June 14th, 2018

Determinantal Representations and Numerical Ranges

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based on joint works with Prof. M. T. Chien

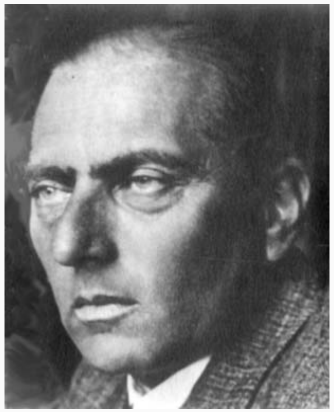
Abstract. In 1981, M. Fiedler posed a question concerning the characterization of the numerical range of a matrix via the hyperbolicity of the Kippenhahn curve. This question was partly solved by himself and generaly affirmatively slved by Helton, Vinnikov in 2007. Some related recent results are introduced in this talk.

Outline of this talk

1. Determinantal representation?

Let A be an $n \times n$ complex matrix. The numerical range of A is defined as

$W(A) = \{\xi^* A \xi : \xi \in \mathbb{C}^n, \xi^* \xi = 1\}$.




Toeplitz, Hausdorff
[P1]

In 1918 Toeplitz introduced this set improving Bendixson's result (1902)
In 1919 Hausdorff proved the convexity of the set $W(A)$.
Kippenhahn (1951) characterize that $W(A)$ is the convex hull of the real affine part of the dual curve of the curve $F_A(t,x,y)=0$, where

$$F_A(t,x,y) = \det(tI_n + x\Re(A) + y\Im(A)),$$
$$\Re(A) = (A+A^*)/2,\ \Im(A) = (A-A^*)/(2i).$$

[P2]

In 2013, Gau, Wu proved that
$F_A(t,x,y) = F_B(t,x,y)$
$\Longleftrightarrow \Lambda_k(A) = \Lambda_k(B)(k = 1, 2, \ldots, n)$.

The homogeneous polynomial $F_A(t,x,y)$ enjoys the property: the equation $F(t, \cos\theta, \sin\theta) = 0$ in t has n real solutions for every angle θ and $F(1,0,0) = 1$. In 1981, Fiedler asked the converse. Let $F(t,x,y)$ be a polynomial satisfying this property. Is there a pair of Hermitian matrices H, K satisfying $F(t,x,y) = \det(tI_n + xH + yK)$? [P3]

Such an expression of F by (H, K) is a determinantal representation of F. In 1990, Fiedler provided a partial answer: If $F(t, x, y) = 0$ is a rational curve, there exists a pair of real symmetric matrices S_1, S_2 satisfying $F(t, x, y) = \det(tI_n + xS_1 + yS_2)$.

[P4]

MiroslavFiedler

(April 7th, 1926 -November 20th, 2015)

[P5]

Related problems of this subject

The numerical range $W(A)$ or higher-rank numerical range $\Lambda_k(A)$

$\Longrightarrow A$ (up to unitary similarity) or

a model of A,

Appli. to the inverse numerical range problem, Fiedler's rational interpolation method

[P6]

2. Adjugate matrix

In 2013, Plaumann, Vinzant proved:

Fiedler's conjecture is true.

We introduce the outline of their proof. Let $F(t, x, y)$ be a non-singular homogeneous polynomial of degree $n \geq 3$ satisfying $\{t \in \mathbb{C} : F(t, \cos\theta, \sin\theta) = 0\} = \{\lambda_n(\theta) < \ldots < \lambda_2(\theta) < \lambda_1(\theta)\}$ ($0 \leq \theta < 2\pi$), $F(1, 0, 0) = 1$. [P7]

We take a degree $n-1$ real homogeneous polynomial $G(t,x,y)$ with $G(1,0,0)>0$ enjoying the interlacing property:

$\{t \in \mathbb{C} : G(t, \cos\theta, \sin\theta) = 0\} =$
$\{\mu_{n-1}(\theta) \leq \ldots \leq \mu_2(\theta) \leq \mu_1(\theta)\}$,

$$\lambda_n(\theta) \leq \mu_{n-1}(\theta) \leq \lambda_{n-1}(\theta) \leq \mu_{n-2}(\theta) \leq \lambda_{n-2}(\theta) \leq \ldots \lambda_2(\theta) \leq \mu_1(\theta) \leq \lambda_1(\theta).$$

The simplicity of their construction: We can choose $G(t,x,y) = F_t(t,x,y)$ however we can choose another interlacing polynomial. [P8]

Divide the $n(n-1)$ intersection points of the two curves $F(t,x,y)=0, G(t,x,y)=0$ in the complex projective plane as $S \cup \overline{S}$. Take a basis $G(t,x,y)$, $G_2(t,x,y),\ldots,G_n(t,x,y)\}$ of linear space of polynomials of degree $n-1$ for which G_j vanish on S. Let G_j^* be the conjugate of G_j:
$G_j = \sum_{k+\ell+m=n-1} \alpha_{k\ell m} t^k x^\ell y^m$,
$G_j^* = \sum_{k+\ell+m=n-1} \overline{\alpha_{k\ell m}} t^k x^\ell y^m$.[P9]

Let $M = \begin{pmatrix} G & G_k \\ G_j^* & G_{jk} \end{pmatrix}$ $(j, k = 2, \ldots, n)$ where $GG_{jk} - G_j^* G_k = F \cdot H_{jk}$ for some degree $n-2$ polynomial H_{jk}. Take the adjugate matrix $L(t, x, y) = L_0 t + L_1 x + L_2 y$ of the matrix polynomial $M = M(t, x, y)$: $L = M(t, x, y)^{-1} * \det(M(t, x, y))$: Then $F(t, x, y) = \det(L_0)^{-1} \det(L_0 t + L_1 x + L_2 y)$ where L_j are Hermitian, L_0: positive definite, hence $F(t, x, y) = \det(tI_n + L_0^{-1/2}(xL_1 + yL_2)L_0^{-1/2})$. [P10]

3. Helton, Vinnikov theorem

Historically P. Lax (1958) conjectured the expression $F(t,x,y)=\det(tI_n+xS_1+yS_2)$ by real symmetric matrices S_1,S_2 in the relation with hyperbolic differential equations. In 2007, Helton, Vinnikov proved:Lax conjecture is true.

Assume that the form $F(t,x,y)$ is irreducible in the polynomial ring.

Assume that $F(t,0,-1)=0$ has n non-zero distinct roots $\beta_1,\ldots,\beta_n$. [P11]

So we may take $S_2 = \mathrm{diag}(\beta_1, \ldots, \beta_n)$. They express the real symmetric matrix $S_1 = (s_{ij})$ by using Riemann-Jacobi theta functions θ_δ on the Riemann surface with genus g where g is the genus of the curve $F(t, x, y) = 0$. The diagonal entry s_{jj} is determined by the implicit function theorem : $\quad s_{jj} = \beta_j \frac{F_x(\beta_j, 0, -1)}{F_y(\beta_j, 0, -1)}$.

[Helton, Spitkovsky(2007)] Let A be an $n \times n$ matrix. There exists an $n \times n$ complex symmetric matrix S with $\Lambda_k(A) = \Lambda_k(S)$. [P12]

4. Elliptic curves

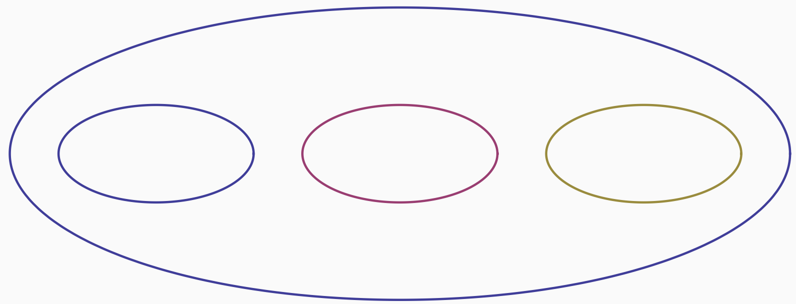
Take a degree n homogeneous polynomial $F(t, x, y)$ in 3 variables t, x, y. If the curve $F(t, x, y) = 0$ has no singular point, the projective curve $\{[(t, x, y)] \in \mathbb{C}P^2 : F(t, x, y) = 0\}$ itself a Riemann surface with genus $g = (n-1)(n-2)/2$. For instance if $n = 4$, the Riemann surface (topologically realized in $\mathbb{R}^3$) has 3 holes. [P13]

In the case the curve $F(t, x, y) = 0$ has singular point. The (complex projective) curve is birationally equivalent to a non-singular curve in a higher-dimensional space by considering "resolutions of singularities".

[P14]

The genus g of the curve $F(t,x,y)=0$ is the number of holes of the new object constructed in this way.

[P15]



triple torus

In the case $g = 1$, the new object is realized as a complex torus $\mathbb{C}/\Gamma$ where Γ is a discrete subgroup of $\mathbb{C}$ isomorphic to $\mathbb{Z}^2$ generated by ω_1, ω_2 which are linearly independent over $\mathbb{R}$. Suppose that $F(t, x, y) = 0$ is an irreducible curve with $g = 1$. By a birational transformation $F(t, x, y) = 0$ is transformed to a non-singular cubic curve $Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3$ and the curve $F(1, x, y) = 0$ is parametrized by

$$x = R_1(\mathcal{P}(s), \mathcal{P}'(s)), \quad y = R_2(\mathcal{P}(s), \mathcal{P}'(s)).$$

[P16]

Theorem[Chien-N. 2018, ELA] The off diagonal entry $s_{jk}(\ j \neq k)$ of S_1 is given as

$$\epsilon(\beta_k - \beta_j)\sqrt{\mathcal{P}(Q_k - Q_j) - e_\delta}$$
$$*[\sqrt{d(R_1/R_2)(Q_j)d(R_1/R_2)(Q_k)}]^{-1}$$

where $\epsilon = \pm 1$, Q_j is the point on the torus $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ corresponding to $(\beta_j, 0, -1)$, $\delta = 1, 2$ and
$4X^3 - g_2X - g_3 = 4(X - e_1)(X - e_2)(X - e_3)$, $e_1 > e_2 > e_3, e_1 + e_2 + e_3 = 0$. [P17]

Weierstrass $\mathcal{P}$-function is a doubly periodic meromorphic function satisfying the differential equation $(\mathcal{P}'(s))^2 = 4\mathcal{P}(s)^3 - g_2\mathcal{P}(s) - g_3$, and it satisfies $\mathcal{P}(s+\omega_1) = \mathcal{P}(s)$, $\mathcal{P}(s+\omega_2) = \mathcal{P}(s)$. This function is defined by

$\mathcal{P}(s) = \frac{1}{s^2} + \sum_{(n,m)\neq(0,0)}\left[\frac{1}{(s-n\omega_1-m\omega_2)^2} - \frac{1}{(n\omega_1+m\omega_2)^2}\right].$

[P18]

5. Symmetric determinantal representations

We shall provide some special matrices which are unitarily similar to symmetric matrices.

(i) Chien-Liu-N-T.Y.Tam (2017)
Toeplitz matrices.

(ii) Chien-N (2015), Unitary bordering matrices, that is, completely non-unitary contractions with defect number 1. [P19]

(iii)Huang-N (2018), Some special cyclic weighted shift matrices.
(iv) Chien-N (2018), Some special cyclic weighted shift matrices.

(iii) For instance $n = 5$, $a - b - c - b - a$,
$n = 6$: $a - b - c - c - b - a$,
$n = 8$: $a - b - c - d - d - c - b - a$
(iv) $n = 6$: $a - b - c - d - c - b$,
$n = 8$: $a - b - c - d - e - d - c - b$. [P20]

If S is a special type matrix in the above list, one of the symmetric matrix $\tilde{S}$ constructed from F_S via the Helton-Vinnikov formula is unitarily similar to the original one S?

Plauman-Sturmfels-Vinzant(2012): If $F_S(t, x, y) = 0$ has no singular point, the answer is "Yes". [P21]

[Chien-N](2018) Let S be a 4×4 unitary bordering matrix with pure imaginary eigenvalues $i/2, i/2, i/2, 0$. Then the curve $F_S(t, x, y) = 0$ is an irreducible elliptic curve. Any of the two symmetric matrices $\tilde{S}$ constructed by the Helton-Vinnikov formula is not unitarily similar to the original S. [P22]

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Thank you

for your attention!

News

M. T. Chien, H. Nakazato: Lecture Notes,

”Matrix Topics and Numerical Ranges”,

105 pages, 15 sections,

Yoshioka Shoten (Kyoto), 20 EURO.