ADDITIVE HERMITIAN IDEMPOTENT PRESERVERS BETWEEN OPERATOR ALGEBRAS

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Abstract. Let L be an additive map between (real or complex) matrix algebras sending $n\times n$ Hermitian idempotent matrices to $m\times m$ Hermitian idempotent matrices. We show that there are nonnegative integers p,q with $n(p+q)=r\le m$ and an $m\times m$ unitary matrix U such that

$$L(A)=U[(I_p\otimes A)\oplus(I_q\otimes A^t)\oplus 0_{m-r}]U^*,\quad \text{for any } n\times n \text{ Hermitian } A \text{ with rational trace.}$$

We also extend this result to the (complex) von Neumann algebra setting, and provide a supplement to the Dye-Bunce-Wright Theorem asserting that every additive map of Hermitian idempotents extends to a Jordan $*$- homomorphism.

1. Introduction

Let $\mathbf{M}_n$ be the set of real or complex $n\times n$ matrices with identity I_n, and let $\mathbf{H}_n=\{A\in\mathbf{M}_n : A=A^*\}$ be the set of Hermitian matrices, and $\mathbf{U}_n=\{A\in\mathbf{M}_n : A^*A=I_n\}$ be the set of unitary matrices in $\mathbf{M}_n$. We say that two matrices A,B in $\mathbf{M}_n$ are *orthogonal* if $A^*B=AB^*=0$, and A is an *idempotent* if $A^2=A$.

There has been considerable interest in the study of preserver problems on matrices, which concerns the characterizations of maps on matrices with some special properties. In early study, researchers often impose linearity or bijective assumption on the maps. For example, Frobenius shows that a linear map $L:\mathbf{M}_n\to\mathbf{M}_n$ satisfies $L(A)=\det(L(A))$ for all $A\in\mathbf{M}_n$ if and only if there are $M,N\in\mathbf{M}_n$ satisfying $\det(MN)=1$ such that L has the form $A\mapsto MAN$ or $A\mapsto MA^tN$. Dieudonné shows that a bijective linear map $L:\mathbf{M}_n\to\mathbf{M}_n$ sends singular matrices to singular matrices if and only if there are invertible $M,N\in\mathbf{M}_n$ such that L has the form $A\mapsto MAN$ or $A\mapsto MA^tN$. One may see [11] and its references for some general background.

In recent study, researchers try to relax the linearity and/or the bijectivity assumption. See, e.g., [10,12]. It follows from the spectral theory that every Hermitian matrix is a real linear combination of Hermitian idempotents, also known as projections. Therefore, a linear map between matrix algebras is determined by its action on Hermitian idempotents. More precisely, let $P(\mathbf{M}_n)$ be the lattice of $n\times n$ Hermitian idempotents (in the positive semi-definite order), and let $\mathrm{span}_{\mathbb{Q}}\, P(\mathbf{M}_n)$ be the rational linear span of Hermitian idempotents in $\mathbf{H}_n$. By [6, Theorem 1] we see that $\mathrm{span}_{\mathbb{Q}}\, P(\mathbf{M}_n)$ consists of those $n\times n$ Hermitian matrices with rational trace. Suppose an additive map $L:\mathbf{H}_n\to\mathbf{H}_m$ sends Hermitian idempotents to Hermitian idempotents. We will see in Theorem 2.1 that L agrees with a Jordan $*$-homomorphism on $\mathrm{span}_{\mathbb{Q}}\, P(\mathbf{M}_n)$.

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We extend this result to the (complex) von Neumann algebra setting. Let $\mathrm{span}_{\mathbb{Q}}\, P(M)$ be the rational span of Hermitian idempotents in a von Neumann algebra M, where $P(M)$ is the lattice of Hermitian idempotents in M. We know that the complex linear span of $P(M)$ is norm dense in M. The seminal Dye Theorem [5] says that if M, N are von Neumann algebras such that M does not contain a direct type I_2 summand, then any bijective map $\Phi : P(M) \to P(N)$ sending orthogonal Hermitian idempotents to orthogonal Hermitian idempotents extends uniquely to a Jordan $*$-isomorphism between the whole algebras. The non-bijective version of the Dye Theorem is provided by Bunce and Wright in [2] (see Theorem 3.8), in which they assume a stronger condition, namely, Φ is an *orthomorphism* in the sense that Φ sends every pair P, Q of orthogonal Hermitian idempotents to orthogonal Hermitian idempotents $\Phi(P), \Phi(Q)$ such that $\Phi(P \vee Q) = \Phi(P) \vee \Phi(Q)$. It is easy to see that Φ is an orthomorphism if and only if it is *orthogonally additive*, that is, $\Phi(P + Q) = \Phi(P) + \Phi(Q)$ whenever P, Q are Hermitian idempotents orthogonal to each other. However, the conclusion does not hold when M contains a direct type I_2 summand, as demonstrated by the following well known example.

Example 1.1. All nontrivial projections in the von Neumann algebra $\mathbf{M}_2$ have rank one, which are in one-to-one correspondence with the elements in the sphere in $\mathbb{R}^3$ of radius $1/2$, that is,

$$P(\mathbf{M}_2) \setminus \{0, I_2\} \cong \left\{ \begin{pmatrix} 1/2 + x & y + iz \\ y - iz & 1/2 - x \end{pmatrix} : x, y, z \in \mathbb{R} \text{ such that } x^2 + y^2 + z^2 = (1/2)^2 \right\}.$$

In this way, the orthogonal complement of

$$P = \begin{pmatrix} 1/2 + x & y + iz \\ y - iz & 1/2 - x \end{pmatrix} \quad \text{is} \quad I_2 - P = \begin{pmatrix} 1/2 - x & -y - iz \\ -y + iz & 1/2 + x \end{pmatrix}.$$

Consider the bijective map $\Phi : P(\mathbf{M}_2) \to P(\mathbf{M}_2)$ fixing every Hermitian idempotent, but exchanging $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ with its orthogonal complement $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. It is clear that Φ preserves orthogonality, and indeed, Φ is orthogonally additive. But the discontinuous map Φ cannot extend to any (continuous) Jordan $*$-homomorphism (indeed, any linear map either) of the whole matrix algebra $\mathbf{M}_2$.

To obtain a full version of our expected result, we need to assume a stronger condition on Φ. It turns out that if Φ is an additive map between the rational spans of the Hermitian idempotents then everything works. We can include the type I_2 case in our main result below, which can be considered as a supplement to both the Dye Theorem and the Bunce-Wright Theorem.

Theorem 1.2. *Let M be a von Neumann algebra and B be a C^*-algebra. Let $L : \mathrm{span}_{\mathbb{Q}}\, P(M) \to \mathrm{span}_{\mathbb{Q}}\, P(B)$ be an additive map sending Hermitian idempotents to Hermitian idempotents. Then L extends to a Jordan $*$-homomorphism $J : M \to B$.*

2. Additive preservers of Hermitian idempotent matrices

In the following, we determine the structure of additive maps that send Hermitian idempotents to Hermitian idempotents. Our approach is different from the one in [3, 13], in which additive (not Hermitian) idempotent preservers are studied through a pure ring theoretical argument. It is also different from [4] in which the additivity is replaced by a weaker assumption, while the surjectivity is assumed.

Theorem 2.1. *Let $L : \mathbf{H}_n \to \mathbf{M}_m$ be an additive map. Then L sends Hermitian idempotent matrices (up to rank 2) to Hermitian idempotent matrices if and only if there are nonnegative integers p, q with $n(p+q) = r \le m$ and there is $U \in \mathbf{U}_m$ such that*

$$L(A) = U[(I_p \otimes A) \oplus (I_q \otimes A^t) \oplus 0_{m-r}]U^*, \quad \textit{for any } A \in \mathbf{H}_n \textit{ with rational trace.} \tag{2.1}$$

Remark 2.2. (a) It is clear that an additive map $L : \mathbf{H}_n \to \mathbf{M}_m$ is exactly rational linear. In many preserver problems, the additive assumption will ensure real linearity. However, it is not the case for Theorem 2.1. On the other hand, a careful reader will see that condition (2.1) is equivalent to the following two separated conditions:

$$L(\alpha I_n) = U(\alpha I_r \oplus 0_{m-r})U^* \text{ for any rational number } \alpha, \tag{2.2}$$

and

$$L(A) = U[(I_p \otimes A) \oplus (I_q \otimes A^t) \oplus 0_{m-r}]U^* \quad \text{for any } A \in \mathbf{H}_n \text{ with zero trace.} \tag{2.3}$$

So, the map L is linear on the real linear subspace of trace zero Hermitian matrices in $\mathbf{M}_n$, and it is rational homogeneous on the identity matrix.

(b) Note that we have no control on the image $L(A)$ when A does not have rational trace. In fact, one can define an additive map $T : \mathbf{H}_n \to \mathbf{M}_m$ by setting $T(A) = L(A) + L_0(A)$, where L_0 is an arbitrary additive map such that $L_0(A) = 0$ whenever $\operatorname{tr} A$ is zero. For instance, we can let $L_0(A) = \tau(\operatorname{tr} A)$, where $\tau : \mathbb{R} \to \mathbf{M}_m$ is any rational linear map such that $\tau(1) = 0$. Clearly, T agrees with L at every Hermitian matrix with rational trace, but $T(A)$ can be arbitrary when A does not have rational trace.

We establish some lemmas to prove Theorem 2.1. For two Hermitian matrices A, B, we write $A \ge B$ if $A - B$ is positive semi-definite.

Lemma 2.3. *If $L : \mathbf{H}_n \to \mathbf{M}_m$ is additive and preserves Hermitian idempotents, then $L(P) \le L(Q)$ whenever $P, Q \in \mathbf{H}_n$ are idempotents satisfying $P \le Q$.*

Proof. If $P, Q \in \mathbf{H}_n$ are idempotents such that $Q \ge P$, then $Q = P + \hat{P}$ for some Hermitian idempotent $\hat{P}$ orthogonal to P. So, $L(Q) = L(P) + L(\hat{P}) \ge L(P)$. ■

Lemma 2.4. *If $L : \mathbf{H}_n \to \mathbf{M}_m$ is additive and sends Hermitian idempotents (up to rank 2) to Hermitian idempotents, then L is real linear on the set of trace zero Hermitian matrices.*

Proof. Suppose x, y are unit vectors orthogonal to each other. Since $L(x^*x+y^*y) = L(x^*x)+L(y^*y)$ is a Hermitian idempotent, the Hermitian idempotents $L(x^*x)$ and $L(y^*y)$ are orthogonal to each other. We may assume that there is $U \in \mathbf{U}_m$ such that $L(xx^*) = U(I_r \oplus 0_s \oplus 0_{m-r-s})U^*$ and $L(yy^*) = U(0_r \oplus I_s \oplus 0_{m-r-s})U^*$. For notation simplicity, we assume that $U = I_m$, $xx^* = E_{11}$, and $yy^* = E_{22}$, where E_{ij} denotes the matrix unit with 1 at the (i,j)th entry and zero elsewhere.

Consider a positive real sequence $\{\alpha_n\} \to 0$. Let $Q_n = \alpha_n(E_{11} - E_{22})$. We will show that $L(Q_n) \to 0$. For each n, let r_n be a rational number with $0 < \alpha_n < r_n \le 2\alpha_n$, let

$$R_n = r_n(E_{11} + E_{22}), \quad \text{and} \quad A_n^{\pm} = \alpha_n(E_{11} - E_{22}) \pm \sqrt{r_n^2 - \alpha_n^2}(E_{12} + E_{21}).$$

Then $(R_n \pm A_n^{\pm})/(2r_n)$ are rank one Hermitian idempotents with range space lying in that of $E_{11}+E_{22}$. By Lemma 2.3,

$$\frac{1}{2r_n}(L(R_n) \pm L(A_n^{\pm})) \leq I_{r+s} \oplus 0_{m-r-s}.$$

Since $L(R_n) = r_n(I_{r+s} \oplus 0_{m-r-s})$, we see that $L(A_n^{\pm}) \in \mathbf{H}_m$ and

$$\pm L(A_n^{\pm}) \leq r_n(I_{r+s} \oplus 0_{m-r-s}).$$

Consequently, $L(Q_n) = \frac{1}{2}(L(A_n^+) + L(A_n^-))$ satisfies

$$\pm L(Q_n) \leq r_n(I_{r+s} \oplus 0_{m-r-s}) \leq 2\alpha_n(I_{r+s} \oplus 0_{m-r-s}).$$

We conclude that $L(\alpha_n(E_{11} - E_{22})) = L(Q_n) \to 0$.

We thus see that for any matrix $A \in \mathbf{H}_n$ of the form $xx^* - yy^*$ for an orthonormal pair x, y and any positive real number α, we can choose a sequence $\{q_n\}$ of positive and rational real numbers converging to α. Then

$$L(\alpha A) = L(q_n A) + L((\alpha - q_n)A) = q_n L(A) + L((\alpha - q_n)A) \to \alpha L(A).$$

Note that every trace zero matrix in $\mathbf{H}_n$ is a real linear combination of Hermitian matrices of the form $xx^* - yy^*$ for an orthonormal set $\{x, y\}$. To see this, if A is Hermitian with $\operatorname{tr} A = 0$, then $A = VDV^*$ for some $V \in \mathbf{U}_n$ and a real diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$ with $\sum_{j=1}^n d_j = 0$. Then we can write $D = \sum_{j=1}^{n-1}(\alpha_j E_{jj} - \alpha_j E_{j+1,j+1})$ for a suitable choice of $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$. So, $A = \alpha_1 A_1 + \cdots + \alpha_{n-1}A_{n-1}$ with $A_j = V(E_{jj} - E_{j+1,j+1})V^* = x_j x_j^* - y_j y_j^*$ for some orthonormal pairs $\{x_j, y_j\}$, $j = 1, \ldots, n-1$. In this way, we see that $L(\alpha A) = \alpha L(A)$ for any $\alpha \in \mathbb{R}$ and trace zero Hermitian matrix A. ∎

Proof of Theorem 2.1. The sufficiency is clear.

To prove the necessity, we note that $\mathbf{H}_n$ is the direct sum of the real linear subspace of trace zero matrices and the one dimensional span of the identity matrix. Indeed, $A = (A - \operatorname{tr}(A)I_n) + \operatorname{tr}(A)I_n$ is the decomposition of any matrix A. We may consider the real linear map $\tilde{L} : \mathbf{H}_n \to \mathbf{M}_m$ defined by $\tilde{L}(A) = L(A)$ if $A \in \mathbf{H}_n$ with $\operatorname{tr} A = 0$ and $\tilde{L}(\alpha I_n) = \alpha L(I_n)$ for any $\alpha \in \mathbb{R}$.

Assume that the underlying field is the complex numbers. Extend $\tilde{L}$ to a complex linear map from $\mathbf{M}_n$ into $\mathbf{M}_m$ by setting $\tilde{L}(H + iG) = \tilde{L}(H) + i\tilde{L}(G)$ for $H, G \in \mathbf{H}_n$. By the spectral theory, every square complex matrix is the complex linear sum of two Hermitian matrices, and every Hermitian matrix is the orthogonal real linear sum of orthogonal rank one Hermitian idempotents. Since $\tilde{L}$ sends orthogonal rank one Hermitian idempotents to orthogonal idempotents, it preserves orthogonal Hermitian idempotents. Consequently, $\tilde{L}$ preserves squares of matrices. Therefore, $\tilde{L}$ is a Jordan $*$-homomorphism, and indeed a direct sum of an algebra $*$-homomorphism and an algebra $*$-anti-homomorphism sending $\mathbf{M}_n$ into $\mathbf{M}_m$. It follows that $\tilde{L}$ has the asserted form $A \mapsto U[(I_p \otimes A) \oplus (I_q \otimes A^t) \oplus 0_{m-p-q}]U^*$. Since L and $\tilde{L}$ agree on any Hermitian idempotent, $L(A) = \tilde{L}(A)$ for any $A \in \mathbf{H}_n$ with rational trace as stated in (2.1).

The case when the underlying field is the real numbers demands an other approach. Below, we borrow an idea from the proof of [10, Theorem 4.10]. We now assume that $\tilde{L}$ is a real linear map sending real $n \times n$ matrices to $m \times m$ matrices, which sends orthogonal real symmetric idempotents to orthogonal Hermitian idempotents. Since every real symmetric matrix is a real linear sum of orthogonal real symmetric idempotents, we see that $\tilde{L}$ preserves zero products; namely, $\tilde{L}(A)\tilde{L}(B) = 0$ whenever A, B are real symmetric matrices with $AB = 0$. Without loss of generality, let $\tilde{L}(I_n) = I_s \oplus 0_{m-s}$. We can further assume that $m = s$, $\tilde{L}(I_n) = I_s$ and

$$\tilde{L}(E_{ii}) = 0_{k_1} \oplus \cdots \oplus I_{k_i} \oplus 0_{k_{i+1}} \oplus \cdots \oplus 0_{k_n}, \quad i = 1, \dots, n.$$

Here, $k_1 + k_2 + \cdots + k_n = s$.

Let

$$B = \tilde{L}(E_{12} + E_{21}) = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \oplus 0_{s'},$$

where B_{ij} are $k_i \times k_j$ matrices for $i, j = 1, 2$, and $s' = s - k_1 - k_2$. For any nonzero real γ, consider

$$X_1 = \begin{pmatrix} \gamma & 1 \\ 1 & 1/\gamma \end{pmatrix} \oplus 0_{n-2} \quad \text{and} \quad X_2 = \begin{pmatrix} 1/\gamma & -1 \\ -1 & \gamma \end{pmatrix} \oplus 0_{n-2}.$$

Because $X_1X_2 = 0_n$, we see that

$$\begin{aligned}
0_s &= \tilde{L}(X_1)\tilde{L}(X_2) = (\tilde{L}(\gamma E_{11} + E_{22}/\gamma) + B)(\tilde{L}(E_{11}/\gamma + \gamma E_{22}) - B) \\
&= \left[\begin{pmatrix} \gamma I_{k_1} & 0 \\ 0 & I_{k_2}/\gamma \end{pmatrix} \oplus 0_{s'} + B\right]\left[\begin{pmatrix} I_{k_1}/\gamma & 0 \\ 0 & \gamma I_{k_2} \end{pmatrix} \oplus 0_{s'} - B\right] \\
&= \begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'} - B^2 - \begin{pmatrix} \gamma B_{11} & \gamma B_{12} \\ B_{21}/\gamma & B_{22}/\gamma \end{pmatrix} \oplus 0_{s'} + \begin{pmatrix} B_{11}/\gamma & \gamma B_{12} \\ B_{21}/\gamma & \gamma B_{22} \end{pmatrix} \oplus 0_{s'} \\
&= \begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'} - B^2 - \begin{pmatrix} (\gamma - 1/\gamma)B_{11} & 0 \\ 0 & (1/\gamma - \gamma)B_{22} \end{pmatrix} \oplus 0_{s'}.
\end{aligned}$$

Since this is true for all nonzero real γ, we see that B_{11} and B_{22} are zero blocks. Because the $(1,1)$ and $(2,2)$ blocks of B are zero, we get

$$B_{12}B_{21} = I_{k_1} \quad \text{and} \quad B_{21}B_{12} = I_{k_2}.$$

Hence, $k_1 = k_2$ and $B_{21} = B_{12}^{-1} = B_{12}^*$, noting that $B = \tilde{L}(E_{12} + E_{21})$ is a Hermitian unitary. Similarly, we get all $k_1 = k_2 = \cdots = k_n$, and we set this common value to be k. It follows $s = nk$.

Now, we may replace $\tilde{L}$ with the map $(B_{12}^* \oplus I_k \oplus I_{s-2k})\tilde{L}(X)(B_{12} \oplus I_k \oplus I_{s-2k})$ so that B_{12} is changed to I_k. Consequently, we can assume

$$B = \tilde{L}(E_{12} + E_{21}) = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix} \oplus 0_{s-2k}.$$

In a similar manner, we can assume, up to similarity,

$$\tilde{L}(E_{1j} + E_{j1}) = (E_{1j} + E_{j1}) \otimes I_k \quad \text{for all } j = 1, \dots, n.$$

Notice that E_{11} and $E_{ij} + E_{ji}$ are orthogonal for all $i, j = 2, \dots, n$. It follows that $\tilde{L}(E_{11}) = I_k \oplus 0_{s-k}$ and $\tilde{L}(E_{ij} + E_{ji})$ are orthogonal for all $i, j = 2, \dots, n$. Consequently, all $\tilde{L}(E_{ij} + E_{ji})$ are contained in $(0_k \oplus I_{s-k})M_s(0_k \oplus I_{s-k})$.

With an induction argument we can show that

$$\tilde{L}(E_{ij} + E_{ji}) = (E_{ij} + E_{ji}) \otimes I_k \quad \text{for all } i, j = 1, \dots, n.$$

After a permutation similarity, we can assume instead

$$\tilde{L}(E_{ij} + E_{ji}) = I_k \otimes (E_{ij} + E_{ji}) \quad \text{for all } i, j = 1, \dots, n.$$

Since $\{E_{ij}+E_{ji} : i,j=1,\ldots,n\}$ is a basis for the real symmetric $n\times n$ matrices, we arrive at the asserted representation (2.1) in which $A^t = A$ since all A are real symmetric. ■

Corollary 2.5. *Let $L : \mathbf{H}_n \to \mathbf{M}_n$ be a nonzero additive map. Then L sends Hermitian idempotents (up to rank 2) to Hermitian idempotents if and only if there is $U \in \mathbf{U}_n$ such that L assumes either the form*

$$A \mapsto UAU^* \quad \text{or} \quad A \mapsto UA^tU^*, \quad \text{for all } A \in \mathbf{H}_n \text{ with rational trace.}$$

In view of Remark 2.2, we may rewrite the necessary condition of the corollary as: $L(I) = I$ and there is $U \in \mathbf{U}_m$ satisfying either

(1) $L(A) = UAU^*$ for all trace zero $A \in \mathbf{H}_n$, or (2) $L(A) = UA^tU^*$ for all trace zero $A \in \mathbf{H}_n$.

3. Additive Hermitian idempotent preservers of von Neumann algebras

Let M be a (complex) von Neumann algebra (resp. C^*-algebra), which can be considered as a $*$-subalgebra of the $*$-algebra $B(H)$ of bounded linear operators on a complex Hilbert space H closed in the weak operator topology (resp. norm topology). When H has finite dimension, M is a direct sum of matrix algebras. Let $P(M)$ and M_{sa} denote the set of Hermitian idempotents and the self-adjoint part of M, respectively.

Let M be a von Neumann algebra, let B be a C^*-algebra, and let $L : M_{\mathrm{sa}} \to B$ be an additive map. Assume that L sends Hermitian idempotents in M to Hermitian idempotents in B. We want to study the structure of L. Our expectation is that L extends to a (complex linear) Jordan $*$-homomorphism $J : M \to B$.

Let $\operatorname{span}_{\mathbb{Q}} P(M)$ be the rational linear span of the Hermitian idempotents in M. We can consider $\operatorname{span}_{\mathbb{Q}} P(M)$ as a $\mathbb{Q}$-linear subspace of M_{sa}. Similarly, $\operatorname{span}_{\mathbb{Q}} P(B)$ is a $\mathbb{Q}$-linear subspace of B_{sa}, and also B. Then any $\mathbb{Q}$-linear map $L : M_{\mathrm{sa}} \to B$ factorizing through the $\mathbb{Q}$-quotient spaces $M_{\mathrm{sa}}/\operatorname{span}_{\mathbb{Q}} P(M)$ and $B/\operatorname{span}_{\mathbb{Q}} P(B)$ is an additive map preserving Hermitian idempotents. There are indeed many of them.

Example 3.1. (a) Let M be any nonzero von Neumann algebra. Let $\mathcal{B}_P$ be a Hamel basis of the $\mathbb{Q}$-linear space $\operatorname{span}_{\mathbb{Q}} P(M)$, and let $\mathcal{B}$ be any Hamel basis of the $\mathbb{Q}$-linear space M_{sa} extending $\mathcal{B}_P$. Suppose $l : \mathcal{B} \to \mathcal{B}$ is any map fixing every basic element in $\mathcal{B}_P$. Then l induces a $\mathbb{Q}$-linear map $L : M_{\mathrm{sa}} \to M$ sending Hermitian idempotents to Hermitian idempotents. For such a map l, choose a sequence $\{v_k\}$ from $\mathcal{B}\setminus\mathcal{B}_P$ and redefine $l(v_k) = kv_k$ for $k = 1, 2, \ldots$. In this way, l defines an unbounded additive Hermitian idempotent preserver L of M_{sa}.

(b) Let $\mathfrak{U}$ be a free ultrafilter on the natural numbers $\mathbb{N}$. Consider the bounded linear functional $L : \ell^\infty \to \mathbb{C}$ of the abelian von Neumann algebra ℓ^∞ of bounded scalar sequences sending (x_n) to $\lim_{\mathfrak{U}} x_n$. A Hermitian idempotent of ℓ^∞ is the indicator function $\mathbf{1}_A$ of a subset A of $\mathbb{N}$. Now, $L(\mathbf{1}_A) = 1$ exactly when $A \in \mathfrak{U}$, and it is zero otherwise. Therefore, L sends Hermitian idempotents to Hermitian idempotents. Note that L is not a normal state, that is, it is not continuous for the weak* topology of ℓ^∞.

(c) Consider a commutative von Neumann algebra $M = L^\infty(\Omega,\mu)$. The Hermitian idempotent lattice $P(M)$ of M consists of indicator functions $\mathbf{1}_X$ of measurable subsets X of Ω. Moreover,

$\mathrm{span}_{\mathbb{Q}}\, P(M)$ consists of all measurable functions which assume finitely many rational values, or equivalently those Hermitian elements with finite rational spectrum.

Let $L : L^\infty(\Omega, \mu)_{\mathrm{sa}} \to L^\infty(\Omega', \mu')$ be an additive map sending Hermitian idempotents to Hermitian idempotents. In other words, L sends indicator functions $\mathbf{1}_X$ of measurable sets X in Ω to indicator functions $\mathbf{1}_Y$ of measurable sets Y in Ω'. In this way, L induces a map Φ between the underlying σ-algebras such that $L(\mathbf{1}_X) = \mathbf{1}_{\Phi(X)}$ for every measurable set X in (Ω, μ). Here, $\Phi(X)$ is a measurable set in (Ω', μ') determined uniquely up to a measure zero error. Indeed, Φ preserves null sets, differences, finite unions and finite intersections, all up to measure zero errors. But Φ might not preserve countable unions or countable intersections, as shown in the example in (b) in which $\Phi(\mathbb{N}) = \Omega'$ while $\Phi(\{n\}) = \emptyset$ for $n = 1, 2, \ldots$.

We can define an algebra $*$-homomorphism $J : L^\infty(\Omega, \mu) \to L^\infty(\Omega', \mu')$ which extends the correspondence $J(\mathbf{1}_X) = \mathbf{1}_{\Phi(X)}$ by (complex) linearity and (norm) continuity. Clearly, L agrees with J on $\mathrm{span}_{\mathbb{Q}}\, P(L^\infty(\Omega, \mu))$; namely, $L(f) = J(f)$ whenever f assumes essentially finitely many rational values.

(d) In general, let B be a C^*-algebra and let $L : L^\infty(\Omega, \mu)_{\mathrm{sa}} \to B$ be an additive map sending Hermitian idempotents to Hermitian idempotents. Arguing as above, we see that L induces a rational linear map from $\mathrm{span}_{\mathbb{Q}}\, P(L^\infty(\Omega, \mu))$ into $\mathrm{span}_{\mathbb{Q}}\, P(B)$ preserving squares. Extending this map by linearity and continuity, we have an algebra $*$-homomorphism $J : L^\infty(\Omega, \mu) \to B$ which agrees with L at every function f in $L^\infty(\Omega, \mu)_{\mathrm{sa}}$ assuming essentially finitely many rational values.

We note that $M_{\mathrm{sa}} = \overline{\mathrm{span}}_{\mathbb{Q}} P(M)$ is the norm closure of the rational linear space spanned by the Hermitian idempotents in M. The following well-known lemma helps us to tell which additive Hermitian idempotent preserver arises from a Jordan $*$-homomorphism.

Lemma 3.2. *Let $T : E \to F$ be an additive map between two real normed space. The following conditions are equivalent.*

(a) *T is (locally) bounded, that is, T sends the unit ball of E into a norm bounded set in F.*

(b) *T is norm-norm continuous.*

In this case, T is real linear.

Recall that a C^*-algebra A has *real rank zero* when every Hermitian element in A can be approximated in norm by Hermitian elements with finite spectrum [1]. In particular, von Neumann algebras have real rank zero.

Lemma 3.3. *Let A, B be C^*-algebras such that A has real rank zero. Let $L : A_{\mathrm{sa}} \to B$ be an additive, or equivalently, a $\mathbb{Q}$-linear map. If L sends Hermitian idempotents to Hermitian idempotents, then L extends to a (complex linear) Jordan $*$-homomorphism $J : A \to B$ exactly when L is bounded.*

Proof. We provide here an easy proof for completeness. We only need to consider the case L is a bounded additive Hermitian idempotent preserver. If p, q are orthogonal Hermitian idempotents in A, then all $L(p)$, $L(q)$ and $L(p+q)$ are Hermitian idempotents in B. It amounts to saying that $L(p)L(q) + L(q)L(p) = 0$. Since both $L(p), L(q)$ are Hermitian idempotents, we have

$$-L(q)L(p) = L(p)L(q) = L(p)(L(p)L(q)) = -L(p)(L(q)L(p)) = (L(q)L(p))L(p) = L(q)L(p) = 0.$$

In other words, $L(p), L(q)$ are orthogonal to each other. For any Hermitian x in A_{sa}, we can approximate x in norm by a rational linear sum of orthogonal Hermitian idempotents $\sum_k \alpha_k p_k$. Since

$$[L(\sum_k \alpha_k p_k)]^2 = \sum_k \alpha_k^2 L(p_k) + \sum_{r\neq s} \alpha_r \alpha_s L(p_r)L(p_s) = \sum_k \alpha_k^2 L(p_k) = L(\sum_k \alpha_k^2 p_k),$$

we see that $L(x^2) = L(x)^2$. We also note that, L is real linear and sends Hermitian elements to Hermitian elements. Define $J : A \to B$ by $J(x+iy) = L(x) + iL(y)$ for $x, y \in A_{\mathrm{sa}}$. It is easy to see that J is a Jordan $*$-homomorphism extending L. ∎

Lemma 3.4. *For a von Neumann algebra M without finite type* I *summand, any additive Hermitian idempotent preserver $L : M_{\mathrm{sa}} \to B$ into a C^*-algebra B extends to a Jordan $*$-homomorphism $J : M \to B$.*

Proof. By a result of Goldstein and Paszkiewicz [7, Theorem 3(3)], any Hermitian element of M of norm not greater than one can be represented in the form

$$p_1 + \cdots + p_{12} - p_{13} - \cdots - p_{24}$$

for some (maybe zero) Hermitian idempotents $p_1, \ldots, p_{24}$ in M. For any x in M_{sa}, we see that x/α is an algebraic sum of 24 Hermitian idempotents for any rational number $\alpha \geq \|x\|$. Consequently, $L(x) = \alpha L(x/\alpha)$ has norm not greater than 24α. Therefore, M_{sa} coincides with the $\mathbb{Q}$-linear span $\mathrm{span}_{\mathbb{Q}}\, P(M)$ of its Hermitian idempotents and L is an additive map from M_{sa} into B of norm bounded by 24. It follows from Lemma 3.3 that L extends to a Jordan $*$-homomorphism from M into B. ∎

Recall that we can embed a C^*-algebra B into $B(K)$ as a C^*-subalgebra for some complex Hilbert space K, by the Gelfand-Naimark theorem.

Corollary 3.5. *Let H, K be complex Hilbert spaces of infinite dimension. Let $L : B(H)_{\mathrm{sa}} \to B(K)$ be a nonzero additive map preserving Hermitian idempotents. Then $\dim H \leq \dim K$, and thus we can consider H as a closed subspace of K. Moreover, there are a unitary U and orthogonal projections I_r, I_s in $B(K)$ such that in the operator block matrix form*

$$L(A) = U \begin{bmatrix} A \otimes I_r & 0 & 0 \\ 0 & A^t \otimes I_s & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \quad \forall A \in B(H)_{\mathrm{sa}}.$$

Here, A^t is the transpose of A with respect to some fixed orthonormal basis of K.

Proof. By Lemma 3.4 we see that L extends to a Jordan $*$-homomorphism $J : B(H) \to B(K)$. Since $B(H)$ is simple, we see that J is injective and thus $\dim H \leq \dim K$. It is well known that $J = \Phi_1 + \Phi_2$ is a sum of an injective algebra $*$-homomorphism Φ_1 and an injective algebra *-anti-homomorphism Φ_2, which have orthogonal ranges, and every $*$-algebra homomorphism of $B(H)$ is a direct sum of inner homomorphisms (see, e.g., [12, Section 2]). The assertion follows. ∎

We now consider the case when M is a finite type I_n factor.

Lemma 3.6. *Let $n > 1$ and $L : \mathbf{H}_n \to B(K)$ be a nonzero additive Hermitian idempotent preserver for some Hilbert space K. Then $\dim K \geq n$ and there is a unitary U and orthogonal Hermitian idempotents I_r, I_s in $B(K)$ such that in the operator matrix block form*

$$L(A) = U \begin{bmatrix} A \otimes I_r & 0 & 0 \\ 0 & A^t \otimes I_s & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \quad \forall A \in \operatorname{span}_{\mathbb{Q}} P(\mathbf{M}_n).$$

Proof. By (the proof of) Lemma 2.4, we see that L is real linear on the real linear subspace of $\mathbf{H}_n$ of Hermitian matrices with zero trace. The conclusion follows from the proof of Theorem 2.1. ∎

Recall that for any von Neumann algebra M of type I_n with center $Z(M)$ for any finite integer $n \geq 2$, we can think of $M \cong Z(M)\overline{\otimes}\mathbf{M}_n \cong \mathbf{M}_n(Z(M))$. The commutative von Neumann algebra $Z(M)$ is in general a direct sum of $L^\infty(\Omega_j, \mu_j)$ spaces for some (can be uncountably many) localizable measure spaces (Ω_j, μ_j). Therefore, we can write the type I_n von Neumann algebra $M \cong \bigoplus_j \mathbf{M}_n(L^\infty(\Omega_j, \mu_j))$.

An element $A(\omega) = [a_1(\omega)\ a_2(\omega)\ \ldots\ a_n(\omega)]$ in $\mathbf{M}_n(L^\infty(\Omega, \mu))$ is a measurable $n \times n$ matrix valued function with measurable column vectors $a_1, a_2, \ldots, a_n$ on the (not necessarily localizable) measure space (Ω, μ). Note that when (Ω, μ) is not localizable, $L^\infty(\Omega, \mu)$ might not be a von Neumann algebra since it might not have a predual, i.e., $L^1(\Omega, \mu)^* \neq L^\infty(\Omega, \mu)$. However, $L^\infty(\Omega, \mu)$ is still a commutative C^*-algebra full of Hermitian idempotents.

Lemma 3.7. *Let B be a C^*-algebra, let (Ω, μ) be a measure space and let n be an integer $n \geq 2$. Let $L : \mathbf{M}_n(L^\infty(\Omega, \mu))_{\mathrm{sa}} \to B$ be an additive map sending Hermitian idempotents to Hermitian idempotents. Then there is a Jordan $*$-homomorphism $J : \mathbf{M}_n(L^\infty(\Omega, \mu)) \to B$ such that $J(A) = L(A)$ whenever $A \in \operatorname{span}_{\mathbb{Q}} P(\mathbf{M}_n(L^\infty(\Omega, \mu)))$.*

Proof. Let $A(\omega) = [a_1(\omega)\ a_2(\omega)\ \ldots\ a_n(\omega)]$ in $\mathbf{M}_n(L^\infty(\Omega, \mu))$ be a measurable $n \times n$ Hermitian matrix valued function on (Ω, μ). By [9, Theorem 2.4] (although it is stated for real symmetric random matrices, the statement and the proof work also for the complex Hermitian matrix case), we can order the real eigenvalues $d_1(\omega) \leq d_2(\omega) \leq \cdots \leq d_n(\omega)$ of $A(\omega)$ as measurable functions on Ω with corresponding measurable eigenvector functions $u_1(\omega), u_2(\omega), \ldots, u_n(\omega)$ which form an orthonormal basis of $\mathbb{C}^n$ for every ω in Ω. Then $U(\omega) = [u_1(\omega), u_2(\omega), \ldots, u_n(\omega)]$ is a unitary element in $\mathbf{M}_n(L^\infty(\Omega, \mu))$.

Arguing as in the proof of Lemma 2.4, if $A(\omega)$ is Hermitian with $\operatorname{tr} A(\omega) = 0$ almost everywhere on Ω, then we can write $A = \alpha_1 A_1 + \cdots + \alpha_{n-1} A_{n-1}$ with $A_j = U(\alpha_j E_{jj} - \alpha_j E_{j+1,j+1})U^* = \alpha_j(x_j x_j^* - y_j y_j^*)$ for some $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$, and measurable orthonormal pairs of $\mathbb{C}^n$ vector valued functions $\{x_j, y_j\}$ on (Ω, μ), for $j = 1, \ldots, n-1$. Moreover, we can show that L is homogeneous on all A_j's, and thus real linear on the set of matrix-valued functions on Ω with zero central traces almost everywhere.

On the other hand, consider the additive map $L_a : L^\infty(\Omega, \mu)_{\mathrm{sa}} \to B$ defined by $L_a(k) = L(k \otimes I_n)$. Then L_a sends Hermitian idempotents to Hermitian idempotents. By Example 3.1(d), we have an algebra $*$-homomorphism $J_a : L^\infty(\Omega, \mu) \to B$ agreeing with L_a at every function in $L^\infty(\Omega, \mu)_{\mathrm{sa}}$ assuming essentially finitely many rational values.

Let $A \in \operatorname{span}_{\mathbb{Q}} P(\mathbf{M}_n(L^\infty(\Omega,\mu)))$. It amounts to say that the central trace $g(\omega) = \operatorname{tr} A(\omega)$ of A assumes essentially finitely many rational values on Ω. Then $C(\omega) = A(\omega) - \frac{g(\omega)}{n} I_n$ is an element in $\mathbf{M}_n(L^\infty(\Omega,\mu))_{\mathrm{sa}}$ of zero central trace. Define $\tilde{L} : \mathbf{M}_n(L^\infty(\Omega,\mu))_{\mathrm{sa}} \to B_{\mathrm{sa}}$ by $\tilde{L}(C) = L(C)$ whenever C has almost everywhere zero central trace, and $\tilde{L}(k \otimes I_n) = J_a(k) L(\mathbf{1}_\Omega \otimes I_n)$ for any real valued function $k \in L^\infty(\Omega,\mu)_{\mathrm{sa}}$. It is clear that $\tilde{L}$ is bounded, real linear and preserves Hermitian idempotents. Thus $\tilde{L}$ extends to the desired Jordan $*$-homomorphism $J : \mathbf{M}_n(L^\infty(\Omega,\mu)) \to B$. ∎

Although an additive Hermitian idempotent preserver $L : M_{\mathrm{sa}} \to B$ might not arise from a Jordan $*$-homomorphism when M has a nonzero finite type I summand, it is very close to the case as suggested by Lemmas 3.6 and 3.7. Indeed, we have the following Dye-Bunce-Wright Theorem. Here, an *orthomorphism* is a map Φ between Hermitian idempotent lattices such that Φ sends orthogonal Hermitian idempotents p, q to orthogonal Hermitian idempotents $\Phi(p), \Phi(q)$ such that $\Phi(p \vee q) = \Phi(p) \vee \Phi(q)$; in other words, Φ is orthogonally additive in the sense that $\Phi(p+q) = \Phi(p) + \Phi(q)$ for orthogonal Hermitian idempotents p, q.

Theorem 3.8 ([2], see also [8, Theorem 8.1.1])**.** *Let M be a von Neumann algebra without type* I_2 *direct summand and B be a C^*-algebra. For every orthomorphism $\Phi : P(M) \to P(B)$, there is a Jordan $*$-homomorphism $J : M \to B$ extending Φ.*

While some of our previous results can be derived from Theorem 3.8, it is known that Theorem 3.8 does not hold if M has a type I_2 direct summand (see Example 1.1). Recall that for a general von Neumann algebra M, there is a central Hermitian idempotent $z_{\mathrm{I_f}}$ (resp. z_{I_2}) in M such that the weak* closed ideal $z_{\mathrm{I_f}} M$ (resp. $z_{\mathrm{I}_2} M$) has finite type $\mathrm{I_f}$ (resp. type I_2), while the weak* closed ideal $(1 - z_{\mathrm{I_f}})M$ (resp. $(1 - z_{\mathrm{I}_2})M$) has no finite type I (resp. type I_2) summand.

Theorem 3.9. *Let M be a von Neumann algebra and B be a C^*-algebra. Let $L : M_{\mathrm{sa}} \to B$ be an additive map sending Hermitian idempotents to Hermitian idempotents. Then there is a (complex linear) Jordan $*$-homomorphism $J : M \to B$ agreeing with L on the non finite type* I *part; namely,*

$$L((1 - z_{\mathrm{I_f}})x) = J((1 - z_{\mathrm{I_f}})x), \quad \forall x \in M_{\mathrm{sa}}.$$

In general, we have

$$L(x) = J(x), \quad \forall x \in \operatorname{span}_{\mathbb{Q}} P(M).$$

Proof. Suppose first that M has no type I_2 direct summand. We note that L restricts to an orthomorphism $\Psi : P(M) \to P(B)$ between the Hermitian idempotent lattices, i.e., $\Psi(p+q) = \Psi(p) + \Psi(q)$ is an orthogonal sum of Hermitian idempotents whenever p, q are orthogonal Hermitian idempotents. Consequently, Ψ sends orthogonal Hermitian idempotents to orthogonal Hermitian idempotents. By Theorem 3.8, Ψ extends to a Jordan $*$-homomorphism $J : M \to B$. It is then clear that J and L agree on $\operatorname{span}_{\mathbb{Q}} P(M)$. Moreover, by Lemma 3.4, J and L agree on the non finite type I part $(1 - z_{\mathrm{I_f}})M$.

Suppose next $M \cong \mathbf{M}_2(Z(M))$ is a type I_2 von Neumann algebra with center $Z(M) \cong C(\Omega)$ for some hyperstonean space Ω. Let $A \in \mathbf{M}_2(C(\Omega))_{\mathrm{sa}}$ have central trace $\operatorname{tr} A(\omega) = 0$ on Ω. Write

$$A(\omega) = \begin{bmatrix} a(\omega) & b(\omega) \\ \overline{b(\omega)} & -a(\omega) \end{bmatrix}, \quad \forall \omega \in \Omega,$$

where a, b are continuous scalar functions on Ω such that $a(\omega) \in \mathbb{R}$ for all $\omega \in \Omega$. Since all entry functions of A are continuous on the compact space Ω, by adding a zero trace constant Hermitian matrix function kT to A for some big integer $k > 0$, where T is the constant field $\omega \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we can assume that $A(\omega) + kT(\omega)$ is always invertible with a continuous eigenvalue function $\alpha(\omega) = \sqrt{a(\omega)^2 + |k + b(\omega)|^2}$ while the other one is $-\alpha(\omega)$, such that

$$|k + b(\omega)| \geq 1, \quad \forall \omega \in \Omega.$$

We can choose two continuous orthonormal vector fields $u_1, u_2 : \Omega \to \mathbb{C}^2$ such that both $u_1(\omega), u_2(\omega)$ are eigenvectors of $A(\omega) + kT(\omega)$ on Ω. For example,

$$u_1(\omega) = \frac{1}{\sqrt{(a(\omega) - \alpha(\omega))^2 + |k + b(\omega)|^2}} \begin{bmatrix} -k - b(\omega) \\ a(\omega) - \alpha(\omega) \end{bmatrix}$$

and

$$u_2(\omega) = \frac{1}{\sqrt{(a(\omega) + \alpha(\omega))^2 + |k + b(\omega)|^2}} \begin{bmatrix} -k - b(\omega) \\ a(\omega) + \alpha(\omega) \end{bmatrix}.$$

Define $U(\omega) = [u_1(\omega)\ u_2(\omega)]$ for all $\omega \in \Omega$. Then U is a unitary in M. As in the proof of Lemma 3.7, we see that $A + kT = U \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix} U^* = \alpha(u_1 u_1^* - u_2 u_2^*)$, and L is real homogeneous on $A + kT$. Arguing as in the proof of Lemma 2.4, we see that L is real homogeneous on T, and thus also real homogeneous on A. Therefore, L is real linear on the real linear span of Hermitian elements in $\mathbf{M}_2(C(\Omega))_{\mathrm{sa}}$ with zero central trace. It then follows that we can define a real linear map $\tilde{L} : \mathbf{M}_2(C(\Omega))_{\mathrm{sa}} \to B$ by setting $\tilde{L}(A) = L(A)$ if A has zero central trace, and $\tilde{L}(\beta I) = \beta L(I)$ for all $\beta \in \mathbb{R}$, where I is the identity of $\mathbf{M}_2(C(\Omega))$. Then $\tilde{L}$ extends to a Jordan $*$-homomorphism J satisfying the assertions.

Finally, if M is a general von Neumann algebra, write

$$M = z_{\mathrm{I_f}} M \oplus (1 - z_{\mathrm{I_f}}) M = z_{\mathrm{I_2}} M \oplus (1 - z_{\mathrm{I_2}}) M.$$

The restriction of L to $(1 - z_{\mathrm{I_2}})M_{\mathrm{sa}}$ extends to a Jordan $*$-homomorphism $J_1 : (1 - z_{\mathrm{I_2}})M \to B$, while the restriction of L to $z_{\mathrm{I_2}} M$ also extends to a Jordan $*$-homomorphism $J_2 : z_{\mathrm{I_2}} M \to B$ by the above argument. We have seen that J_1 agrees with L on $(1 - z_{\mathrm{I_f}})M_{\mathrm{sa}}$. Note that J_1 and J_2 have disjoint ranges since $z_{\mathrm{I_2}}$ and $1 - z_{\mathrm{I_2}}$ are orthogonal. Therefore, the sum $J = J_1 + J_2$ is again a Jordan $*$-homomorphism. Clearly, J satisfies the asserted conclusions. ■

Notice that all arguments above for the additive Hermitian idempotent preserver L of M_{sa} concern only the action of L on $\mathrm{span}_{\mathbb{Q}}\, P(M)$. Theorem 3.9 thus provides us a supplement of the Dye-Bunce-Wright Theorem 3.8 which also covers the type I_2 case as stated in Theorem 1.2.

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