Operators with numerical range in a closed halfplane

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Abstract

Various characterizations are given of real and complex Hilbert space operators with the numerical range in a closed halfplane.

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1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space, and $B(\mathcal{H})$ be the algebra of bounded linear operators acting on $\mathcal{H}$. The *numerical range* of an operator $A \in B(\mathcal{H})$ is defined and denoted by

$$W(A) = \{(Ax, x) \in \mathsf{C} \ : \ x \in \mathcal{H}, \quad \|x\| = 1\}.$$

It is easy to see that

$$W(A) = \{((Px, x), (Qx, x)) \in \mathsf{R}^2 \ : x \in \mathcal{H}, \quad \|x\| = 1\}, \tag{1.1}$$

where

$$P = \frac{1}{2}(A + A^*) \quad \text{and} \quad Q = \frac{1}{2i}(A - A^*)$$

are the real and imaginary parts of A, respectively.

In this note we characterize the operators A whose numerical range lies in a closed halfplane. By applying a shift $A \mapsto A + \lambda I$ for a suitable scalar λ, we may assume that the halfplane is defined by a line passing through the origin. Thus, we are interested in the following property:

$$W(A) \subset \{(x, y) \in \mathsf{R}^2 \ : \ ax + by \geq 0\}, \tag{1.2}$$

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for some fixed $a, b \in \mathsf{R}$ not both zero. Note that the sectorial operators, that is the operators A with the property that (1.2) holds with 0 replaced by some positive number ε, are well known and widely used in differential equations (see, for example, [4], [2]). By analogy, we say that an operator A with the property (1.2) is *weakly sectorial.*

We consider also a real Hilbert space $\mathcal{H}_r$, and the real algebra $B(\mathcal{H}_r)$ of linear bounded operators on $\mathcal{H}_r$. In this case one defines the *real joint numerical range* $W(P, Q)$ of an ordered pair of selfadjoint operators $P, Q \in B(\mathcal{H}_r)$ by a formula analogous to (1.1), i. e.,

$$W_r(P, Q) = \{((Px, x), (Qx, x)) \in \mathsf{R}^2 \ : x \in \mathcal{H}_r, \quad \|x\| = 1\}.$$

We characterize the pairs (P, Q) for which the real joint numerical range lies in a closed halfplane that passes through the origin, in other words, the following inclusion holds true:

$$W_r(P, Q) \subset \{(x, y) \in \mathsf{R}^2 \ : \ ax + by \geq 0\}, \tag{1.3}$$

for some fixed $a, b \in \mathsf{R}$ not both zero. As in the complex case, we say that the pair (P, Q) of bounded selfadjoint operators in $B(\mathcal{H}_r)$ is *real weakly sectorial* if (1.3) holds true.

We give various characterizations of real weakly sectorial pairs of selfadjoint operators. As it turns out, the characterizations in the real case are not entirely analogous to the complex case.

2 Results and proofs: The complex case

In this section, $\mathcal{H}$ is a complex Hilbert space. We begin with the following well-known equivalent conditions of weakly sectorial operators.

Proposition 1 *Let $A \in B(\mathcal{H})$, and let P and Q be the real and imaginary parts of A, respectively. The following statements are equivalent:*

(1) *A is weakly sectorial.*

(2) *There exist $a, b \in \mathsf{R}$ not both zero such that $aP + bQ$ is positive semidefinite.*

(3) *There exists a complex unimodular number μ such that $\mu A + \bar{\mu} A^*$ is positive semidefinite.*

Next, we characterize weakly sectorial operators $A \in B(\mathcal{H})$ in terms of low dimension compressions of A, i.e., operators of the form X^*AX for some $X : \mathcal{H}_1 \to \mathcal{H}$ where $\mathcal{H}_1$ is a low dimension subspace of $\mathcal{H}$. Note that if $W(X^*AX)$ lies in the same closed half plane, then it is easy to show that $W(A)$ lies in the same halfplane. Our theorem shows that even if we only know that $W(X^*AX)$ lies in a closed half plane which may a priori depend on X, it will follow that $W(A)$ and hence all $W(X^*AX)$ lie in the same closed half plane. It is also shown that weakly sectorial operators A can be characterized in terms of the linearly dependence of the vectors $(A+A^*)x$ and $(A-A^*)x$ if $(Ax,x)=0$.

Theorem 2 *Let $A \in B(\mathcal{H})$, and let P and Q be the real and imaginary parts of A, respectively. Then A is weakly sectorial if and only if any one of the following four equivalent statements holds:*

(4) *For any subspace $\mathcal{H}_1$ of $\mathcal{H}$, and any bounded linear map $X : \mathcal{H}_1 \to \mathcal{H}$, the operator X^*AX is weakly sectorial.*

(5) *For all integers $k \geq 2$ we have the property that for every linear operator $X : \mathsf{C}^k \to \mathcal{H}$ satisfying $X^*X = I_k$, the operator $X^*AX \in B(\mathsf{C}^k)$ is weakly sectorial.*

(5′) *For every linear operator $X : \mathsf{C}^2 \to \mathcal{H}$ satisfying $X^*X = I_2$, the operator $X^*AX \in B(\mathsf{C}^2)$ is weakly sectorial.*

(6) *For every $x \in \mathcal{H}$ such that $(Px,x) = (Qx,x) = 0$, the two vectors Px and Qx are R-linearly dependent.*

In this statement, the equivalence of the weak sectoriality and condition (4) is trivial, and is presented only for convenience. Conditions (5) and (5') express the equivalence of the "local weak sectoriality" and the "global weak sectoriality".

Proof. Suppose $\mathcal{H}_1$ is a subspace of $\mathcal{H}$ and $X : \mathcal{H}_1 \to \mathcal{H}$. If $aP + bQ$ is positive semidefinite, then so is $aX^*PX + bX^*QX$. By condition (2) in Proposition 1, if A is weakly sectorial, then so is X^*AX. Thus, condition (4) holds.

Using the same arguments as in the last paragraph, one sees that condition (4) implies condition (5). In particular, the weaker condition (5') follows as well.

Next, we assume the condition (5'). We prove that A is weakly sectorial. We assume that the dimension of $\mathcal{H}$ is larger than 2 to avoid trivial consideration. If A is not weakly sectorial, then 0 is an interior point of $W(A)$. So, there exist unit vectors $x, y, z \in \mathcal{H}$ such that 0 lies in the interior of the convex hull of $\{(Ax,x), (Ay,y), (Az,z)\}$. Let B be the compression of A on the space spanned by $\{x,y,z\}$. Then $W(B)$ contains $\{(Ax,x), (Ay,y), (Az,z)\}$, and hence B is not weakly sectorial. Hence, there exists $r > 0$ such that $re^{i\theta} \in W(B)$ for every $\theta \in [0, 2\pi)$. In other words, for every complex unimodular number μ, there are unit vectors u and v (depending on μ) in C^3 such that $(Bu,u) = r\bar{\mu}$ and $(Bv,v) = -r\bar{\mu}$. Thus, there exists a 3×2 matrix X with $X^*X = I_2$ and $W(X^*BX)$ containing $\pm r\bar{\mu}$. Since $W(X^*BX)$ lies in a half space defined by a line passing through the origin, we see that μX^*BX is self-adjoint. So, $X^*(\mu B - \bar{\mu}B^*)X = 0_2$, and hence $\mu B - \bar{\mu}B^*$ is singular for all complex numbers μ. As a result, if $B = H + iG$ with $H = (B+B^*)/2$, we have $\det(H + rG) = 0$ for all $r \in \mathsf{R}$. So, $\det(B) = \det(H + iG) = 0$, i.e., 0 is an eigenvalue of B. If 0 is not an orthogonally reducing eigenvalue of B, then there is a 3×2 matrix X with $X^*X = I_2$

such that $X^*BX = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$ with $a \neq 0$, and hence 0 lies in the interior of $W(X^*BX)$ by the elliptical range theorem (see, for example, [6]), which contradicts condition (5'). If 0 is an orthogonally reducing eigenvalue of B, then B is unitarily similar to $[0] \oplus B_0$. Since $W(B)$ is the convex hull of $\{0\} \cup W(B_0)$ (a well known property of the numerical range, see, e.g., [3, Section 1.2]) containing 0 in its interior, it follows that 0 lies in the interior of $W(B_0)$, which is again a contradiction.

Next, we turn to condition (6). Suppose A is weakly sectorial. Then condition (2) of Proposition 1 holds. Let $x \in \mathcal{H}$ be such that $(Px, x) = (Qx, x) = 0$. Then $((aP + bQ)x, x) = 0$, and since $aP + bQ$ is positive semidefinite, we must have $a(Px) + b(Qx) = 0$, i.e., Px and Qx are R-linearly dependent.

Conversely, suppose condition (6) holds. Assume that A is not weakly sectorial. By condition (4), there is $X : \mathsf{C}^2 \to \mathcal{H}$ such that $X^*X = I_2$ and $B = X^*AX$ is not weakly sectorial. Hence, 0 is an interior point of $W(B)$ and there is a unit vector $v \in \mathsf{C}^2$ such that $(Bv, v) = 0$. Let $x = Xv$. Then

$$(Ax, x) = (Bv, v) = 0.$$

By condition (6), there exists a real vector $(\cos\theta, \sin\theta)$ such that $\cos\theta Px + \sin\theta Qx = 0$. Hence,

$$(e^{-i\theta}B + e^{i\theta}B^*)v = \cos\theta X^*PXv + \sin\theta X^*QXv = 0.$$

Consequently, the real part of $e^{-i\theta}B$ is either positive semidefinite, or negative semidefinite. So, B is weakly sectorial, which is a contradiction. $\square$

In [5] an independent proof of the equivalence of Theorem 2 (6) and Proposition 1 (3) (for finite dimensional $\mathcal{H}$) was given using canonical forms of pairs of hermitian matrices.

An operator $A \in B(\mathcal{H})$ is said to be *essentially self-adjoint* if there exist $a, b \in \mathsf{C}$ and $H = H^* \in B(\mathcal{H})$ such that $a \neq 0$ and $A = aH + bI$. Clearly, A is essentially self-adjoint, if and only if $W(A)$ is a subset of a straight line. Similarly to Theorem 2, we have the following result on equivalence of "local" and "global" essential selfadjointness, whose finite dimensional version was proved in [7].

Theorem 3 *Let $A \in B(\mathcal{H})$. Then A is essentially self-adjoint if and only if any one of the following two equivalent statements holds true:*

(7) *For all integers $k \geq 2$ we have that for every linear operator $X : \mathsf{C}^k \to \mathcal{H}$ satisfying $X^*X = I_k$, $W(X^*AX)$ is a subset of a straight line.*

(8) *For every linear operator $X : \mathsf{C}^2 \to \mathcal{H}$ satisfying $X^*X = I_2$, $W(X^*AX)$ is a subset of a straight line.*

The proof is analogous to that of (part of) Theorem 2, and therefore is omitted.

3 Results and proofs: The real case

In this section $\mathcal{H}_r$ is a real Hilbert space. As in the complex case one verifies that a pair (P, Q) of selfadjoint operators in $B(\mathcal{H}_r)$ is real weakly sectorial if and only if some linear combination $aP + bQ$ is positive semidefinite, where a, b are real numbers not both zero.

The real analogue of Theorem 2 reads as follows:

Theorem 4 *Let there be given a pair of selfadjoint operators (P, Q), $P, Q \in B(\mathcal{H}_r)$. Then the following five statements are equivalent:*

(9) (P, Q) *is real weakly sectorial;*

(10) *For any subspace $\mathcal{H}_{r1}$ of $\mathcal{H}_r$, and any bounded linear map $X : \mathcal{H}_{r1} \to \mathcal{H}_r$, the pair of operators (X^*PX, X^*QX) is real weakly sectorial.*

(11) *For all integers $k \geq 2$ we have the property that for every linear operator $X : \mathsf{R}^k \to \mathcal{H}_r$ satisfying $X^*X = I_k$, the pair (X^*PX, X^*QX) is real weakly sectorial.*

(12) *For every linear operator $X : \mathsf{R}^2 \to \mathcal{H}_2$ satisfying $X^*X = I_2$, the pair (X^*PX, X^*QX) is real weakly sectorial.*

(13) *For every $x \in \mathcal{H}_r$ such that $(Px, x) = (Qx, x) = 0$, the two vectors Px and Qx are R-linearly dependent, and at least one of the following conditions (i) and (ii) fails:*

  (i) $\operatorname{Ker} P = \operatorname{Ker} Q$ *has codimension* 2*;*

  (ii) *the dimension of* $\operatorname{Range}(aP + bQ)$ *is equal to* 2 *for every $a, b \in \mathsf{R}$ not both zero.*

Note that in contrast to the complex case, the condition

(14) For every $x \in \mathcal{H}_r$ such that $(Px, x) = (Qx, x) = 0$, the two vectors Px and Qx are R-linearly dependent,

is generally not equivalent to (9) - (12), as it was observed in [5]. Namely, if

$$P = \begin{bmatrix} \mu & \nu \\ \nu & -\mu \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mu, \nu \in \mathsf{R},\ \mu \neq 0,$$

then (14) holds true (because only the zero vector x satisfies $(Px, x) = (Qx, x) = 0$), but the pair (P, Q) is not real weakly sectorial. See also the proof of Proposition 5 below.

The proof of equivalence of (9) - (12) is basically the same as the proof of the corresponding parts of Theorem 2, using the well-known real analogue (proved by

Brickman [1]) of the Toeplitz-Hausdorff theorem on the convexity of the numerical range, namely, that $W_r(P,Q)$ is convex if $\dim \mathcal{H}_r \geq 3$ and is an ellipse if $\dim \mathcal{H}_r = 2$.

The equivalence of (9) and (13) was proved in [5] for the case when $\mathcal{H}_r$ is finite dimensional using the canonical form for pairs of hermitian matrices.

To proceed with the proof of Theorem 4, we start with a proposition.

Proposition 5 *Assume that both conditions* (*i*) *and* (*ii*) *hold true. Then the pair* (P,Q) *is not real weakly sectorial. Moreover, in this case*

$$((Px,x),(Qx,x)) = (0,0), \qquad x \in \mathcal{H}_r, \tag{3.1}$$

if and only if

$$Px = Qx = 0.$$

Proof. Suppose (i) and (ii) hold. With respect to the orthogonal decomposition $\mathcal{H}_r = \operatorname{Ker} P \oplus \operatorname{Span}\{x\} \oplus \operatorname{Span}\{y\}$, for a suitable choice of orthonormal vectors $x, y \in \mathcal{H}_r$, we have

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}, \qquad \text{where } \alpha, \beta \in \mathsf{R} \setminus \{0\}.$$

Replacing if necessary P by $P + \mu Q$ for a suitable real μ we may assume that P has the following form with respect to the same decomposition:

$$P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & z & \gamma \end{bmatrix}, \qquad \text{where } z, \gamma \in \mathsf{R}.$$

Here $z \neq 0$, otherwise a contradiction with (i) occurs. We claim that $aP + bQ$ is indefinite for all $a, b \in \mathsf{R}$ not both zero. It will then follow that the pair (P,Q) is not weakly sectorial. If our claim is not true, then for the continuous curve

$$R(t) = (\cos t)P + (\sin t)Q, \qquad t \in \mathsf{R}$$

there exists $t_0 \in \mathsf{R}$ such that $R(t_0)$ is positive semidefinite, and $R(t_0 + \pi)$ is negative semidefinite. By the continuity of eigenvalues, $\operatorname{Ker} R(t_1)$ has codimenion less than 2 for some $t_1 \in [t_0, t_0 + \pi]$, a contradiction with (ii). So, our claim holds.

Clearly, if $Px = Qx = 0$ then (3.1) holds. Conversely, assume that (3.1) holds true for some x such that $x \notin \operatorname{Ker} P = \operatorname{Ker} Q$. By the reduction in the proof at the beginning of our proof, there exists $\widehat{x} = [x_1, x_2]^t \in \mathsf{R}^2 \setminus \{0\}$ such that

$$0 = \left(\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \widehat{x}, \widehat{x} \right) = \alpha x_1^2 + \beta x_2^2 \quad \text{and} \quad 0 = \left(\begin{bmatrix} 0 & z \\ z & \gamma \end{bmatrix} \widehat{x}, \widehat{x} \right) = 2z x_1 x_2 + \gamma x_2^2.$$

It follows that

$$\det \left((z x_1) \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} + (\beta x_2) \begin{bmatrix} 0 & z \\ z & \gamma \end{bmatrix} \right)$$

$$= z\alpha\beta\gamma x_1 x_2 + 2z^2\alpha\beta x_1^2 - z^2\alpha\beta x_1^2 - z^2\beta^2 x_2^2 = 0,$$

a contradiction with (ii). □

The following lemma will be needed in this paper only in a particular situation, but it is sufficiently interesting to be stated and proved in the more general case. We denote by $\mathrm{Proj}_{\mathcal{N}}$ the orthogonal projection on a subspace $\mathcal{N}$. The set of subspaces of $\mathcal{H}_r$ of a fixed finite dimension m will be denoted by $\mathrm{Gras}_m(\mathcal{H}_r)$; it is a complete metric space in the *gap topology*, i.e., with metric defined by the *gap*

$$\theta(\mathcal{M}, \mathcal{N}) = \|\mathrm{Proj}_{\mathcal{N}} - \mathrm{Proj}_{\mathcal{M}}\|, \quad \mathcal{M}, \mathcal{N} \in \mathrm{Gras}_m(\mathcal{H}_r).$$

See, for example, [4] for more details on this topology.

Lemma 6 *Let m and n be given positive integers, and let $A \in B(\mathcal{H}_r)$. Then the set $\Xi(A; m, n)$ of all subspaces $\mathcal{M} \in \mathrm{Gras}_m(\mathcal{H}_r)$ such that the range of the operator $\mathrm{Proj}_{\mathcal{M}} A \mathrm{Proj}_{\mathcal{M}}$ has dimension at least n is either empty or dense in $\mathrm{Gras}_m(\mathcal{H}_r)$, in the gap topology.*

The result of Lemma 6 also holds (with the same proof) for operators and subspaces of a complex Hilbert space.

Proof. We can assume that

$$\dim \left(\mathrm{Range}\, \mathrm{Proj}_{\mathcal{M}_0} A \mathrm{Proj}_{\mathcal{M}_0}\right) \geq n \tag{3.2}$$

for some $\mathcal{M}_0 \in \mathrm{Gras}_m(\mathcal{H}_r)$. By [8, Theorem 1], the set of subspaces of $\mathcal{H}_r$ which are direct complements to $\mathcal{M}_0^{\perp}$ is dense in $\mathrm{Gras}_m(\mathcal{H}_r)$. (The statement and proof of [8, Theorem 1] are given in [8] for the complex case; the proof applies without change to the real case as well.) Denote

$$D_{\mathcal{M}_0} := \{\mathcal{N} \in \mathrm{Gras}_m(\mathcal{H}_r) \, : \, \mathcal{N} \dot{+} \mathcal{M}_0^{\perp} = \mathcal{H}_r\}.$$

Thus, it suffices to show that

$$D_{\mathcal{M}_0} \cap \Xi(A; m, n) \quad \text{is dense in } D_{\mathcal{M}_0}. \tag{3.3}$$

Note that every $\mathcal{N} \in D_{\mathcal{M}_0}$ has the form

$$\mathcal{N} = \mathrm{Range} \begin{bmatrix} I \\ X \end{bmatrix}, \quad \begin{bmatrix} I \\ X \end{bmatrix} : \mathcal{M}_0 \; \to \; \mathcal{M}_0 \oplus \mathcal{M}_0^{\perp}, \tag{3.4}$$

for some bounded linear operator $X : \mathcal{M}_0 \; \to \; \mathcal{M}_0^{\perp}$, which is uniquely defined by $\mathcal{N}$. Moreover, the formula (3.4) establishes a homeomorphism between $D_{\mathcal{M}_0}$ as a subset of $\mathrm{Gras}_m(\mathcal{H}_r)$ with the induced topology and the Banach space $B(\mathcal{M}_0, \mathcal{M}_0^{\perp})$ of bounded linear operators from $\mathcal{M}_0$ to $\mathcal{M}_0^{\perp}$ with the operator topology. To verify the homeomorphism property, use the formula for the orthogonal projection on $\mathcal{N} \in D_{\mathcal{M}_0}$:

$$\mathrm{Proj}_{\mathcal{N}} = \begin{bmatrix} Y & YX^* \\ XY & XYX^* \end{bmatrix}, \quad Y = (I + X^*X)^{-1}, \tag{3.5}$$

where X is taken from (3.4).

We have reduced the proof to the following claim: The set of all operators $X \in B(\mathcal{M}_0, \mathcal{M}_0^{\perp})$ for which

$$\dim \mathrm{Range} \begin{bmatrix} Y & YX^* \\ XY & XYX^* \end{bmatrix} A \begin{bmatrix} Y & YX^* \\ XY & XYX^* \end{bmatrix} \geq n, \quad Y = (I + X^*X)^{-1}, \tag{3.6}$$

is dense in $B(\mathcal{M}_0, \mathcal{M}_0^{\perp})$. Let $X_0 \in B(\mathcal{M}_0, \mathcal{M}_0^{\perp})$, and consider a family of operators on the m-dimensional space $\mathcal{M}_0$:

$$A(t) = [I \quad tX_0^*] A \begin{bmatrix} I \\ tX_0 \end{bmatrix}, t \in \mathsf{R}.$$

We will represent $A(t)$ as $m \times m$ matrices with respect to a fixed orthonormal basis in $\mathcal{M}_0$. By (3.2), the dimension of the range of $A(0)$ is at least n. Thus, there exists an $n \times n$ submatrix, call it $A_s(0)$, in $A(0)$ with a nonzero determinant. Clearly, the determinant of $A_s(t)$ is a polynomial of t, and therefore for $t \neq 1$ arbitrarily close to 1, the determinant of $A_s(t)$ is also nonzero, hence the dimension of the range of $A(t)$ is at least n. A fortiori

$$\dim \mathrm{Range} \begin{bmatrix} Y_0 & Y_0tX_0^* \\ tX_0Y_0 & t^2X_0Y_0X_0^* \end{bmatrix} A \begin{bmatrix} Y_0 & Y_0tX_0^* \\ tX_0Y_0 & t^2X_0Y_0X_0^* \end{bmatrix} \geq n, \; Y_0 = (I + t^2X_0^*X_0)^{-1},$$

for $t \neq 1$ arbitrarily close to 1, and the claim is proved. □

Proof of Theorem 4. In view of Proposition 5 and the remarks made after Theorem 4, we need only to prove that (9) implies (14), and that if (13) holds then (P, Q) is real weakly sectorial. The implication (9) $\Longrightarrow$ (14) is easy, because if $aP + bQ$ is positive semidefinite for some real a and b not both zero, and if $(Px, x) = (Qx, x) = 0$, then $((aP + bQ)x, x) = 0$, and in view of positive semidefiniteness of $aP + bQ$, we have $(aP + bQ)x = 0$, i.e., Px and Qx are linearly dependent.

Assuming that (13) holds, we will prove that the pair (P, Q) is real weakly sectorial. We suppose that $\mathcal{H}$ is infinite dimensional, as the result in the finite dimensional case was proved in [5]. It is easy to check, and will be used in the sequel, that for every subspace $\mathcal{N} \subseteq \mathcal{H}_r$, the pair

$$(\mathrm{Proj}_{\mathcal{N}} P \mathrm{Proj}_{\mathcal{N}}, \mathrm{Proj}_{\mathcal{N}} Q \mathrm{Proj}_{\mathcal{N}}),$$

considered as a pair of selfadjoint operators on $\mathcal{N}$, also satisfies condition (13).

Suppose first that the codimension of $\operatorname{Ker} P$ is larger than 2. Then there exists a subspace $\mathcal{M}_0$ of $\mathcal{H}_r$ of dimension at most six and at least three (spanned by three linearly independent vectors x_1, x_2, x_3 in the range of P, and by three vectors y_1, y_2, y_3 such that $Py_1 = x_1, Py_2 = x_2, Py_3 = x_3$) such that the range of the linear transformation $\operatorname{Proj}_{\mathcal{M}_0} P \operatorname{Proj}_{\mathcal{M}_0}$ on $\mathcal{M}_0$ has dimension at least 3. Denote by κ the dimension of $\mathcal{M}_0$. By the finite dimensional result of [5], the pair

$$(\operatorname{Proj}_{\mathcal{M}_0} P \operatorname{Proj}_{\mathcal{M}_0}, \operatorname{Proj}_{\mathcal{M}_0} Q \operatorname{Proj}_{\mathcal{M}_0}) \tag{3.7}$$

is real weakly sectorial. By Lemma 6, the set of all κ dimensional subspaces $\mathcal{M}$ such that

$$\dim\,(\operatorname{Range} \operatorname{Proj}_{\mathcal{M}} P \operatorname{Proj}_{\mathcal{M}}) \geq 3$$

is dense in $\operatorname{Gras}_\kappa(\mathcal{H}_r)$. Therefore, by continuity we obtain that the pair (3.7) is real weakly sectorial for every κ dimensional subspace of $\mathcal{H}_r$. Now (P, Q) is real weakly sectorial by the equivalence of (12) and (9). We are similarly done if the codimension of $\operatorname{Ker} Q$ is larger than 2.

If the codimension of both $\operatorname{Ker} Q$ and $\operatorname{Ker} P$ is equal 2, but

$$\operatorname{Ker} Q \neq \operatorname{Ker} P,$$

or if (i) holds true but (ii) does not, then with respect to the orthogonal decomposition

$$\mathcal{H}_r = (\operatorname{Ker} P \cap \operatorname{Ker} Q) \oplus (\operatorname{Ker} P \cap \operatorname{Ker} Q)^\perp$$

write the operators in the block matrix form

$$P = \begin{bmatrix} 0 & 0 \\ 0 & \widehat{P} \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & \widehat{Q} \end{bmatrix}.$$

Thus, it will suffice to prove that the pair $(\widehat{P}, \widehat{Q})$ is real weakly sectorial. But since $(\operatorname{Ker} P \cap \operatorname{Ker} Q)^\perp$ is finite dimensional we are done by the result of [5]. $\square$

References

[1] L. Brickman. On the field of values of a matrix. *Proc. of Amer. Math. Soc.*, 12 (1961), 61-66.

[2] D. E. Edmunds, and W. D. Evans, *Spectral Theory and Differential Operators*, Oxford University Press, New York, 1987.

[3] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[4] T. Kato, *Perturbation Theory for Linear Operators*, Springer Verlag, 1995. (Reprint of 1980 edition.)

[5] P. Lancaster and L. Rodman, Canonical forms for hermitian matrix pairs under strict equivalence and congruence, submitted for publication.

[6] C. K. Li, A simple proof of the elliptical range theorem. *Proc. Amer. Math. Soc.* 124 (1996), 1985-1986.

[7] C. K. Li and N. K. Tsing, On the k-th matrix range, *Linear and Multilinear Algebra* 28 (1991), 229-239.

[8] L. Rodman, On global geometric properties of subspaces in Hilbert space, *Journal of Functional Analysis*, 45 (1982), 226-235.