Toeplitz-Hausdorff Theorem - Convexity and Connectedness

Tin-Yau Tam

Department of Mathematics and Statistics
Auburn University, USA &
University of Nevada, Reno, USA, ttam@unr.edu

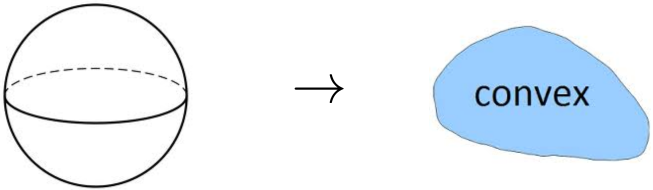
WONRA 2018
Technical University, Münich, Germany

June 13-17, 2018

The *numerical range* of $A \in \mathbb{C}_{n\times n}$ is the set

$$W(A) = \{x^* A x : x \in \mathbb{C}^n, x^* x = 1\} \subset \mathbb{C}.$$

$W(A)$ is the image of the unit sphere of $\mathbb{C}^n$ under the map $x \mapsto x^* A x$.



Theorem (Toeplitz 1918, Hausdorff 1919)

$W(A)$ is convex.

- O. Toeplitz, Das algebraische Analogen zu einem Sätze von Fejer, Math. Z. **2** (1918), 187-197.
- F. Hausdorff, Der Wertvorrat einer Bilinearform, Math. Z. **3** (1919), 314-316.

Das algebraische Analogon zu einem Satze von Fejér.

Von

Otto Toeplitz in Kiel.

Herr Fejér hat[1]) den folgenden Satz bewiesen:

Ist $f(t)$, $g(t)$ ein Paar reeller, stetiger Funktionen der reellen Variabeln t mit der Periode 2π, sind $f_n(t)$, $g_n(t)$ die n-ten arithmetischen Mittel ihrer Fourierschen Entwicklungen, und ist ferner $\mathfrak{K}$ der kleinste konvexe Bereich, der die durch die Parameterdarstellung $x = f(t)$, $y = g(t)$ gegebene Kurve ganz in sich enthält, $\mathfrak{K}_n$ der entsprechende Bereich für die Kurve $x = f_n(t)$, $y = g_n(t)$, so ist $\mathfrak{K}_n$ in $\mathfrak{K}$ enthalten.

Zu diesem Satze soll im folgenden ein Analogon aus der Algebra gewonnen werden. Zur Aufstellung eines solchen Analogons wurde ich durch dasselbe Übertragungsprinzip geführt, durch das ich in meiner Habilitationsschrift[2]) eine Verbindung zwischen der Theorie der Fourierschen Reihen und einer gewissen Klasse bilinearer Formen hergestellt habe. Aber während dieses Übertragungsprinzip wie überhaupt die Theorie der unendlichvielen Veränderlichen bisher meist dazu gedient hat, um gewisse Hilfsmittel der Algebra den Problemen der Analysis nutzbar zu machen, will ich hier an einem einfachen Beispiel zeigen, wie man auch umgekehrt dieses Prinzip verwenden kann, um, von Sätzen der Analysis ausgehend, zu algebraischen Tatsachen zu gelangen.

Die Arbeit setzt die Kenntnis der Theorie der unendlichvielen Veränderlichen nicht voraus, mit Ausnahme des letzten Paragraphen; sie handelt im übrigen nur von Bilinearformen mit n Variabelnpaaren.

---

[1]) Über gewisse durch die Fouriersche und Laplacesche Reihe definierten Mittelkurven und Mittelflächen, Rendiconti del Circolo Matematico di Palermo 38 (1914), S. 79–97.

[2]) Math. Annalen 70 (1911), S. 351–376; Gött. Nachr. 1910, math.-phys. Kl. S. 489–506; Rendiconti del Circolo Matematico di Palermo 32 (1911), S. 191–192.

Der Wertvorrat einer Bilinearform.

Von

Felix Hausdorff in Greifswald.

Herr Toeplitz hat in dieser Zeitschrift kürzlich gezeigt[1]), daß der Wertvorrat W einer Bilinearform, als Punktmenge in der komplexen Zahlenebene, außen von einer konvexen Kurve begrenzt ist, aber die Frage offen gelassen, ob er das ganze Innere dieser Kurve erfüllt. Es ist nun sehr einfach zu zeigen, daß diese Frage zu bejahen, nämlich W selbst eine konvexe Menge ist.

Wenn

$$C = \begin{pmatrix} c_{11} \dots c_{1n} \\ \dots\dots\dots \\ c_{n1} \dots c_{nn} \end{pmatrix}, \qquad C^* = \begin{pmatrix} \bar{c}_{11} \dots \bar{c}_{n1} \\ \dots\dots\dots \\ \bar{c}_{1n} \dots \bar{c}_{nn} \end{pmatrix}$$

eine n-reihige Matrix und ihre begleitende (konjugiert-transponierte) Matrix bedeuten, so sind durch

$$A = \frac{1}{2}(C + C^*), \qquad B = \frac{1}{2i}(C - C^*)$$

oder umgekehrt

$$C = A + iB, \qquad C^* = A - iB$$

zwei Hermitesche Matrizen A, B, die *Komponenten* von C, definiert. Eine lineare Substitution (nichtverschwindender Determinante) mit der Matrix P, die C in

$$\mathfrak{C} = PCP^*$$

überführt, führt gleichzeitig C^* in $\mathfrak{C}^*$ und die Komponenten A, B in die Komponenten $\mathfrak{A}$, $\mathfrak{B}$ über.

Der Wertvorrat W von C ist, nach Herrn Toeplitz, die Menge der komplexen Zahlen w, welche die Form

[1]) O. Toeplitz, Das algebraische Analogon zu einem Satze von Fejér, Math. Zeitschrift 2 (1918), S. 187–197.

Toeplitz and Hausdorff - From Wiki

Otto Toeplitz (1 August, 1881 - 15 February 1940) was a German mathematician working in functional analysis. ... In 1928 Toeplitz succeeded Eduard Study at Bonn University. In 1933, the Civil Service Law came into effect and professors of Jewish origin were removed from teaching. Initially, Toeplitz was able to retain his position due to an exception for those who had been appointed before 1914, but he was nonetheless dismissed in 1935. In 1939 he emigrated to Palestine, where he was scientific advisor to the rector of the Hebrew University of Jerusalem. He died in Jerusalem from tuberculosis a year later.

Felix Hausdorff (November 8, 1868 - January 26, 1942) was a German mathematician who is considered to be one of the founders of modern topology and who contributed significantly to set theory, descriptive set theory, measure theory, function theory, and functional analysis.
Life became difficult for Hausdorff and his family after Kristallnacht in 1938. ... On 26 January 1942, Felix Hausdorff, along with his wife and his sister-in-law, committed suicide by taking an overdose of veronal, rather than comply with German orders to move to the Endenich camp, ...

Some generalized numerical ranges

- (Halmos, Berger 1963) The *k-numerical range* $(1 \leq k \leq n)$

$$W_k(A) = \left\{ \sum_{i=1}^{k} x_i^* A x_i : x_1, \ldots, x_k \text{ are orthonormal in } \mathbb{C}^n \right\}$$

  is convex.

- (Westwick 1975) The *c-numerical range* $(c \in \mathbb{R}^n)$

$$W_c(A) = \left\{ \sum_{i=1}^{n} c_i x_i^* A x_i : x_1, \ldots, x_n \text{ are orthonormal in } \mathbb{C}^n \right\}$$
$$= \{\mathrm{tr}\,(CU^*AU) : U \in \mathrm{U}(n)\}, \text{ where } C = \mathrm{diag}\,(c_1, \ldots, c_n),$$

  is convex.

- (Cheung and Tsing 1996) The *C-numerical range* $(C \in \mathbb{C}_{n\times n})$

$$W_C(A) = \{\mathrm{tr}\,(CU^*AU) : U \in \mathrm{U}(n)\}$$

  is star-shaped.

Westwick's proof

$A = A_1 + iA_2$ Hermitian decomposition.
$C \in \mathbb{C}_{n\times n}$ Hermitian with eigenvalues $c \in \mathbb{R}^n$.
Then $W_c(A)$ can be identified with

$$W_C(A_1, A_2) = \{(\operatorname{tr} CU^*A_1U, \operatorname{tr} CU^*A_2U) : U \in \mathrm{U}(n)\} \subset \mathbb{R}^2.$$

Note that

$$W_C(e^{i\theta}A) = e^{i\theta}W_C(A), \quad \forall\theta \in \mathbb{R}.$$

and

$$e^{i\theta}A = (\cos\theta A_1 - \sin\theta A_2) + i(\sin\theta A_1 + \cos\theta A_2) := (A_1(\theta), A_2(\theta)).$$

So $W_c(e^{i\theta}A)$ is identified with $W_C(A_1(\theta), A_2(\theta))$.

Note that

$$W_C(A) = W_A(C), \quad W_C(U^*AU) = W_C(A),\ \forall U \in \mathrm{U}(n)$$

Hausdorff connectedness argument:

Let $D(n)$ be the subgroup of diagonal matrices in $\mathrm{U}(n)$.

May assume that B and C are diagonal matrices.

Westwick considered $f_{B,C} : \mathrm{U}(n)/D(n) \to \mathbb{R}$

$$f_{B,C}([U]) = \operatorname{tr} CU^*BU,$$

where $B, C \in \mathbb{C}_{n\times n}$ are Hermitian and $[U] = U/D(n)$ for $U \in \mathrm{U}(n)$.

Main ideas in Westwick's proof:

- When B and C have distinct eigenvalues, $f_{B,C}$ is a Morse function and its Hessian has even index at each critical point.
- make use of differential topology to conclude that $f_{B,C}^{-1}(c)$ is connected for any $c \in \mathbb{R}$.
- Claim: Connectedness holds for arbitrary Hermitian B and C.

Poon (1980) gave the first elementary and complete proof of Westwick's result.

Raïs' setting for compact Lie groups

- K **compact** Lie group with Lie algebra $\mathfrak{k}$
- $\langle \cdot, \cdot \rangle$ is a Ad K-invariant inner product on $\mathfrak{k}$.
- (Raïs) For $X, Y, C \in \mathfrak{k}$, the *C-numerical range* of (X, Y) is

$$W_C(X, Y) = \{(\langle X, \mathrm{Ad}\,(k)C \rangle, \langle Y, \mathrm{Ad}\,(k)C \rangle) : k \in K\}.$$

- Adjoint orbit of C:

$$\mathrm{Ad}\,(K)C := \{\mathrm{Ad}\,(k)C : k \in K\}.$$

Theorem (Tam 2002)

Given $X, Y, C \in \mathfrak{k}$, $W_C(X, Y)$ is convex.

Corollary

1. (Westwick) Let $G = \mathrm{U}(n)$ or $\mathrm{SU}(n)$. The C-numerical range

$$W_C(A_1, A_2) = \{(\operatorname{tr} A_1 UCU^*, \operatorname{tr} A_2 UCU^*) : U \in G\}$$

is convex, where A_1, A_2 and C are Hermitian matrices.

2. The set

$$W_C(A_1, A_2) = \{(\operatorname{tr} A_1 OCO^T, \operatorname{tr} A_2 OCO^T) : O \in \mathrm{SO}(n)\}$$

is convex, where A_1, A_2, and C are real skew symmetric matrices.

3. The set

$$W_C(A_1, A_2) = \{(\operatorname{tr} A_1 UCU^*, \operatorname{tr} A_2 UCU^*) : U \in \mathrm{Sp}(n)\}$$

is convex, where $A_1, A_2, C \in \mathfrak{sp}(n)$.

The ingredients in Tam's proof are

- Kirillov-Kostant-Souriau: Co-adjoint orbits are symplectic manifold.
- Atiyah: Let M be a compact connected symplectic manifold and let $f : M \to \mathbb{R}$ whose Hamiltonian vector field is almost periodic. Then all fibres $f^{-1}(c)$ are connected.

The connectedness of the fibres of the map $\pi_C : \mathrm{Ad}\,(K)X \to \mathbb{R}$ defined by

$$\pi_C(Y) = \langle C, Y \rangle, \qquad \text{for all } Y \in \mathrm{Ad}\,(K)X$$

is established. The convexity of $W_C(X_1, X_2)$ then follows through rotation.

• T.Y. Tam, Convexity of generalized numerical range associated with a compact Lie group, J. Austral. Math. Soc. **72** (2002), 57-66.

Markus-Tam's result

Markus-Tam gave another proof of the convexity of $W_C(X_1, X_2)$ via connectedness. Without symplectic technique, they proved

Theorem (Markus-Tam, 2011)

The connectedness of the fibres of $f_{C,X} : K \to \mathbb{R}$ for all $C, X \in \mathfrak{k}$:

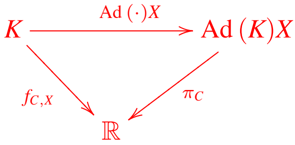
$$f_{C,X}(k) = \langle C, \mathrm{Ad}\,(k)X \rangle, \qquad \text{for all } k \in K.$$

are connected.

Again, the convexity of $W_C(X_1, X_2)$ then follows through rotation.

• A. Markus and T.Y. Tam, Connectedness of some fibers on a compact connected Lie group, Linear and Multilinear Algebra, **59** (2011), 1121–1126.

- Markus-Tam's fibre connectedness result (2011) in K is stronger than Tam's fibre connectedness result (2002) in the adjoint orbit $\mathrm{Ad}\,(K)X$:



since the map $\mathrm{Ad}\,(\cdot)X : K \to \mathrm{Ad}\,(K)X$ is continuous.

- We will give a third convexity proof via a connectedness result (on $\mathrm{Ad}\,(K)X$) of Atiyah and a Hessian index result of Duistermaat, Kolk and Varadarajan.

Complex semisimple Lie algebras and star-shapedness conjecture

- G **complex semisimple** Lie group with Lie algebra $\mathfrak{g}$
- $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ Cartan decomposition
- K connected subgroup of G with Lie algebra $\mathfrak{k}$
- $B_\theta(\cdot, \cdot)$ inner product on $\mathfrak{g}$ induced by the Killing form $B(\cdot, \cdot)$ and θ
- For $X, C \in \mathfrak{g}$, the *C-numerical range* of X is

$$W_C(X) = \{B_\theta(C, \mathrm{Ad}\,(k)X) : k \in K\}.$$

Recall

$$W_C(X) := \{B_\theta(C, \mathrm{Ad}\,(k)X) : k \in K\}, \quad C, X \in \mathfrak{g}.$$

Conjecture (Tam 2001, still open)

Given $C, X \in \mathfrak{g}$, $W_C(X)$ is star-shaped with origin as the star center for all complex semisimple Lie algebras $\mathfrak{g}$.

• T.Y. Tam, On the shape of numerical ranges associated with Lie groups, Taiwanese J. Math., **5** (2001), 497-506.

Development of the conjecture

Theorem (Cheung-Tsing 1996, Djoković-Tam 2003, Cheung-Tam 2011)

$W_C(X)$ is star-shaped for complex simple Lie algebras of type A, B, D, E_6 and E_7.

The conjecture is valid for simple Lie algebras of

1. type A ($\mathfrak{sl}_n(\mathbb{C})$, Cheung-Tsing, 1996).
2. type D, i.e., $\mathfrak{so}_{2n}(\mathbb{C})$, E_6 and E_7 (Djokovic and Tam, 2003).
3. type B, i.e., $\mathfrak{so}_{2n+1}(\mathbb{C})$, type D (Cheung and Tam, 2011).
4. Unknown for type C, i.e., $\mathfrak{sp}_n(\mathbb{C})$, E_8, F_4, G_2.

Question: Is the conjecture true?

• W.S. Cheung and N.K. Tsing, The C-numerical range of matrices is star-shaped, Linear Multilinear Algebra, **41** (1996), 245-250.
• D.Z. Djoković and T.Y. Tam, Some questions about semisimple Lie groups originating in matrix theory, Canad. Math. Bull., **46** (2003), 332-343.
• W.S. Cheung and T.Y. Tam, Star-shapedness and K-orbits in complex semisimple Lie algebras, Canad. Math. Bull., **54** (2011), 44-55.

Setting for real semisimple Lie algebras

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition of **real semisimple** $\mathfrak{g}$
- The Killing form B is positive definite on $\mathfrak{p}$
- G connected Lie group with Lie algebra $\mathfrak{g}$
- K connected subgroup of G with Lie algebra $\mathfrak{k}$
- For $C, X, Y \in \mathfrak{p}$, the *C-numerical range* of (X, Y) is

$$W_C(X, Y) = \{(B(C, \mathrm{Ad}\,(k)X), B(C, \mathrm{Ad}\,(k)Y)) : k \in K\}.$$

Example: When $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$, up to a scalar multiple,

$$W_C(X, Y) = \{(\mathrm{tr}\, COXO^T, \mathrm{tr}\, COYO^T) : O \in \mathrm{SO}(n)\}$$

where $C, X, Y \in \mathbb{R}_{n\times n}$ are symmetric matrices.

Li-Tam's results

Theorem (Li-Tam 2000)

$W_C(X, Y)$ is convex for all classical real simple Lie algebras except $\mathfrak{sl}_2(\mathbb{R})$.

Li-Tam's approach is case-by-case computation.

Questions: Is there a unified proof without case-by-case computation? How about exceptional cases?

• C.K. Li and T.Y. Tam, Numerical ranges arising from simple Lie algebras, J. Canad. Math. Soc., **52** (2000), 141–171.

The classical real simple Lie algebras are isomorphic to one of the following real forms $\mathfrak{h} \subset \mathfrak{g}$ and $\mathfrak{g}^{\mathbb{R}}$ (the realification of $\mathfrak{g}$).

1. $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}), n \geq 2$
   1. $\mathfrak{h} = \mathfrak{sl}_n(\mathbb{R})$
   2. $\mathfrak{h} = \mathfrak{sl}_m(\mathbb{H}), n = 2m$
   3. $\mathfrak{h} = \mathfrak{su}_{p,q}$ $(p = 0, 1, \ldots, [\frac{n}{2}], p + q = n)$
2. $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C}), n \geq 2$
   1. $\mathfrak{h} = \mathfrak{so}_{p,q}$ $(p = 0, 1, \ldots, n, p + q = 2n + 1)$
3. $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C}), n = 2m, m \geq 3$
   1. $\mathfrak{h} = \mathfrak{sp}_{2n}(\mathbb{R}), n = 2m$
   2. $\mathfrak{h} = \mathfrak{sp}_{p,q}, (p = 0, 1, \ldots, [m/2], p + q = m)$
4. $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C}), n \geq 4$
   1. $\mathfrak{h} = \mathfrak{so}_{p,q}, (p = 0, 1, \ldots, n, p + q = 2n)$
   2. $\mathfrak{h} = \mathfrak{so}^*(2n)$.

Example

- $B(X, Y) = \operatorname{tr} XY$
- $\theta(X) = -X^\top$
- $\mathfrak{k} = \mathfrak{so}(2)$ and $K = \mathrm{SO}(2)$
- $\mathfrak{p} = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} : a, b \in \mathbb{R} \right\}$ and $\mathfrak{a} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : a \in \mathbb{R} \right\}$
- Pick $C = X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a}$ and $Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{p}$
- For $k = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K, f_{C,X}(k) = \operatorname{tr} CkXk^{-1} = 2\cos(2\theta)$.
- The fibre $f_{C,X}^{-1}(2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ is not connected.
- $W_C(X, Y)$ is a circle in $\mathbb{R}^2$.

Duistermaat-Kolk-Varadarajan

- For $C, X \in \mathfrak{p}$, define $f_{C,X} : K \to \mathbb{R}$ by

$$f_{C,X}(k) = B(C, \mathrm{Ad}\,(k)X).$$

- One may assume $C, X \in \mathfrak{a}$ since $\mathfrak{p} = \cup_{k \in K} \mathrm{Ad}\,(k)\mathfrak{a}$.
- $W = M'/M$ Weyl group of G, where M centralizer of $\mathfrak{a}$ in K and M' is the normalizer of $\mathfrak{a}$ in K.
- For each $X \in \mathfrak{a}$, let K_X and W_X denote the centralizers of X in K and in W, respectively.

Lemma (DKV 1983)

The critical set of $f_{C,X}$ is $K_{C,X} = K_C W K_X = \bigcup_{w \in W_C \backslash W / W_X} K_C w K_X$, where the union is disjoint.

Theorem (DKV 1983)

Let $k = ux_w v$ with $u \in K_C$, $v \in K_X$, and x_w a representative of w in K. The Hessian H_k of $f_{C,X}$ at k

$$H_k(Z,Z) = -\sum_{\alpha\in\Sigma^+} \alpha(X)(w\cdot\alpha)(C)\|F_\alpha(\mathrm{Ad}\,(v)Z)\|^2, \qquad \forall Z \in \mathfrak{k}$$

where $F_\alpha : \mathfrak{k} \to \mathfrak{k} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ is an orthogonal projection.
In particular, $f_{C,X}$ is a Morse-Bott function and its index at k is

$$\sum_{\alpha\in\Sigma^+, \alpha(X)(w\cdot\alpha)(C)>0} \dim \mathfrak{g}_\alpha .$$

• J.J. Duistermaat, J.A.C. Kolk, and V.S. Varadarajan, Functions, flows and oscillatory integrals on flag manifolds and conjugacy classes in real semisimple Lie groups, Compositio Math., **49** (1983), 309–398.

Example: $\mathfrak{sl}_n(\mathbb{C})$

View $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ as a real semisimple Lie algebra.

- $B(X, Y) = 4n \operatorname{Re} \operatorname{tr} XY$ for all $X, Y \in \mathfrak{g}$.
- The Hermitian decomposition is the Cartan decomposition.
- Let $\mathfrak{a} \subset \mathfrak{p}$ be the subspace of (real) diagonal matrices.
- Root space decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = (\mathfrak{a} \oplus i\mathfrak{a}) \oplus \bigoplus_{i \neq j} \mathbb{C}E_{ij},$$

  where E_{ij} is the matrix with 1 at the (i, j)-entry and 0 elsewhere.
- The root system is $\Sigma = \{e_i - e_j : 1 \leq i \neq j \leq n\}$, where $e_i \in \mathfrak{a}^*$ sends $A \in \mathfrak{a}$ to the i-th diagonal entry of A.

- The Weyl group $W \cong P_n$ group of permutation matrices. The centralizers K_C (resp., K_X) of C (resp., X) in K consists of all matrices in $\mathrm{SU}(n)$ that commute with C (resp., X).
- The critical set of $f_{C,X}$ is $K_{C,X} = K_C P_n K_X$.
- For each $k = UPV$ with $U \in K_C$, $P \in P_n$, and $V \in K_X$, the Hessian of $f_{C,X}$ at k is

$$\begin{aligned} & - \sum_{\alpha \in \Sigma^+} (w \cdot \alpha)(C)\alpha(X) \|F_\alpha(\mathrm{Ad}\,(v)Z)\|^2 \\ = \quad & -8n \sum_{i<j} \mathrm{tr}\, P(c_i E_{ii} - c_j E_{jj})P^{-1}(x_i E_{ii} - x_j E_{jj}) \cdot |(VZV^{-1})_{ij}|^2 \end{aligned}$$

  for any $Z \in \mathfrak{su}(n)$.
- The index of $f_{C,X}$ at k is

$$\begin{aligned} & \sum_{\substack{((PCP^{-1})_{ii} - (PCP^{-1})_{jj})(x_i - x_j) > 0 \\ i<j}} \dim_{\mathbb{R}} \mathbb{C} E_{ij} \\ = \quad & \text{even integer} \end{aligned}$$

Atiyah's connectedness result

Theorem (Atiyah 1982)

Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a compact connected manifold M. If neither f nor $-f$ has a critical manifold of index 1, then $f^{-1}(c)$ is connected (or empty) for every $c \in \mathbb{R}$.

- M.F. Atiyah, Convexity and commuting Hamiltonians, Bull. London Math. Soc., **308** (1982), 1–15.

$\mathfrak{sl}_3(\mathbb{R})$ even index sufficient but not necessary

- $G = \mathrm{SL}_3(\mathbb{R}), \mathfrak{g} = \mathfrak{sl}_3(\mathbb{R}), \theta(X) = -X^\top$
- $\mathfrak{k} = \mathfrak{so}(3), K = \mathrm{SO}(3)$
- $\mathfrak{p}$ = traceless real symmetric matrices
- $\mathfrak{a}$ = traceless real diagonal matrices
- M = real diagonal matrices in $\mathrm{SO}(3)$
- M' = generalized permutation matrices in $\mathrm{SO}(3)$ whose nonzero entries are ± 1
- Weyl group $W = M'/M \cong S_3 \cong P_3$

- Pick $C = \operatorname{diag}(c_1, c_2, c_3) \in \mathfrak{a}_+$ and $X = \operatorname{diag}(x_1, x_2, x_3) \in \mathfrak{a}_+$
- $K_C = K_X = M$, $K_{C,X} = K_C W K_X = M'$
- The index

$$\sum_{\alpha \in \Sigma^+, \alpha(X)(w\cdot\alpha)(C) > 0} \dim \mathfrak{g}_\alpha$$

  is equal to the number of positive roots sent to positive roots by $w \in W$.
- Since $W_X = W_C$ is trivial, each $w \in W$ can appear for some $k \in K_{C,X}$.
- Indices are 3, 2, 2, 0, 1, 1 for the six Weyl group elements.
- Since $\dim K = \dim \mathrm{SO}(3) = 3$, neither even index condition nor Atiyah's condition is satisfied, but $W_C(X, Y)$ is still convex.

• X. Liu and T.Y. Tam, Connectedness, convexity and generalized numerical range, The Natalia Bebiano Anniversary Volume, Textos de Matematica, **44** (2013), 107-119.

Connectedness is sufficient but not necessary: $\mathfrak{sl}_3(\mathbb{R})$

Example

Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$ and let $C = \operatorname{diag}(c, 0, -c)$ with $c > 0$ and $X = \operatorname{diag}(1, 0, -1)$.

Then the image of $f_{C,X} : \mathrm{SO}(3) \to \mathbb{R}$ is the interval $[-2c, 2c] \subset \mathbb{R}$.

$$f_{C,X}^{-1}(2c) = \{I_3, \operatorname{diag}(1, -1, -1), \operatorname{diag}(-1, 1, -1), \operatorname{diag}(-1, -1, 1)\}$$

is evidently disconnected in $\mathrm{SO}(3)$.

However if we consider $\mathfrak{sl}_3(\mathbb{C})$ with the same X and C, the corresponding $f_{C,X}^{-1}(2c)$ is

$$\{\operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) : \theta_1 + \theta_2 + \theta_3 = 0\} \subset \mathrm{SU}(3)$$

which is connected.

Thank You!

Questions?