Im(A) = (A - A^*)/(2i)$.

Kippenhahn 1951 proved $W(A)$ is the convex hull of the real affine part of the dual curve of the algebraic curve $F_A(t, x, y) = 0$.

1. Introduction

A ternary form $F(t,x,y)$ is hyperbolic with respect to (1,0,0) if $F(1,0,0)=1$, and for any real pairs x,y, the equation $F(t,x,y)=0$ has only real roots.

Peter Lax conjecture(1958): For any hyperbolic ternary form $F(t,x,y)$ w.r.t (1,0,0), there exist real symmetric matrices S_1, S_2 so that $F(t,x,y) = F_{S_1+iS_2}(t,x,y) = \det(tI_n + xS_1 + yS_2)$.

Helton and Vinnikov 2007 proved the conjecture is true using Riemann theta functions.

Hyperbolic ternary forms completely determine numerical ranges

1. Introduction

The inverse numerical range problem:
Given a point $z \in W(A)$, find a unit vector x so that $z = x^* A x$.

N. K. Tsing, 1984
solved the inverse numerical range problem for 2×2 matrices.

Given $z \in W(A)$. The chord passing through z and the center of $W(A)$, intersects the ellipse at $u^* A u$ and $v^* A v$. Find t so that $z = t\, u^* A u + (1 - t)\, v^* A v$. Then the unit vector

$$x = \sqrt{t}\, u + \sqrt{1 - t} e^{i\theta}\, v$$

satisfies $z = x^* A x$, where $v^* A u = \rho\, e^{i\theta}$

1. Introduction

Geometry algorithms

C. R. Johnson, 1978
F. Uhlig, 2008
R. Carden, 2009
C. Chorianopoulos, P. Psarrakos, F. Uhlig, 2010
N. Bebiano, et al., 2014

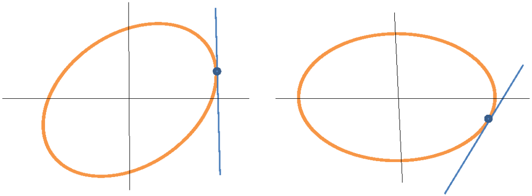
For $0 \leq \theta \leq 2\pi$, $\Re(e^{-i\theta}A) = \cos\theta\, \Re(A) + \sin\theta\, \Im(A)$.
The line

$$\{z \in \mathbb{C}, \Re(z) = \lambda_{\max}(\Re(e^{-i\theta}A))\}$$

is the right vertical support line of $W(e^{-i\theta}A)$.
Assume

$$\Re(e^{-i\theta}A)\,\xi_\theta = \lambda_{\max}(\Re(e^{-i\theta}A))\,\xi_\theta \tag{1}$$

Then the point $\xi_\theta^* A \xi_\theta \in \partial W(A)$.



The eigen condition

$$\Re(e^{-i\theta}A)\,\xi_\theta = \lambda_{\max}(\Re(e^{-i\theta}A))\,\xi_\theta \qquad (1)$$

is equivalent to

$$\Big(\lambda_{\max}(\Re(e^{-i\theta}A))I_n - \cos\theta\,\Re(A) - \sin\theta\,\Im(A)\Big)\xi_\theta = 0. \qquad (2)$$

Then

$$\begin{aligned} & F_A\Big(\lambda_{\max}(\Re(e^{-i\theta}A)), -\cos\theta, -\sin\theta\Big) \\ = \ & \det\Big(\lambda_{\max}(\Re(e^{-i\theta}A)), -\cos\theta, -\sin\theta\Big) \\ = \ & 0. \end{aligned}$$

1. Introduction

Starting an $n \times n$ matrix A, define

$$M_A(t, x, y) = tI_n + x\Re(A) + y\Im(A).$$

Consider non-zero vectors $(t(s), x(s), y(s))$ on the algebraic curve $F_A(t, x, y) = 0$, we construct kernel vector function $\xi(s)$ satisfying

$$M_A\Big(t(s), x(s), y(s)\Big)\,\xi(s) = 0.$$

In this sense, the kernel vector function $\xi(s)$ gives a new direction to do the inverse numerical range problem.

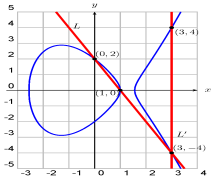
2. Elliptic curves

Let $F(t,x,y)$ be an irreducible hyperbolic ternary form of degree n. Assume the genus of its algebraic curve $F(t,x,y)=0$ is 1, in other words, the algebraic curve is elliptic. Birationally transforms the elliptic curve $F(t,x,y)=0$ to a cubic curve of the Weierstrass canonical equation

$$Y^2 = 4X^3 - g_2X - g_3$$

for some real constants g_2, g_3 with $g_2^3 - 27g_3^2 > 0$.

Elliptic curve group structure: $P+Q$: $(0,2)+(1,0)=(3,4)$



2. Elliptic curves

The complex elliptic curve

$$Y^2 = 4X^3 - g_2X - g_3 = 4(X - e_1)(X - e_2)(X - e_3)$$

is parametrized as

$$X = \wp(s), \quad Y = \wp'(s),$$

where $\wp(s)$ and $\wp'(s)$ are the Weierstrass $\wp$-functions and its derivative.

The function $\wp(s)$ has two half-periods ω_1 and ω_2 with $\omega_1 > 0$ and $\omega_2/i > 0$, i.e.,

$$\wp(s + 2\omega_1) = \wp(s + 2\omega_2) = \wp(s).$$

The τ-invariant of the curve is defined by $\tau = \omega_2/\omega_1$.

The real affine part $F(1,x,y)=0$ is then parametrized as

$$\{(1,x,y)=(1,R_1(\wp(u),\wp'(u)),R_2(\wp,\wp'(u))):$$

$$\Im(u)=0,0<\Re(u)<2\omega_1 \text{ or}$$

$$\Im(u)=\Im(\omega_2),0\leq\Re(u)\leq 2\omega_1\}$$

by real rational functions R_1, R_2 of $\wp$ and $\wp'$ over the torus $\mathbb{T}=\mathbb{C}/(2\mathbb{Z}\omega_1+2\mathbb{Z}\omega_2)$ and its normalized Abel-Jacobi variety $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$.
This parametrization $s\mapsto(1,x,y)$ is the inverse of the Abel-Jacobi map $\phi:\{F(1,x,y)=0\}\to\mathbb{T}$.

Define

$$\theta(u) = \sum_{m\in\mathbb{Z}} \exp(\pi i(m^2\tau + 2mu)),\ \ u \in \mathbb{C}.$$

The Riemann theta function $\theta[\epsilon](u)$ with 2^{2g} characteristics ϵ:

$$\theta[\epsilon](u) = \exp(\pi i(a^2\tau + 2au + 2ab))\theta(u + \tau a + b, \tau),$$

where $\epsilon = a + \tau b$. For $(a, b) = (0, 0), (1/2, 0), (0, 1/2), (1/2, 1/2)$, the four respective Riemann theta functions are defined by

$$\theta_1(u) = -\theta[1/2, 1/2](u),\ \ \theta_2(u) = \theta[1/2, 0](u),$$

$$\theta_3(u) = \theta[0, 0](u),\ \ \theta_4(u) = \theta[0, 1/2](u).$$

Elliptic hyperbolic ternary form representation theorm

Theorem 3.1 Let $F(t,x,y)$ be an irreducible hyperbolic ternary form. Assume that the genus of the curve $F(t,x,y)=0$ is 1, and the curve $F(t,x,y)=0$ intersects the line $x=0$ at n distinct points $Q_j=(\beta_j,0,-1)$, $\beta_j\in\mathrm{R}$, $\beta_j\neq 0$. Denote by ϕ the Abel-Jacobi map from the curve $F(t,x,y)=0$ onto the torus $\mathrm{C}/(2\mathbb{Z}\omega_1+2\mathbb{Z}\omega_2)$. Let $Q_j'=\phi(Q_j)$. For the Riemann theta functions θ_δ, $\delta=2,3$, the symmetric matrix $S=C+i\,\mathrm{diag}(\beta_1,\dots,\beta_n)$ satisfying

$$F(t,x,y)=F_S(t,x,y)=\det(tI_n+xC+y\,\mathrm{diag}(\beta_1,\dots,\beta_n))$$

are given by

$$c_{jk}=\frac{(\beta_k-\beta_j)\theta_1'(0)}{2\omega_1\theta_\delta(0)}\times\frac{\theta_\delta((Q_k'-Q_j')/(2\omega_1))}{\theta_1((Q_k'-Q_j')/(2\omega_1))}\times\frac{1}{\sqrt{d(R_1/R_2)(Q_j')}\sqrt{d(R_1/R_2)(Q_k')}},$$

$$c_{jj}=\beta_j\frac{F_x(\beta_j,0,-1)}{F_y(\beta_j,0,-1)}.$$

3. Kernel and theta functions

We propose the following conjecture:

Let A be an $n \times n$ matrix, and $F_A(t, x, y) = 0$ be elliptic. Assume a linear pencil

$$M(t, x, y) = t\, I_n + x\, C + y\, \mathrm{diag}(b_1, b_2, \ldots, b_n)$$

represents $F_A(t, x, y) = \det(M(t, x, y))$.
Then

(1) there exist n points $P_1', P_2', \ldots, P_n'$, obtained from the kernel vector function $(\xi_1, \xi_2, \ldots, \xi_n)^T$ of $M(x, y, z)$, and the kernel vector function can be expressed as

$$\xi_k(s) = \alpha_k \Big(\prod_{1 \le j \le n, j \ne k} \theta_1(s - Q_j') \Big) \theta_1(s - P_k'), k = 1, 2, \ldots, n$$

for some non-zero constants α_k.

(2) With respect to the Abel group structure of this variety, the points $P_1' - Q_1', P_2' - Q_2', \ldots, P_n' - Q_n'$ satisfy the equation

$$P_1' - Q_1' \equiv P_2' - Q_2' \equiv \cdots \equiv P_n' - Q_n'.$$

(3) If the linear pencil $M(x, y, z)$ is unitarily equivalent to the pencil via θ_δ-representation in Theorem 3.1 for $\delta = 2$ or $\delta = 3$, then the point $P_1' - Q_1'$ satisfies the equation

$$(P_1' - Q_1') + (P_1' - Q_1') = 0.$$

(4) Moreover, if $\delta = 2$, the point $P_1' - Q_1' = 1/2$ of the normalized Abel-Jacobi variety,
and if $\delta = 3$, the point $P_1' - Q_1' = (1 + \tau)/2$.

Where the equivalence relation $z_1 \equiv z_2$ on the normalized Abel-Jacobi variety $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ means $z_1 - z_2 = n + m\tau$ for some integers n, m.

Lemma 4.1 Let T be a 3×3 complex symmetric matrix in the standard form

$$T = \begin{pmatrix} a_{11} + ib_1 & a_{12} & a_{13} \\ a_{12} & a_{22} + ib_2 & a_{23} \\ a_{13} & a_{23} & a_{33} + ib_3 \end{pmatrix}, \tag{4.1}$$

where (a_{ij}) is a real symmetric matrix and b_1, b_2, b_3 are mutually distinct real numbers. Let $\xi = (\xi_1, \xi_2, \xi_3)^T$ be the third column of the adjugate matrix of the linear pencil

$$M(t,x,y) = tI_3 + x\Re(T) + y\Im(T) = tI_3 + x\,(a_{ij}) + y\,\mathrm{diag}(b_1, b_2, b_3).$$

Then ξ is a kernel vector function of $M(t,x,y)$, and

$$\xi_1 = x\,(a_{12}a_{23}x - a_{13}a_{22}x - a_{13}b_2y - a_{13}t),$$

$$\xi_2 = x\,(a_{12}a_{13}x - a_{11}a_{23}x - a_{23}b_1y - a_{23}t),$$

$$\xi_3 = (a_{11}a_{22} - a_{12}^2)x^2 + b_1b_2y^2 + t^2 + (a_{22}b_1 + a_{11}b_2)xy$$
$$+(a_{11} + a_{22})xt + (b_1 + b_2)yt.$$

Lemma 4.2 Let T be a 3×3 complex symmetric matrix defined in (4.1). Assume $\xi = (\xi_1, \xi_2, \xi_3)^T$ is the third column of the adjugate matrix of the linear pencil $M(t, x, y) = tI_3 + x\Re(T) + y\Im(T)$. Then the intersection points of the two curves $\mathcal{V}_{\mathbb{C}}(F_T)$ and $\mathcal{V}_{\mathbb{C}}(\xi_j)$ are characterized by the following divisors on the curve $\mathcal{V}_{\mathbb{C}}(F_T)$:

$$F_T \cdot \xi_1 = Q_1 + 2Q_2 + Q_3 + P_1 + P_3,$$

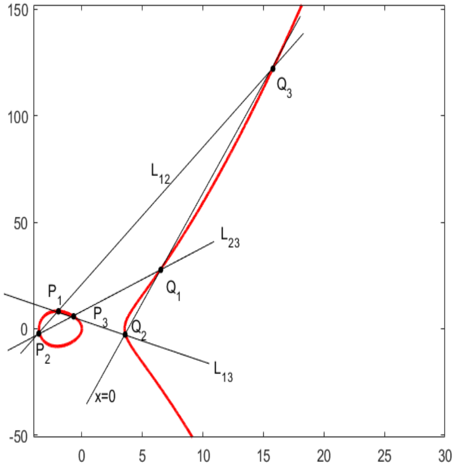
$$F_T \cdot \xi_2 = 2Q_1 + Q_2 + Q_3 + P_2 + P_3,$$

$$F_T \cdot \xi_3 = 2Q_1 + 2Q_2 + 2P_3.$$

$$\xi_1 \leftrightarrow x = 0, \ \ L_{13} : a_{12}a_{23}x - a_{13}a_{22}x - a_{13}b_2 y - a_{13}t = 0$$

$$\xi_2 \leftrightarrow x = 0, \ \ L_{23} : a_{12}a_{13}x - a_{11}a_{23}x - a_{23}b_1 y - a_{23}t = 0$$

4. Results



Theorem 4.3 Let T be a 3×3 complex symmetric matrix defined in (4.1). Assume the the ternary form $F_T(t, x, y)$ is irreducible and the cubic curve $\mathcal{V}_{\mathbb{C}}(F_T)$ is elliptic. Denote $Q' = \phi(Q) \in \mathbb{C}/(\mathbb{Z} + \tau Z)$ corresponding to a point $Q \in \mathcal{V}_{\mathbb{C}}(F_T)$. Then

$$P_1' - Q_1' \equiv P_2' - Q_2' \equiv P_3' - Q_3'$$

and

$$(P_1' - Q_1') + (P_1' - Q_1') \equiv 0.$$

Furthermore, the linear pencil $M(t, x, y) = tI_3 + x\,(a_{ij}) + y\,\mathrm{diag}(b_1, b_2, b_3)$ is unitarily equivalent to the linear pencil via the θ_δ-representation, $\delta = 2$ or 3, in Theorem 3.1 for the hyperbolic form $F_T(t, x, y)$.

If $\delta = 2$ then $P_1' - Q_1' = 1/2$ on the Abel-Jacobi variety, and $P_1' - Q_1' = (1 + \tau)/2$ if $\delta = 3$.

The point $P_1' - Q_1' \equiv P_2' - Q_2' \equiv P_3' - Q_3'$, $2(P_3' - Q_3') = 0$, must be one of the points

$$\{1/2, (1+\tau)/2\}$$

of the normalized Abel-Jacobi variety $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ which corresponds to a point $(t_1, x_1, y_1) = (1, e_j, 0)$ of the cubic curve. Assume $Q_3' = 0$. Then $P_3' = (1, e_j, 0)$, $j = 1, 2$.

A. Hurwitz and R. Courant, 1964

$$\left(\frac{\theta_1'(0)}{2\omega_1\theta_{k+1}(0)}\frac{\theta_{k+1}(u)}{\theta_1(u)}\right)^2 = \wp(2\omega_1 u) - e_k, \quad k = 1, 2, 3.$$

Therefore, θ_2, θ_3-representations.

Example 1

We give an example to illustrate the relationship between the kernel vector functions and the Riemann theta functions for $\eta = 1/2$ and $\eta = (1+\tau)/2$ on the normalized Abel-Jacobi varieties.

$$T = \begin{pmatrix} 3+3i & -2 & 1 \\ -2 & 3+i & 1 \\ 1 & 1 & 2i \end{pmatrix}$$

$$F_T(t,x,y) = t^3 + 6xt^2 + 6yt^2 + 3x^2t + 24xyt + 11y^2t - 10x^3$$
$$+6x^2y + 24xy^2 + 6y^3.$$

Example 1

Changing the variables

$$x_1 = x - \frac{1}{5}y - \frac{1}{10}t, \quad y_1 = y, \quad t_1 = -\frac{2}{5}(t + 2y),$$

$F_T(t, x, y) = 0$ is expressed in the canonical form:

$$y_1^2 = 4x_1^3 - g_2 x_1 - g_3 = 4(x_1 - e_1)(x_1 - e_2)(x_1 - e_3),$$

where

$$g_2 = \frac{63}{4}, \quad g_3 = -\frac{81}{8}, \quad e_1 = \frac{3}{2}, \quad e_2 = \frac{3}{4}, \quad e_3 = -\frac{9}{4}.$$

Example 1

Then the three line L_{12}, L_{13}, L_{23} for the matrix T are respectively given by

$$L_{12} = x + 4y + 2t, \quad L_{13} = -(5x + y + t), \quad L_{23} = -(5x + 3y + t).$$

In this case, the 6 points Q_j and P_k are given as follows:

$$Q_1 : (t_1, x_1, y_1) = (4, 1, 10), Q_2 = (4, 1, -10), Q_3 = (0, 0, 1),$$

$$P_1 : (t_1, x_1, y_1) = (4, 21, 90), P_2 = (4, 21, -90), P_3 = (4, 3, 0).$$

The point Q_3 is the neutral element of the elliptic curve group structure. The point P_3 on the line $y_1 = 0$ satisfies $2P_3 = 0$. Since $P_3 - Q_3 = P_3$ is a point $(t_1, x_1, y_1) = (1, e_2, 0)$, it follows that $\eta = (1 + \tau)/2$ for the matrix T. θ_3-representation.

Example 2

$$T = \begin{pmatrix} 3+3i & 1 & \sqrt{5/2} \\ 1 & 3+i & \sqrt{5/2} \\ \sqrt{5/2} & \sqrt{5/2} & 2i \end{pmatrix}.$$

$$F_T(t,x,y) = t^3 + 6xt^2 + 6yt^2 + 3x^2t + 24xyt + 11y^2t - 10x^3 + 6x^2y + 24xy^2 + 6y^3.$$

Changing the variables

$$x_1 = x - \frac{1}{5}y - \frac{1}{10}t, \quad y_1 = y, \quad t_1 = -\frac{2}{5}(t+2y),$$

$F_T(t,x,y) = 0$ is expressed in the canonical form:

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where

$$g_2 = \frac{63}{4}, \quad g_3 = -\frac{81}{8}, \quad e_1 = \frac{3}{2}, \quad e_2 = \frac{3}{4}, \quad e_3 = -\frac{9}{4}.$$

Example 2

Then the three line L_{12}, L_{13}, L_{23} for the matrix T are respectively given by

$$L_{12} = \frac{1}{2}(5x - 4y - 2t), \quad L_{13} = -\sqrt{5/2}(2x + y + t), \quad L_{23} = -\sqrt{5/2}(2x + 3y + t).$$

In this case, the 6 points Q_j and P_k are given as follows:

$$Q_1 : (t, x, y) = (3, 0, -1), Q_2 = (1, 0, -1, 1), Q_3 = (2, 0, -1),$$

$$P_1 : (t_1, x_1, y_1) = (4, -3, -18), P_2 = (4, -3, 18), P_3 = (2, 3, 0).$$

The point P_3 lies on the pseudo line part of the cubic curve and satisfies $P_3 + P_3 = 0$, that is, the intersection point of the cubic curve and the line $y = 0$ corresponding to the invariant e_1, and thus $\eta = 1/2$. θ_2-representation.

Conclusion

We verify that the conjecture is true for $n = 3$ symmetric matrices.

Given an $n \times n$ symmetric matrix A. The Helton-Vinnikov theorem gives a symmetric matrix S so that $F_A(t, x, y) = F_S(t, x, y)$. The construction of S involves Riemann theta function.

We express the kernel vector function ξ of the linear pencil $tI_n + x\Re(A) + y\Im(A)$ as a function on the Abel-Jacobi variety of the associated elliptic curve of A. The intersection points of the curves $F_A(t, x, y) = 0$ and $\xi = 0$ provide informations for determining the Riemann theta representation.

We have tried $n = 4$. Examples suggest the conjecture is also true for quartic elliptic curves.

Thank you