Multiplicative Preservers and Induced Operators

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Dedicated to Professor Graciano de Oliveira on the occasion of his retirement.

Abstract

Let V be an n-dimensional Hilbert space. Suppose H is a subgroup of the symmetric group of degree m, and $\chi : H \to \mathbb{C}$ is a character of degree 1 on H. Consider the symmetrizer on the tensor space $\otimes^m V$

$$S(v_1 \otimes \cdots \otimes v_m) = \frac{1}{|H|} \sum_{\sigma \in H} \chi(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}$$

defined by H and χ. The subspace $V_\chi^m(H)$ of $\otimes^m V$ spanned by $S(\otimes^m V)$ is called the symmetry class of tensors over V associated with H and χ. The elements in $V_\chi^m(H)$ of the form $S(v_1 \otimes \cdots \otimes v_m)$ are called decomposable tensors and are denoted by $v_1 * \cdots * v_m$. For any linear operator T acting on V, there is an (unique) induced operator $K_\chi(T)$ (or just $K(T)$ for notational simplicity) acting on $V_\chi^m(H)$ satisfying

$$K(T) v_1 * \ldots * v_m = T v_1 * \cdots * T v_m.$$

We characterize multiplicative maps ϕ such that $F(\phi(T)) = F(T)$ for all operators T acting on V, where F are various scalar or set valued functions including the spectral radius, (decomposable) numerical radius, spectral norm, spectrum, (decomposable) numerical range of T or $K(T)$.

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1 Introduction

Let V be an n-dimensional Hilbert space. Suppose H is a subgroup of the symmetric group of degree m, and $\chi : H \to \mathbb{C}$ is a character of degree 1 on H. Consider the symmetrizer on the tensor space $\otimes^m V$

$$S(v_1 \otimes \cdots \otimes v_m) = \frac{1}{|H|} \sum_{\sigma \in H} \chi(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}$$

defined by H and χ. The subspace $V_\chi^m(H)$ of $\otimes^m V$ spanned by $S(\otimes^m V)$ is called the symmetry class of tensors over V associated with H and χ. The elements in $V_\chi^m(H)$ of the form $S(v_1 \otimes \cdots \otimes v_m)$ are called decomposable tensors and are denoted by $v_1 * \cdots * v_m$.

The study of symmetry classes of tensors is motivated by many branches of both pure and applied mathematics: combinatorial theory, matrix theory, operator theory, group representation theory, differential geometry, partial differential equations, quantum mechanics and other areas. One may see [4, 13, 14, 16] for some general background.

For any linear operator T acting on V, there is a (unique) induced operator $K(T)$ acting on $V_\chi^m(H)$ satisfying

$$K(T) v_1 * \ldots * v_m = Tv_1 * \cdots * Tv_m.$$

The induced operator is a useful object in the study of symmetry classes of tensors. In [13, 14, 12] many basic properties and problems concerning induced operators were studied. In particular, the authors in [12] studied linear maps ϕ on operators acting on V such that $F(\phi(T)) = F(T)$ for various scalar or set valued functions F such as the spectral radius, (decomposable) numerical radius, spectral norm, spectrum, (decomposable) numerical range of T or $K(T)$. In this paper, we study multiplicative maps ϕ having these properties.

We shall present some preliminaries in Section 2, and prove our multiplicative preserver results in Section 3 and Section 4.

In the subsequent discussion, we always assume that χ is a linear character on a subgroup H of the symmetric group of degree m. As shown in Section 2, we can identify V with $\mathbb{C}^n$, and identify the algebra of linear operators on V with the algebra of $n \times n$ complex matrices M_n.

2 Preliminaries

Define the generalized matrix function $d_\chi : M_m \to \mathbb{C}$ associated with χ by

$$d_\chi(X) = \sum_{\sigma \in H} \chi(\sigma) \prod_{j=1}^{m} X_{j,\sigma(j)}, \qquad X = (X_{jk}) \in M_m.$$

One can use this concept to facilitate the study of symmetry classes of tensors and induced operators. For instance, the inner product on $V_\chi^m(H)$ can be expressed in terms of the inner product (u, v) on V by the following formula:

$$(u_1 * \cdots * u_m, v_1 * \cdots * v_m) = \frac{1}{|H|} d_\chi[(u_i, v_j)].$$

In fact, if we identify V with $\mathbb{C}^n$ with respect to a fixed orthonormal basis

$$\mathcal{B} = \{e_1, \ldots, e_n\},$$

then a decomposable tensor $v^* = v_1 * \cdots * v_m \in V_\chi^m(H)$ can be identified with the $n \times m$ matrix X such that the jth column of X is the co-ordinate vector of v_j with respect to $\mathcal{B}$ and

$$(v^*, v^*) = \frac{1}{|H|} d_\chi(X^* X).$$

Let $\Gamma_{m,n}$ be the set of sequences $\alpha = (\alpha(1), \ldots, \alpha(m))$ with $1 \le \alpha(j) \le n$ for $j = 1, \ldots, m$. Then

$$\{e_{\alpha(1)} \otimes \cdots \otimes e_{\alpha(m)} : \alpha \in \Gamma_{m,n}\}$$

is a basis for $\otimes^m \mathbb{C}^n$. Furthermore, one can generate an orthonormal basis for $V_\chi^m(H)$ from $\mathcal{B}$. We need some more notations to do that. For $r = 1, \ldots, n$ and $\alpha \in \Gamma_{m,n}$, let $m_r(\alpha)$ be the number of times the integer r appears in α. Two sequences α and β in $\Gamma_{m,n}$ are said to be equivalent modulo H, denoted by $\alpha \sim \beta$, if there exists $\sigma \in H$ such that $\beta = \alpha\sigma$. Evidently, this equivalence relation partitions $\Gamma_{m,n}$ into equivalence classes. Let Δ be a system of representatives for the equivalence classes so that each sequence in Δ is first in lexicographic order in its equivalence class. Define $\bar{\Delta}$ as the subset of Δ consisting of those sequences $w \in \Delta$ such that

$$\nu(w) = \sum_{\sigma \in H_w} \chi(\sigma) \neq 0,$$

where H_w is the stabilizer of w, i.e., $H_w = \{\sigma \in H : w\sigma = w\}$. Then

$$\{e_{\alpha(1)} * \cdots * e_{\alpha(m)} : \alpha \in \bar{\Delta}\}$$

is an orthogonal basis for $V_\chi^m(H)$, and one can get an orthonormal basis $\tilde{\mathcal{B}}$ after normalization. Furthermore, for any $A = (a_{jk}) \in M_n$, denote by $A[\beta|\alpha]$ the $m \times m$ matrix with (r, s) entry equal to $a_{\beta(r),\alpha(s)}$. If $A \in M_n$ is the matrix representation of T with respect to $\mathcal{B}$, then the induced operator $K(T)$ has a matrix representation with respect to the basis $\tilde{\mathcal{B}}$, denoted by $K(A)$. In fact (see e.g. [13, p.126]), $K(A)$ is an $|\bar{\Delta}| \times |\bar{\Delta}|$ matrix with rows and columns indexed lexicographically by the set $\bar{\Delta}$ so that the entry of $K(A)$ labeled by (α, β) in $\bar{\Delta} \times \bar{\Delta}$ is equal to

$$\frac{1}{\sqrt{\nu(\alpha)\nu(\beta)}} d_\chi(A^t[\beta|\alpha]).$$

By the above discussion, we can identify V with $\mathbb{C}^n$, T with $A \in M_n$ and $K(T)$ with $K(A)$, etc. From this point, we shall work on these matrix formulations of the induced operators. Furthermore, we shall always assume that $\bar{\Delta} \neq \emptyset$ so that $K(A)$ is well-defined.

We give several common examples of symmetry classes of tensors and induced operators in the following.

Example 2.1 *Let $1 \leq m \leq n$, $H = S_m$ and χ be the alternate character. Then $V_\chi^m(H)$ is the mth exterior space over $V = \mathbb{C}^n$, $\bar{\Delta}$ is the set of strictly increasing sequences in $\Gamma_{m,n}$, $d_\chi(B) = \det(B)$ is the determinant for $B \in M_m$, and $K(A)$ is the mth compound matrix of $A \in M_n$.*

Example 2.2 *Let $H = S_m$ and $\chi = 1$ be the principal character. Then $V_\chi^m(H)$ is the mth completely symmetric space over $V = \mathbb{C}^n$, $\bar{\Delta} = G_{m,n}$ is the set of increasing sequences in $\Gamma_{m,n}$, $d_\chi(B) = \mathrm{per}(B)$ is the permanent for $B \in M_m$, and $K(A)$ is the mth induced power of $A \in M_n$.*

Example 2.3 *Let $H = \{e\}$ where e is the identity in S_m and $\chi = 1$ be the principal character. Then $V_\chi^m(H) = \otimes^m V$, $\bar{\Delta} = \Gamma_{m,n}$, $d_\chi(B) = \prod_{j=1}^m b_{jj}$ for $B = (b_{jk}) \in M_m$, and $K(A) = \otimes^m A$ is the mth tensor power of $A \in M_n$.*

We list some basic properties of $K(A)$ in the following (see [13, Chapter 2]).

Proposition 2.4 *The following properties hold for induced matrices.*
(a) $K(I_n) = I_{|\bar{\Delta}|}$.
(b) $K(AB) = K(A)K(B)$ *for any* $A, B \in M_n$.
(c) $K(A^*) = K(A)^*$ *for any* $A \in M_n$.
(d) $A \in M_n$ *is invertible if and only if* $K(A)$ *is. Moreover, we have* $K(A^{-1}) = K(A)^{-1}$.
(e) *If* $A \in M_n$ *is in (lower or upper) triangular or in diagonal form, then so is* $K(A)$.
(f) *If* A *has eigenvalues* $\lambda_1, \ldots, \lambda_n$, *and singular values* $s_1 \geq \cdots \geq s_n$, *then for any* $\sigma \in S_n$, $K(A)$ *has eigenvalues* $\prod_{j=1}^n \lambda_{\sigma(j)}^{m_j(\alpha)}$ *and singular values* $\prod_{j=1}^n s_{\sigma(j)}^{m_j(\alpha)}$, $\alpha \in \bar{\Delta}$.
(g) $\det(K(A)) = \det(A)^k$, *where* $k = |\bar{\Delta}|m/n$.
(h) *If* $\mathrm{rank}\,(A) = r$, *then* $\mathrm{rank}\,(K(A)) = |\Gamma_{m,r} \cap \bar{\Delta}|$.
(i) *If* $A \in M_n$ *is normal, unitary, positive (semi-)definite, hermitian or skew-hermitian (when* m *is odd), then* $K(A)$ *has the corresponding property.*

Note that part (f) in the above proposition is usually stated with σ equal to the identity permutation. In our statement, we emphasis that relabeling of the indices of the eigenvalues or singular values will not affect the conclusion. This observation will be used frequently in our study.

We shall use $\mu(\bar{\Delta})$ to denote the smallest integer r such that $\Gamma_{m,r} \cap \bar{\Delta} \neq \emptyset$. As a result, a matrix $A \in M_n$ satisfies $K(A) = 0$ if and only if $\mathrm{rank}\,(A) < \mu(\bar{\Delta})$.

It is natural to ask whether the converse of Proposition 2.4 (i) holds. Unfortunately, it is not true in general as noted in [13, p.148]. In [12], the authors identified the situation under which the converses of the above proposition hold. In particular, it was shown that the converses only fail for the following types of characters.

Definition 2.5 We say that the character χ is of the *determinant type* if every element $\alpha \in \bar{\Delta}$ satisfies $m_1(\alpha) = \cdots = m_n(\alpha) = m/n$. Furthermore, we say that χ is of the *special type* if every $\alpha \in \Gamma_{m,r} \cap \bar{\Delta}$ satisfies $m_1(\alpha) = \cdots = m_r(\alpha)$ where $r = \mu(\bar{\Delta})$ with $1 < r$.

Note that the alternate character on S_n is of the determinant type; the alternate character on S_m with $m < n$ is of the special type but not of the determinant type, and the principal character is of the general type. Here we give some additional examples of χ that are of the special type and the determinant type.

Example 2.6 *Consider the alternating group in S_4 and use the character χ_2 in* [10, p.181]*, i.e., $\chi_2(\sigma) = 1$ if σ is the identity permutation or a product of two disjoint transpositions, and if $1 \leq i < j < k \leq 4$ then $\chi_2((i,j,k)) = w$ and $\chi_2((i,k,j)) = w^2$, where $w = e^{2\pi i/3}$.*

(a) *If $n = 2$, then*

$$\bar{\Delta} = \{(1,1,2,2)\},$$

and χ_2 is of the determinant type.

(b) *If $n = 3$, then*

$$\bar{\Delta} = \{(j,j,k,k) : 1 \leq j < k \leq 3\} \cup \{(1,1,2,3),(1,2,2,3),(1,2,3,3)\},$$

and χ_2 is of the special type but not of the determinant type.

Let $A \in M_n$. The numerical range of A is defined by

$$W(A) = \{(Ax,x) : x \in \mathbb{C}^n, (x,x) = 1\},$$

and the numerical radius of A is defined by

$$w(A) = \max\{|\eta| : \eta \in W(A)\}.$$

These concepts have been studied extensively because of their connections and applications to many branches of pure and applied mathematics (see e.g., [5, 6, 8, 9]).

In the study of induced operators on symmetry tensors, it is natural to consider the decomposable numerical range of $A \in M_n$ defined by

$$W_\chi(A) = \{(K(A)x^*, x^*) : x^* \text{ is a decomposable unit tensor }\},$$

see [14, 15], and the decomposable numerical radius of $A \in M_n$ defined by

$$w_\chi(A) = \max\{|\eta| : \eta \in W_\chi(A)\}.$$

In terms of the generalized matrix function, we can write

$$W_\chi(A) = \left\{ \frac{d_\chi(X^*AX)}{d_\chi(X^*X)} : X \in M_{n\times m},\ d_\chi(X^*X) \neq 0 \right\}.$$

Evidently, when $m = 1$, $W_\chi(A)$ reduces to the classical numerical range $W(A)$.

Certainly, one can also consider the classical numerical range and radius of the induced matrix $K(A)$. Since the set of decomposable unit tensors is usually a proper subset of the set of unit vectors in $V_\chi^m(H)$, we have

$$W_\chi(A) \subseteq W(K(A)),$$

and the inclusion is often strict. Consequently, we have

$$w_\chi(A) \leq w(K(A)),$$

and again the inequality is usually strict. Thus, the set $W_\chi(A)$ usually contains "less" information than $W(K(A))$, and the quantity $w_\chi(A)$ is different from $w(K(A))$. However, in the study of symmetry classes of tensors and induced operators, one may only have information about $W_\chi(A)$ and $w_\chi(A)$ but not $W(K(A))$ and $w(K(A))$. Nonetheless, it was shown in [12] that one can extract useful information about the operator A based on the limited knowledge on $W_\chi(A)$ and $w_\chi(A)$.

3 Multiplicative Preservers

We will use the following notations:

$\{E_{11}, E_{12}, \ldots, E_{nn}\}$: the standard basis for M_n,

$M_n^{(k)}$: the semigroup of matrices in M_n with rank at most k, where $k \in \{1, \ldots, n\}$.

GL_n: the group of invertible matrices in M_n,

SL_n: the group of matrices in M_n with determinant 1,

$\mathbf{T}$: the group of $z \in \mathbb{C}$ with $|z| = 1$,

$\mathbb{C}^*$: the group of nonzero complex numbers,

$\tau(A) = (A^t)^{-1}$ for any invertible $A \in M_n$,

$\mathrm{Eig}(A)$: the multiset (with n elements) of eigenvalues of $A \in M_n$,

$\mathrm{Sp}(A)$: the spectrum of $A \in M_n$, i.e. the set of distinct eigenvalues of A,

$\rho(A) = \max\{|\lambda| : \lambda \in \mathrm{Sp}(A)\}$ is the spectral radius of A.

We often use the fact that a field automorphism f on $\mathbb{C}$ has the form $z \mapsto z$ or $z \mapsto \bar{z}$ if f satisfies any one of the following conditions:

$$\text{(i) } |f(z)| = |z| \text{ for all } z \in \mathbb{C}, \quad \text{(ii) } |f(z)| = 1 \text{ for all } z \in \mathbb{C} \text{ with } |z| = 1. \tag{3.1}$$

Let $\mathcal{R} = M_n^{(k)}$, SL_n or GL_n. In the following, we determine the structures of multiplicative maps $\phi : \mathcal{R} \to M_n$ on matrices such that

$$F(\phi(A)) = F(A) \qquad \text{for all } A \in \mathcal{R}, \tag{3.2}$$

where $F(A) = K(A)$, $\|K(A)\|$, $\rho(K(A))$, $w(K(A))$, $w_\chi(A)$, $W(K(A))$, $W_\chi(A)$, $\mathrm{Eig}(K(A))$, or $\mathrm{Sp}(K(A))$.

Note that if χ is of the determinant type then

$$|\det(A)|^{m/n} = \|K(A)\| = \rho(K(A)) = w(K(A)) = w_\chi(A),$$

$$\{\det(A)^{m/n}\} = W(K(A)) = W_\chi(A) = \mathrm{Sp}(K(A)) = \mathrm{Eig}(K(A)).$$

There are many degenerate multiplicative maps satisfying (3.2). For examples, the mapping ϕ of the form

$$A \mapsto \mathrm{diag}\,(\det(A), 1, \ldots, 1)$$

will satisfy (3.2). Recall that $\mu(\bar{\Delta})$ is the smallest integer r such that $\Gamma_{m,r} \cap \bar{\Delta}$ is nonempty, and thus $A \in M_n$ satisfies $K(A) = 0$ if and only if $\mathrm{rank}\,(A) < \mu(\bar{\Delta})$. So, condition (3.2) does not carry much information on $\phi(A)$ for matrices with rank less than $\mu(\bar{\Delta})$. Therefore, we only consider multiplicative maps $\phi : \mathcal{R} \to M_n$ with $\mathcal{R} = SL_n$ or GL_n if $\mu(\bar{\Delta}) = n$. If $\mu(\bar{\Delta}) < n$, we can also consider $\mathcal{R} = M_n^{(k)}$ with $k \geq \mu(\bar{\Delta})$ and we have the following.

Theorem 3.1 *Suppose χ is not of the determinant type, $\mu(\bar{\Delta}) < n$ and $k \geq \mu(\bar{\Delta})$. Let $\mathcal{R}$ be a semigroup in M_n containing $M_n^{(k)}$, and let $F(A) = K(A)$, $\|K(A)\|$, $\rho(K(A))$, $w(K(A))$, $w_\chi(A)$, $W(K(A))$, $W_\chi(A)$, $\mathrm{Sp}(K(A))$, or $\mathrm{Eig}(K(A))$. A multiplicative map $\phi : \mathcal{R} \to M_n$ satisfies*

$$F(\phi(A)) = F(A) \qquad \text{for all } A \in \mathcal{R},$$

if and only if one of the following holds.

(a) *$F(A) = K(A)$ and ϕ is the identity map.*

(b) *$F(A) = \rho(K(A))$ and there exists $S \in SL_n$ such that ϕ has the form*

$$A \mapsto S^{-1}AS \qquad \text{or} \qquad A \mapsto S^{-1}\overline{A}S.$$

(c) *$F(A) = \mathrm{Eig}(K(A))$ or $\mathrm{Sp}(K(A))$; there exists $S \in SL_n$ such that ϕ has the form*

$$A \mapsto S^{-1}AS.$$

(d) *$F(A) = \|K(A)\|$, $w(K(A))$ or $w_\chi(A)$; there exists a unitary U such that ϕ has the form*

$$A \mapsto U^*AU \qquad \text{or} \qquad A \mapsto U^*\overline{A}U.$$

(e) *$F(A) = W(K(A))$ or $W_\chi(A)$; there exists a unitary U such that ϕ has the form*

$$A \mapsto U^*AU.$$

Proof. The ($\Leftarrow$) part is clear. We consider the ($\Rightarrow$) part. Note that if $E \in M_n^{(k)}$ is an idempotent, i.e., $E^2 = E$, then $\phi(E) = \phi(E^2) = \phi(E)^2$ is also an idempotent. Let $r = \mu(\bar{\Delta}) < n$. If E is a rank r idempotent, then $K(E) \neq 0_{|\bar{\Delta}|} = K(0_n)$. Thus, for any of our functions F, $F(\phi(E)) = F(E) \neq F(0_n)$ and hence $\phi(E) \neq 0_n$. By the result in [11] (see also Proposition 2.2 and 2.5 in [1]), one of the following holds.

(i) There exist $S \in SL_n$ and a field automorphism $f : \mathbb{C} \to \mathbb{C}$ such that ϕ has the form

$$(a_{ij}) \mapsto S^{-1}(f(a_{ij}))S.$$

(ii) $n \geq 3$, $r = n-1$, $\phi(E) = 0$ for all idempotent E with rank less than $n-1$, and

$$\{\phi(I - E_{jj}) : 1 \leq j \leq n\}$$

is a set of mutually orthogonal rank one idempotents.

If (ii) holds, then $\phi(I - E_{nn}) = Y$ has rank one and $\phi(Y) = 0_n$. But then

$$F(0_n) = F(\phi(Y)) = F(Y) = F(I - E_{jj}) \neq F(0_n),$$

which is a contradiction. So, we conclude that (i) holds.

Suppose $F(A) = K(A)$, $\rho(K(A))$, $\mathrm{Eig}(A)$ or $\mathrm{Sp}(K(A))$. Then for $A = \mu(E_{11} + \cdots + E_{rr})$ with $\mu \in \mathbb{C}$,

$$F(\phi(A)) = F(A)$$

implies that

$$|f(\mu)^m| = \rho(K(\phi(A))) = \rho(K(A)) = |\mu^m|.$$

It follows that $|f(\mu)| = |\mu|$. So f satisfies one of the conditions in (3.1) and has the form $\mu \mapsto \mu$ or $\mu \mapsto \bar{\mu}$. In case $F(A) = K(A)$, $\mathrm{Eig}(A)$ or $\mathrm{Sp}(K(A))$, we see that f is the identity map. In particular, conditions (b) and (c) hold.

If $F(A) = K(A)$, then $K(A) = K(S^{-1}AS)$. If χ is not of the special type, then $\phi(A) = \xi A$ for some mth root of unity $\xi \in \mathbb{C}$ by Theorem 2.17 in [12]. If A has rank r and a nonzero eigenvalue then $\mu = 1$ because ξA and $\phi(A) = S^{-1}AS$ have the same eigenvalues. Since $S^{-1}AS = A$ for all A with rank r and a nonzero eigenvalue, we see that $S = I_n$. Consequently, we have $\phi(A) = A$ for all $A \in \mathcal{R}$.

Suppose χ is of the special type. For $X \in M_n$ and $J \subseteq \{1, \ldots, n\}$, let $X[J]$ be the principal submatrix of X lying in rows and columns with indices in J, and $X(J)$ be the principal submatrix of X lying in rows and columns with indices not in J. By Theorem 2.17 in [12], for any A of the form

$$A = \sum_{j \in J} E_{jj}, \qquad \text{where } J \subseteq \{1, \ldots, n\} \text{ with } |J| = r, \tag{3.3}$$

$S^{-1}AS$ is a direct sum of $S^{-1}AS[J]$ and $S^{-1}AS(J)$, where $S^{-1}AS = A_0$ with $|\det(A_0)| = 1$ and $S^{-1}AS(J) = 0$. So $AS = S(S^{-1}AS)$, and we see that S itself is a direct sum of $S[J]$ and $S(J)$. Since this is true for all $J \subseteq \{1, \ldots, n\}$ with $|J| = r$, we conclude that $S = \mathrm{diag}\,(s_1, \ldots, s_n)$ is a diagonal matrix. We may replace S by S/s_1 and assume that $s_1 = 1$. Now, suppose $s_j \neq 1$ for some $j \geq 2$. Consider $\alpha \in \bar{\Delta}$ with $m_1(\alpha) = \cdots = m_r(\alpha)$ where $r = \mu(\bar{\Delta})$. Let

$$v^* = f_{\alpha(1)} * \cdots * f_{\alpha(m)} \quad \text{and} \quad u^* = f_{\alpha(1)} * \cdots * f_{\alpha(m-1)} * e_1$$

such that $e_1 \notin \{f_1, \ldots, f_r\} = \{e_{i_1}, \ldots, e_{i_r}\}$ with $f_{\alpha(m)} = e_j$. Suppose $A = \sum_{t=1}^r E_{i_t,i_t} + E_{j1}$. Then v^* is a unit eigenvector of $K(A)$ and

$$K(A)v^* = Af_{\alpha(1)} * \cdots * Af_{\alpha(m)} = f_{\alpha(1)} * \cdots * f_{\alpha(m)}.$$

Moreover,

$$(S^{-1}AS)f_{\alpha(t)} = f_{\alpha(t)} \qquad \text{for } t = 1, \ldots, m-1,$$

and

$$(S^{-1}AS)e_1 = (S^{-1}A)e_j/s_j = f_{\alpha(m)}/s_j.$$

As a result,

$$\begin{aligned} f_{\alpha(1)} * \cdots * f_{\alpha(m)} &= Af_{\alpha(1)} * \cdots * Af_{\alpha(m-1)} * Ae_1 \\ &= K(A)u^* \\ &= K(S^{-1}AS)u^* \\ &= (S^{-1}AS)f_{\alpha(1)} * \cdots * (S^{-1}AS)f_{\alpha(m-1)} * (S^{-1}AS)e_1 \\ &= f_{\alpha(1)} * \cdots * f_{\alpha(m)}/s_j. \end{aligned}$$

Since $v^* = f_{\alpha(1)} * \cdots * f_{\alpha(m)}$ is a unit vector, it follows that $s_j = 1$, which is a contradiction. Thus, all diagonal entries of S are the same, and $\phi(A) = S^{-1}AS = A$ for any $A \in M_n$, i.e., ϕ is the identity map. Hence, condition (a) holds.

Suppose $F(X) = \|K(X)\|$, $w(K(X))$, or $w_\chi(X)$. Let $A = \mu(E_{11} + \cdots + E_{rr})$ with $\mu \in \mathbb{C}$ such that $|\mu| = 1$. Then for any positive integer s,

$$\begin{aligned} 1 &= F(A^s) = F(\phi(A^s)) = F(\phi(A)^s) = F(f(\mu)^s S^{-1}(E_{11} + \cdots + E_{rr})S) \\ &\geq \rho(K(f(\mu)^s S^{-1}(E_{11} + \cdots + E_{rr})S)) = |f(\mu)^{sm}|. \end{aligned}$$

So, $1 \geq |f(\mu)|$. If $1 > |f(\mu)|$, then $\lim_{s\to\infty} \phi(A^s) = 0$ implies

$$1 = \lim_{s\to\infty} F(A^s) = \lim_{s\to\infty} F(\phi(A^s)) = 0,$$

which is a contradiction. So, $|f(\mu)| = 1$ whenever $|\mu| = 1$. So one of the conditions in (3.1) holds, and f has the form $\mu \to \mu$ or $\mu \to \bar{\mu}$. Suppose

$$A = \gamma U^*(E_{11} + \cdots + E_{rr})U \qquad \text{with } \gamma > 0 \quad \text{and} \quad U \text{ unitary}. \tag{3.4}$$

We claim that

(1) $\phi(A) = \gamma V^*(E_{11} + \cdots + E_{rr})V$ for some unitary V, and

(2) the matrices of the form (3.4) generate the set $\mathcal{P}$ of all positive semi-definite matrices of rank at most r under multiplication.

To prove (1), note that

$$\gamma^m = F(A) = F(\phi(A)) = F(S^{-1}AS) \geq \rho(K(S^{-1}AS)) = \gamma^m.$$

By Proposition 3.10 in [12], we see that $\phi(A) = S^{-1}AS$ is unitarily similar to $A_1 \oplus A_2$ such that $A_1 \in M_r$ has eigenvalues $\lambda_1, \ldots, \lambda_r$ such that $F(\phi(A)) = |\prod_{j=1}^r \lambda_j^{m_j(\alpha)}|$ for some $\alpha \in \bar{\Delta}$. Observe that $\phi(A) = S^{-1}AS$ has rank r and r nonzero eigenvalues all equal to γ. Moreover, and $\phi(A/\gamma)^2 = \phi(A/\gamma)$. We see that $(A_1/\gamma)^2 = A_1/\gamma = I_r$ and $A_2 = 0_{n-r}$. Thus, (1) holds.

To prove (2), suppose $C = X^*(\sum_{j=1}^r c_j E_{jj})X$ with some unitary X and $c_1 \geq \cdots \geq c_r \geq 0$. We show that C is a product of the matrices in the form (3.4) by induction on the number of elements in $\{c_1, \ldots, c_r\}$ different from c_1. If $c_1 = \cdots = c_r$, then C itself has the form (3.4). Suppose the result is proved for the case when $c_1 = \cdots = c_{r-k}$ with $k \geq 0$. Consider the case when $c_1 = \cdots = c_{r-k-1} > c_{r-k}$. Let $t \in [0, \pi/2)$ such that $c_1(\cos^2 t) = c_r$. By induction assumption, the matrices $B_1 = \sum_{j=1}^{r-1} c_j E_{jj} + c_1 E_{rr}$ is a product of the matrices of the form (3.4). Also, the matrix

$$B_2 = I_{r-1} \oplus \begin{pmatrix} \cos^2 t & \cos t \sin t \\ \cos t \sin t & \sin^2 t \end{pmatrix}$$

is a matrix of the form (3.4). Moreover, $B_2 B_1 B_2$ has the same eigenvalues as C. So, there is a unitary Y such that $(Y^* B_2 Y)(Y^* B_1 Y)(Y^* B_2 Y) = C$. Hence, our induction proof is completed, and (2) holds.

Now, by (1) and (2), we see that $\phi(\mathcal{P}) \subseteq \mathcal{P}$. Note also that $\phi(X) = S^{-1}XS \neq 0_n = \phi(0_n)$ for a rank one X. By Theorem 3.6 of [1], ϕ has the form asserted in (d).

If $W(K(\phi(A))) = W(K(A))$ for all $A \in \mathcal{R}$, then $w(K(\phi(A))) = w(K(A))$ for all $A \in \mathcal{R}$. So, ϕ has one of the two form in (d). Consider $A = \mu(E_{11} + \cdots + E_{rr})$ for nonzero $\mu \in \mathbb{C}$. One sees that the second form cannot hold. Similarly, one can show that ϕ has the same form if $W_\chi(K(\phi(A))) = W_\chi(K(A))$ for all $A \in \mathcal{R}$. Thus, condition (e) holds. ∎

Next we consider multiplicative maps $\phi : G \to M_n$ with $G = SL_n$ or GL_n. We begin with the following.

Theorem 3.2 *Let $G = SL_n$ or GL_n. Suppose χ is not of the determinant type. A multiplicative map $\phi : G \to M_n$ satisfies*

$$K(\phi(A)) = K(A) \qquad \textit{for all } A \in G$$

if and only if ϕ is the identity map.

Proof. For any $A \in G$, there exists B such that $B^m = A$. By Theorem 2.17 in [12], $K(\phi(B)) = K(B)$ implies $\phi(B) = \mu B$ for some $\mu \in \mathbb{C}$ such that $\mu^m = 1$. Thus, $\phi(A) = \phi(B)^m = A$. ∎

To further our study, we need the following result in [3].

Lemma 3.3 *Suppose $G = SL_n$ or GL_n and $\phi : G \to M_n$ is a multiplicative map. Then one of the following holds.*

(1) *$\{\phi(A) : A \in SL_n\}$ is a singleton.*

(2) *There is a field embedding $f : \mathbb{C} \to \mathbb{C}$, a multiplicative map $g : \mathbb{C}^* \to \mathbb{C}^*$, and $S \in SL_n$ such that ϕ has the form*

$$A \mapsto g(\det(A))S^{-1}f(A)S \qquad or \qquad A \mapsto g(\det(A))S^{-1}(f(\tau(A)))S,$$

where $\tau(A) = (A^t)^{-1}$ and $f(A) = (f(a_{ij}))$ for $A = (a_{ij})$.

Next, we consider multiplicative maps on $G = SL_n$ or GL_n preserving other functions.

Theorem 3.4 *Suppose χ is not of the determinant type, and $F(A) = \rho(K(A))$, $\|K(A)\|$, $w(K(A))$, or $w_\chi(A)$. A multiplicative map $\phi : G \to M_n$ satisfies*

$$F(\phi(A)) = F(A) \quad \text{for all } A \in G \tag{3.5}$$

if and only if there exist a multiplicative map $g : \mathbb{C}^* \to \mathbf{T}$ and $S \in SL_n$, where S is unitary if $F(A) \neq \rho(K(A))$, such that one of the following holds.

(a) *ϕ has the form*

$$A \mapsto g(\det(A))S^{-1}AS \quad or \quad A \mapsto g(\det(A))S^{-1}\overline{A}S.$$

(b) *$F(A) = F(\det(A)^{2/n}\tau(A))$ and ϕ has the form*

$$A \mapsto g(\det(A))|\det(A)|^{2/n}S^{-1}\tau(A)S \quad or \quad A \mapsto g(\det(A))|\det(A)|^{2/n}S^{-1}\tau(\overline{A})S.$$

Proof. The $(\Leftarrow)$ can be verified readily. We focus on the converse. Since χ is not of the determinant type, there are nonnegative integers $m_1 \geq \cdots \geq m_n$ with $m_1 + \cdots + m_n = m$ and $m_1 > m_n$ such that $\rho(K(A)) = |\lambda_1(A)^{m_1} \cdots \lambda_n(A)^{m_n}|$, whenever $\lambda_1(A), \ldots, \lambda_n(A)$ are eigenvalues of $A \in M_n$ satisfying $|\lambda_1(A)| \geq \cdots \geq |\lambda_n(A)|$.

Let $G = SL_n$. Then there is $S \in SL_n$ and a field embedding f on $\mathbb{C}$ such that ϕ has the form

$$A \mapsto S^{-1}f(A)S \quad \text{or} \quad A \mapsto S^{-1}f(\tau(A))S.$$

Note that we must have $f(\pm 1) = \pm 1$ and $f(i) = \pm i$. First, we show that if $F(A) = \|K(A)\|$, $w(K(A))$ or $w_\chi(A)$, then S can be chosen to be unitary. To this end, let $\mathcal{T}$ be the set of unitary matrices $A \in SL_n$ such that each row and column has exactly one nonzero entries in $\{1, -1, i, -i\}$. For each A of this form, we have $f(A) = B \in \mathcal{T}$ and

$$|\det(\phi(A))|^{m/n} = |\det(B)|^{m/n} = F(B) = F(\phi(A)).$$

By Theorem 3.11 in [12], we see that $\phi(A)$ is unitary. Thus, $(S^{-1}AS)(S^{-1}AS)^*$ is the identity, and hence $(SS^*)A = A(SS^*)$ for any $A \in \mathcal{T}$. It is easy to show that the linear span of $\mathcal{T}$

equals M_n. Thus, SS^* commutes with all matrices in M_n. It follows that $SS^* = aI_n$ for some $a > 0$. Replacing S by $a^{-1/2}S$, we may assume that S is unitary.

For any $z \in \mathbb{C}^*$, there is a positive integer k such that $|kz| > 1$ and $|kf(z)| > 1$. If $A = \operatorname{diag}(kz, 1, \ldots, 1, 1/(kz))$, then $\phi(A)$ is normal with eigenvalues $kf(z), 1, \ldots, 1, 1/(kf(z))$. Since

$$|kf(z)|^{m_1 - m_n} = \rho(K(\phi(A)) = F(\phi(A)) = F(A) = \rho(K(A)) = |kz|^{m_1 - m_n},$$

we have $|z| = |f(z)|$. Hence f satisfies one of the conditions in (3.1) and has the form $z \mapsto z$ or $\bar{z}$. Thus, ϕ on SL_n has the form (a) – (b). Note that for any $A \in SL_n$, we have $\det(A) = 1$, and thus we always have $g(\det(A)) = g(\det(A))|\det(A)|^{2/n} = 1$.

Now, consider $G = GL_n$. Suppose ϕ on SL_n has the form (a). Then ϕ on GL_n has the form

$$A \mapsto g(\det(A))S^{-1}AS \quad \text{ or } \quad A \mapsto g(\det(A))S^{-1}\overline{A}S.$$

Considering $A = aI$, one see that $|g(z)| = 1$ for all $z \in \mathbb{C}^*$.

Suppose ϕ on SL_n has the form (b). Then ϕ on GL_n has the form

$$A \mapsto g(\det(A))|\det(A)|^{2/n}S^{-1}\tau(A)S \quad \text{ or } \quad A \mapsto g(\det(A))|\det(A)|^{2/n}S^{-1}\tau(\overline{A})S$$

for some multiplicative map g on $\mathbb{C}^*$. We may assume that the former case holds; otherwise, replace ϕ by the mapping $A \mapsto \phi(\overline{A})$. If $n = 2$, this reduces to the operator of the form (a). If $n \geq 3$, then for $A = \operatorname{diag}(r, r, 1, \ldots, 1, 1/r^2)$ with $r > 1$, we have $r^{m_1 + m_2 - 2m_n} = F(A) = F(\phi(A)) = r^{2m_1 - m_{n-1} - m_n}$. Thus, $m_1 + m_n = m_2 + m_{n-1}$. Next, consider $A = \operatorname{diag}(r, r, r, 1, \ldots, 1/r^3)$. Since $r^{m_1 + m_2 + m_3 - 3m_n} = F(A) = F(\phi(A)) = r^{3m_1 - m_{n-2} - m_{n-1} - m_n}$, we see that $m_3 + m_{n-2} = m_1 + m_n$. Repeating these arguments, we conclude that

$$m_j + m_{n-j+1} = 2m/n \qquad \text{for } j = 1, \ldots, n, \tag{3.6}$$

where $m = m_1 + \cdots + m_n$. It is now easy to see that for any normal matrix A,

$$F(A) = \prod_{j=1}^{n} |\lambda_j(A)|^{m_j} = |\det(A)|^{2m/n} \prod_{j=1}^{n} |\lambda_{n-j+1}^{-m_j}(A)| = F(\det(A)^{2/n}\tau(A)).$$

By polar decomposition, every matrix A is the product of two normal matrices; thus the above conclusion holds for every matrix $A \in GL_n$. ∎

Theorem 3.5 *Suppose χ is not of the determinant type, and $F(A) = \operatorname{Eig}(K(A))$, $\operatorname{Sp}(K(A))$, $W(K(A))$, or $W_\chi(A)$. A multiplicative map $\phi : G \to M_n$ satisfies*

$$F(\phi(A)) = F(A) \quad \textit{for all } A \in G$$

if and only if there is $S \in SL_n$, where S is unitary if $F(A) = W(K(A))$ or $W_\chi(A)$, such that ϕ has the form

$$A \mapsto S^{-1}AS.$$

Proof. The ($\Leftarrow$) part is clear. Conversely, if ϕ preserves F for those F in Theorem 3.5, then ϕ will preserve those F in Theorem 3.4. Thus ϕ satisfies the conclusion of Theorem 3.4. Considering diagonal matrices of the form diag $(z, 1, \ldots, 1, 1/z)$, we see that ϕ on SL_n has the asserted form. For GL_n, if $z_1 \in \mathbb{C}^*$ then $\phi(z_1 I) = z_2 I$ with $z_1^m = z_2^m$. It follows that $\phi(zI) = zI$ for any $z \in \mathbb{C}^*$. Thus ϕ has the asserted form as well. ∎

4 Generalized Matrix Function Preservers

In this section, we assume $m = n$ and study multiplicative maps $\phi : M_n \to M_n$ preserving generalized matrix functions on M_n, i.e., $d_\chi(\phi(A)) = d_\chi(A)$ for all $A \in M_n$. We begin with the following observation.

Lemma 4.1 *The equality $\mu(\bar{\Delta}) = n$ holds if and only if $\chi = \epsilon$, the alternating character, i.e., d_χ is the determinant function.*

Proof. If $\chi = \epsilon$, then $\mu(\bar{\Delta}) = n$. Conversely, suppose $\chi \neq \epsilon$. If there exists a transposition $(i, j) \notin H$, then $d_\chi(I + E_{ij} + E_{ji}) = 1$. If H contains all the transpositions, then $H = S_n$. Note that there must exists some $(i, j) \in H$ such that $\chi((i, j)) \neq -1$ otherwise $\chi = \epsilon$ which is impossible. So again $d_\chi(I + E_{ij} + E_{ji}) \neq 0$. Therefore $\mu(\bar{\Delta}) < n$. ∎

Suppose $\mathcal{R} \subseteq M_n$. Define

$$\mathcal{S}_\chi(\mathcal{R}) = \{S \in SL_n : d_\chi(S^{-1}AS) = d_\chi(A) \text{ for all } A \in \mathcal{R}\}.$$

The set $\mathcal{S}_\chi(M_n)$ is completely classified by De Oliveira and Dias Da Silva[17, 18]. Note that $\mathcal{S}_\chi(M_n) = \mathcal{S}_\chi(GL_n) = \mathcal{S}_\chi(SL_n)$ because of the following reasons. The set $\mathcal{R} = GL_n$ is dense in M_n. Since the functions d_χ and the function $A \mapsto S^{-1}AS$ are continuous on M_n, we see that $d_\chi(A) = d_\chi(S^{-1}AS)$ for all $A \in M_n$ and hence $\mathcal{S}_\chi(GL_n) = \mathcal{S}_\chi(M_n)$. In the case of SL_n, every $A \in GL_n$ can be written as $\mu(A/\mu)$ with $\mu = \det(A)^{1/n}$ so that $A/\mu \in SL_n$, and hence

$$d_\chi(A) = d_\chi(\mu A) = \mu^m d_\chi(A/\mu) = \mu^m d_\chi(S^{-1}(A/\mu)S) = d_\chi(\mu S^{-1}(A/\mu)S) = d_\chi(S^{-1}AS).$$

Therefore $\mathcal{S}_\chi(SL_n) = \mathcal{S}_\chi(GL_n)$.

Theorem 4.2 *Suppose d_χ is not the determinant function and $\mathcal{R}$ contains $M_n^{(k)}$ for some positive integer $k \geq \mu(\bar{\Delta})$. A multiplicative map $\phi : \mathcal{R} \to M_n$ satisfies*

$$d_\chi(\phi(A)) = d_\chi(A) \qquad \text{for all } A \in \mathcal{R},$$

if and only if there exists $S \in \mathcal{S}_\chi(\mathcal{R})$ such that ϕ has the form $A \mapsto S^{-1}AS$.

Proof. Let $\alpha \in \Gamma_{n,r} \cap \bar{\Delta}$. We may assume that

$$m_1(\alpha) \geq \cdots \geq m_r(\alpha) > 0 = m_{r+1}(\alpha) = \cdots = m_n(\alpha).$$

Let $A = \mathrm{diag}\,(1/a_1, \ldots, 1/a_r, 0, \ldots, 0) \in M_n$ such that $a_j = m_j(\alpha)$ for $j = 1, \ldots, n$. Then

$$E = A[\alpha|\alpha] \tag{4.1}$$

is permutationally similar to

$$A_1 \oplus \cdots \oplus A_r$$

such that $A_j \in M_{a_j}$ with all entries equal to $1/a_j$ for $j = 1, \ldots, r$. Thus, E is a rank r orthogonal projection. Moreover, $d_\chi(E) = d_\chi(A^t[\alpha, \alpha])$ is a multiple of the (α, α)th entry of $K(A)$ and equals $\nu(\alpha) \prod_{j=1}^r (1/a_j)^{m_j(\alpha)} \neq 0$; see the discussion before Example 2.1. Since $d_\chi(0_n) = 0 \neq d_\chi(E)$, we see that $\phi(E) \neq 0_n$. By the result in [11] (see also [1]), we conclude that one of the following conditions holds.

(i) There exist $S \in SL_n$ and a field automorphism $f : \mathbb{C} \to \mathbb{C}$ such that ϕ has the form

$$(a_{ij}) \mapsto S^{-1}(f(a_{ij}))S. \tag{4.2}$$

(ii) $n \geq 3$, $r = n - 1$, $\phi(F) = 0$ for all idempotent F with rank less than $n - 1$, and

$$\{\phi(I - E_{jj}) : 1 \leq j \leq n\}$$

is a set of mutually orthogonal rank one idempotents.

Suppose (ii) holds. Then the matrix E constructed in (4.1) must have rank at least $n - 1$. Otherwise,

$$0 \neq d_\chi(E) = d_\chi(\phi(E)) = d_\chi(0_n) = 0,$$

which is a contradiction. Since we assume that $\mu(\bar{\Delta}) \neq n$, we see that E has rank $n - 1 = \mu(\bar{\Delta})$. Note that the matirx E can be written as $[T\,|\,0](I - E_{nn})[T\,|\,0]^t$ for some $n \times (n - 1)$ real matrix T such that $T^tT = I_{n-1}$. Since $\phi(I - E_{nn})$ has rank one, we conclude that

$$\phi(E) = \phi([T\,|\,0])\phi(I - E_{nn})\phi([T\,|\,0]^t)$$

has rank at most one, and thus $\phi(\phi(E)) = 0$. But then

$$0 = d_\chi(\phi(\phi(E))) = d_\chi(E) > 0,$$

which is a contradiction.

From the above discussion, we see that condition (i) holds. It remains to prove that f is the identity map. Let us continue to use the matrix E constructed in (4.1). For any $\mu \in \mathbb{C}$, there is $\nu \in \mathbb{C}$ such that $\nu^n = \mu$. Since ϕ has the form (4.2), we see that

$$\phi(\nu E) = f(\nu)\phi(E)$$

and hence

$$\mu d_\chi(E) = d_\chi(\nu E) = d_\chi(\phi(\nu E)) = d_\chi(f(\nu)\phi(E)) = f(\nu)^n d_\chi(\phi(E)) = f(\mu) d_\chi(E).$$

Thus, $f(\mu) = \mu$ for all $\mu \in \mathbb{C}$, and ϕ has the asserted form. ■

Finally we consider $\phi : G \to M_n$ for $G = SL_n$ or GL_n. The following lemma links the d_χ-preserving problem with the induced operators.

Lemma 4.3 *Let $\mathcal{R} = GL_n$, SL_n or M_n. If a multiplicative map $\psi : \mathcal{R} \to M_{|\bar{\Delta}|}$ satisfies $\psi(A)_{11} = d_\chi(A^t)$ then $\psi(A) = S^{-1}K(A)S$ for some invertible matrix $S \in M_{|\bar{\Delta}|}$. Consequently, if $\phi : \mathcal{R} \to M_n$ is a multiplicative map preserving $F(A) = d_\chi(A)$ then $K(\phi(A)) = S^{-1}K(A)S$ for some invertible matrix $S \in M_{|\bar{\Delta}|}$.*

Proof. Up to a similarity by a permutation matrix in $M_{|\bar{\Delta}|}$, $d_\chi(A^t)$ can be viewed as the $(1,1)$ entry of $K(A)$. Hence we can assume that $K(A)$ has the form

$$K(A) = \begin{pmatrix} d_\chi(A^t) & v(A) \\ u(A) & Q(A) \end{pmatrix}.$$

Since $K(I_n) = I_{|\bar{\Delta}|}$, we see that for any $\alpha \in \bar{\Delta}$ not equal to $e = (1, \ldots, n)$, $v(I_n[e|\alpha]) = u(I_n[\alpha|e])^t$ is a nonzero multiple of the standard basis vector in $\mathbb{C}^N$ with 1 at the α-th position, where $N = |\bar{\Delta}| - 1$. For each $\alpha \in \bar{\Delta}$ not equal to $e = (1, \ldots, n)$ we construct A_α such that $\{u(A_\alpha^t) : e \neq \alpha \in \bar{\Delta}\}$ and $\{v(A_\alpha) : e \neq \alpha \in \bar{\Delta}\}$ are two sets of linearly independent vectors as follows:

if $\mathcal{R}$ contains $M_n^{(k)}$, let $A_\alpha = I_n[e|\alpha] \in \mathcal{R}$; if $\mathcal{R} = GL_n$, by continuity of d_χ there exists $\epsilon > 0$ such that $A_\alpha = I_n[e|\alpha] + \epsilon I_n \in \mathcal{R}$ satisfies our requirement; if $\mathcal{R} = SL_n$, we replace the matrices A_α constructed in the case of GL_n by $\det(A_\alpha)^{-1/n}A_\alpha$.

Writing $\psi(A) = \begin{pmatrix} d_\chi(A^t) & x(A) \\ y(A) & R(A) \end{pmatrix}$ and inspecting the $(1,1)$ entry of $\psi(A_\alpha)\psi(A_\beta^t)$, we see that

$$x(A_\alpha)y(A_\beta^t) = d_\chi(A_\beta A_\alpha^t) - d_\chi(A_\alpha^t)d_\chi(A_\beta) = v(A_\alpha)u(A_\beta^t).$$

Hence there exists an invertible $S \in M_N$ such that $x(A_\alpha)S^{-1} = v(A_\alpha)$ and $Sy(A_\beta^t) = u(A_\beta^t)$. We may assume that $S = I_N$ by replacing ψ by $([1] \oplus S)\psi(A)([1] \oplus S^{-1})$. We have $x(A_\alpha) = v(A_\alpha)$ and $y(A_\beta^t) = u(A_\beta^t)$. Now

$$x(A)y(A_\beta^t) = d_\chi(A_\beta A^t) - d_\chi(A^t)d_\chi(A_\beta) = v(A)u(A_\beta^t) = v(A)y(A_\beta^t)$$

and hence $x(A) = v(A)$. Finally we have

$$\begin{aligned} x(A_\alpha)[y(A)\ R(A)] &= [d_\chi(A^t A_\alpha^t) - d_\chi(A_\alpha^t)d_\chi(A^t)\, x(A_\alpha A) - d_\chi(A_\alpha^t)x(A)] \\ &= [d_\chi(A^t A_\alpha^t) - d_\chi(A_\alpha^t)d_\chi(A^t)\, v(A_\alpha A) - d_\chi(A_\alpha^t)v(A)] \\ &= v(A_\alpha)[u(A)\ Q(A)] \\ &= x(A_\alpha)[u(A)\ Q(A)] \end{aligned}$$

and so $y(A) = u(A)$ and $R(A) = Q(A)$, i.e. $\psi(A) = K(A)$.

The last statement in the theorem follows from the fact that $A \mapsto K(\phi(A))$ is a multiplicative map with $d_\chi(A^t)$ in the (1,1)-position. ∎

Theorem 4.4 *Let $G = GL_n$ or SL_n. Assume that d_χ is not the determinant function. Then $\phi : G \to M_n$ satisfies*

$$d_\chi(\phi(A)) = d_\chi(A) \qquad \text{for all } A \in G$$

if and only if there exists $S \in \mathcal{S}_\chi(M_n)$ such that ϕ has the form $A \mapsto S^{-1}AS$.

Proof. Let $\overline{\chi}$ be the irreducible character of H such that $\overline{\chi}(\sigma) = \chi(\sigma^{-1})$. Let ψ be the multiplicative map defined by $\psi(A) = K_{\overline{\chi}}(\phi(A))$. Then ψ satisifes

$$\psi(A)_{ee} = K_{\overline{\chi}}(\phi(A))_{ee} = d_{\overline{\chi}}(\phi(A)) = d_\chi(A) = d_{\overline{\chi}}(A^t).$$

By Lemma 4.3, $\psi(A) = R^{-1}K_{\overline{\chi}}(A)R$, that is, $K_{\overline{\chi}}(\phi(A)) = R^{-1}K_{\overline{\chi}}(A)R$ for all $A \in G$. Then,

$$\mathrm{Eig}(K_{\overline{\chi}}(\phi(A))) = \mathrm{Eig}(R^{-1}K_{\overline{\chi}}(A)R) = \mathrm{Eig}(K_{\overline{\chi}}(A)).$$

By Theorem 3.5 and the comment before Theorem 4.2, there exists $S \in SL_n$ such that ϕ has the form $A \mapsto S^{-1}AS$. ■

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