The diameter and width of higher rank numerical ranges

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Abstract We give a computational criterion for the boundary points of the higher rank numerical range of a matrix, and describe a general scheme for constructing its boundary curve. The results are used to study the diameter and minimal width of the higher rank numerical range. Detailed study on a 6-by-6 matrix is done. While its rank-2-numerical range is not of constant width, it shares interesting properties of convex set with constant width. A conjecture on the classical numerical range is extended to the higher rank numerical range, namely, the higher rank numerical range of a matrix has constant width if and only if it is a circular disk.

2010 Mathematics Subject Classification: 15A60, 52A10
Keywords: Diameter, width, constant width, higher rank numerical range.

1 Introduction

Let A be an $n \times n$ matrix. In connection to the study of quantum error correction, Choi et al. [4, 5] introduced the rank-k-numerical range $\Lambda_k(A)$ of A which is defined by

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} : AP = \lambda P \text{ for some } k-\text{dimensional orthogonal projection } P\},$$

where k is a given integer in $\{1, \ldots, n\}$. If $k = 1$, the rank-k-numerical range reduces to the classical numerical range which is denoted by

$$W(A) = \{\langle A\xi, \xi\rangle : \xi \in \mathbb{C}^n, \langle \xi, \xi\rangle = 1\}.$$

Choi et al. [4, 5] conjectured that $\Lambda_k(A)$ is convex, Woerdeman [12] confirmed the conjecture. Furthermore, Li and Sze [8] showed that

$$\Lambda_k(A) = \cap_{t\in[0,2\pi)}\{\mu \in \mathbb{C} : \Re(e^{-it}\mu) \leq \lambda_k(\Re(e^{-it}A)\},$$

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where $\lambda_1(H) \geq \cdots \geq \lambda_n(H)$ denote the eigenvalues of a Hermitian matrix $H \in M_n$, and $\Re(B) = (B+B^*)/2$ the Hermitian part of a square matrix B. The characterization of $\Lambda_k(A)$ can be rewritten as

$$\Lambda_k(A) = \cap_{-\pi/2 \leq t \leq \pi/2} \{\mu \in \mathbb{C} : \lambda_{n+1-k}(\Re(e^{-it}A)) \leq \Re(e^{-it}\mu) \leq \lambda_k(\Re(e^{-it}A))\}.$$

Note that

$$\max\{\Re(\mu) : \mu \in \Lambda_k(A)\} \leq \lambda_k(\Re(A)).$$

In the case that $k = 1$, it is known that

$$\max\{\Re(\mu) : \mu \in W(A)\} = \lambda_1(\Re(A)).$$

For a compact convex set $K \subset \mathbb{C} \equiv \mathbb{R}^2$, we consider the distance

$$d(t) = \max\{\Re(e^{-it}\mu) : \mu \in K\} - \min\{\Re(e^{-it}\mu) : \mu \in K\} \tag{1.1}$$

of the two parallel support lines with normal vector e^{it} for $t \in [0, \pi]$. The diameter of K is the maximum of the function $d(t)$ on $[0, 2\pi]$. The width of K is the minimum of the function $d(t)$. The convex set K has constant width if diameter and width are equal. The diameter and width of the numerical range are investigated in [3] (see also [11]). The authors provided an algorithm for computing the diameter and width of the numerical range, and examined the numerical ranges of certain nilpotent Toeplitz matrices having constant width. As shown in [11], for the classical numerical range $W(A)$ of a matrix $A \in M_n$, the function $d(t)$ with $t \in [0, \pi]$ can be expressed as

$$d(t) = \lambda_1 \Re(e^{it}A) - \lambda_n \Re(e^{it}).$$

In view of the characterization in [8], one might think that for $\Lambda_k(A)$ the function $d(t)$ with $t \in [0, \pi]$ can be expressed as

$$d(t) = \lambda_k \Re(e^{it}A) - \lambda_{n-k+1} \Re(e^{it}).$$

It turns out that this is not true for $k > 1$. In this paper, we give a computational criterion for the boundary points of $\Lambda_k(A)$, and describe a general scheme for constructing its boundary curve; see Section 2. The results are used to study the diameter and minimal width of the higher rank numerical range that somewhat explains the above phenomenon. Detailed study on a 6-by-6 matrix is done in Section 3. While its rank-2-numerical range is not of constant width, it shares interesting properties of convex set with constant width.

We close this section by extending a conjecture on the classical numerical range in [3] to the higher rank numerical range as follows.

The higher rank numerical range of a matrix has constant width if and only if it is a circular disk.

2 Boundary of the higher rank numerical range

2.1 Boundary points

Suppose $A \in M_n$. For $t \in [0, 2\pi)$, let $f_k(t) = \lambda_k(\Re(e^{-it}A))$, $P_t = \{z \in \mathbb{C} : \Re(e^{-it}z) \leq f_k(t)\}$, and $R_t = \{re^{it} : r \geq 0\}$. We give a computational criterion for the boundary points of the higher rank numerical range. Furthermore, we obtain a modified function $\tilde{f}_k(t)$ of $f_k(t)$, which will be useful in constructing the boundary curve.

Theorem 2.1 *Let $A \in M_n$ and $k \in \{1, \ldots, n\}$ be such that 0 is an interior point of $\Lambda_k(A)$. Suppose $f_k(t) = \lambda_k(\Re(e^{-it}A))$ for $t \in [0, 2\pi)$. Then $r(t)e^{it}$ is a boundary point of $\Lambda_k(A)$ if and only if*

$$r(t) = \min\{\frac{f_k(t+s)}{\cos s} : -\pi/2 < s < \pi/2\}.$$

The modified kth eigenvalue $\tilde{f}_k(t)$ of the Hermitian matrix $\Re(e^{-it}A)$ is given by

$$\begin{aligned}\tilde{f}_k(t) &= \max\{\Re(e^{-it}z) : z \in \Lambda_k(A)\} = \max\{r(t+u)\cos u : -\pi/2 \le u \le \pi/2\} \\ &= \max_{-\pi/2 \le u \le \pi/2} \min_{-\pi/2 < s < \pi/2} \{f_k(t+u+s)\frac{\cos u}{\cos s}\}.\end{aligned}$$

When $k = 1$, we have $\tilde{f}_1(t) = f_1(t)$ for all $t \in [0, 2\pi)$.

Proof. Evidently, the intersection of $\Lambda_k(A)$ and the ray $R_t = \{re^{it} : r \ge 0\}$ can be written as

$$R_t \cap (\cap_{s\in[0,2\pi)} P_s) = \cap_{s\in[0,2\pi)}(R_t \cap P_s) = \cap_{t-\pi/2 \le s \le t+\pi/2}(R_t \cap P_s).$$

If t is a minimal point of the function $f_k(u)$ on $u \in [0, 2\pi)$, $r(t)$ is given as $f_k(t)$. So we may assume that

$$f_k(t) = \alpha \min\{f_k(u) : u \in [0, 2\pi)\}$$

for some $\alpha > 1$. Then the angle $-\pi/2 < s < \pi/2$, $\cos s < 1/\alpha$ satisfies

$$\frac{f_k(t+s)}{\cos s} > f_k(t) = \frac{f_k(t)}{\cos 0}.$$

So the continuous function $f_k(t+s)/\cos s$ on the closed interval $[-\arccos(1/\alpha), \arccos(1/\alpha)]$ attains its minimum. We get the first assertion.

For the second assertion, note that for each $t \in [0, 2\pi)$, there is a unit vector x such that $\Re(e^{-it}A)x = f_1(t)x$ so that $x^*Ax \in W(A)$ with real part equal to $f_1(t)$. So that $\tilde{f}_1(t) = f_1(t)$. ∎

Remark As shown in Section 3, in general one may have $f_2(t) > \tilde{f}_2(t)$. One can modify the example to get examples that $f_k(t) > \tilde{f}_k(t)$ for any $k > 1$; see the final remark in Section 3.

2.2 Boundary generating curve

By Rellich's perturbation theorem [7], the function $f_k(t)$ is differentiable except for a finite number of points. We consider the envelope of the lines

$$\{\mu \in \mathbb{C} : \Re(e^{-it}\mu) = f_k(t)\}$$

for those $0 \le t \le 2\pi$ at which $f_k(t)$ is differentiable (cf. [2]), which we call the boundary generating curve $z_k(t)$ of $\Lambda_k(A)$. So this curve $z_k(t) = X(t) + iY(t)$ is given by

$$z_k(t) = f_k(t)e^{it} + if_k'(t)e^{it}, \tag{2.1}$$

i.e.,

$$\Re(z_k(t)) = f_k(t)\cos t - f_k'(t)\sin t,$$
$$\Im(z_k(t)) = f_k(t)\sin t + f_k'(t)\cos t,$$

for differentiable points t. At non-differentiable points t, the two points $z_k(t^-)$, $z_k(t^+)$ may be different. The line segment $[z_k(t^-), z_k(t^+)]$ forms a flat portion on the boundary of $\Lambda_k(A)$. It is a crucial subject to recognize the relation between the boundary of $\Lambda_k(A)$ and the boundary generating curve $z_k(t)$. Kippenhahn showed that $\Lambda_1(A)$ is the convex hull of the curve $z_1(t)$. Recently, Plaumann et al. [9] provided a Kippenhahn type formulation of the joint numerical range of an m-ple of $n \times n$ Hermitian matrices. A modified Kippenhahn's type theorem for $\Lambda_k(A)$ is formulated in [2] at which the dual convex set $D_k(A)$ of $\Lambda_k(A)$ under the assumption that 0 is an interior point of $\Lambda_k(A)$, or equivalently $f_k(t) > 0$ for $0 \le t \le 2\pi$, is obtained as

$$D_k(A) = \{(x, y) \in \mathbb{R}^2 : xu + yv + 1 \ge 0 \,\text{for every}\, u + iv \in \Lambda_k(A), u, v \text{ are real}\}.$$

This compact convex set $D_k(A)$ is the convex hull of the simple closed curve

$$\Big\{(x(t), y(t)) = (\frac{-\cos t}{f_k(t)}, \frac{-\sin t}{f_k(t)}) : 0 \le t \le 2\pi\Big\}. \tag{2.2}$$

In the case $k = 1$, the curve (2.2) is an oval and is exactly the boundary of $D_1(A)$ (see [1, Corollary 3.23]). At a differentiable point of $f_k(t)$, the tangent of the curve $(x(t), y(t))$ (2.2) is given by

$$X(t)x + Y(t)y + 1 = 0,$$

where

$$X(t) = \frac{-y'(t)}{x(t)y'(t) - y(t)x'(t)}, \quad Y(t) = \frac{x'(t)}{x(t)y'(t) - y(t)x'(t)}.$$

Under the assumption $f_k(t) > 0$ for $t \in [0, 2\pi]$, the curve $(x(t), y(t))$ (2.2) is constructed. Then the set $D_k(A)$ is the convex hull of the curve (2.2), and based on the duality the range $\Lambda_k(A)$ is determined by

$$\Lambda_k(A) = \{u + iv \in \mathbb{C} : xu + yv + 1 \ge 0 \,\text{for any}\, (x, y) \in D_k(A)\}.$$

At a $C^{(1)}$-differentiable point t of $f_k(t)$, the curve $(x(t), y(t))$ (2.2) has the following length element

$$d\ell = \frac{\sqrt{f_k(t)^2 + f_k'(t)^2}}{f_k(t)^2} dt.$$

The signed curvature $\kappa(t)$ of the curve $(x(t), y(t))$ (2.2) is given by

$$\kappa(t) = \frac{f_k(t) + f_k''(t)}{(f_k(t)^2 + f_k'(t)^2)^{3/2}}$$

(cf. [6, Chapter 1]). The signature of $f_k(t) + f_k''(t)$ may change as the typical example given in [2, Example 1],

$$f_2(t) = \Big(2 + \sqrt{2} - \frac{\sqrt{2} + 1}{2}\sqrt{5 - \cos(4t)}\Big)^{1/2},$$

$0 \le t \le 2\pi$. Even if $f_k(t) + f_k''(t) > 0$ for any $C^{(2)}$-differentiable points t, the curve $(x(t), y(t))$ may have interior points in $D_k(A)$ a case which we will treat in Section 2. An angle $t \in [0, 2\pi]$ for which $(x(t), y(t))$ is an interior point of $D_k(A)$ corresponds to $z_k(t) \notin \Lambda_k(A)$. If $f_k(t)$ is $C^{(2)}$-differentiable at every $t \in [0, 2\pi]$ and $f_k(t) + f_k''(t) > 0$ for all t, the total absolute curvature of this curve C (2.2) satisfies

$$\int_C |\kappa(\ell)| d\ell = \int_0^{2\pi} \kappa(t) \frac{d\ell}{dt} = \int_0^{2\pi} \frac{f_k(t)^2 + f_k''(t) f_k(t)}{f_k(t)^2 + f_k'(t)^2} dt = 2\pi$$

and this identity implies that the domain bounded by C is convex (cf. [6, 10]).

3 A case study

In this section, we do a detailed study of the rank-2-numerical range of the following matrix

$$A = A(r) = \begin{pmatrix} 1 & 2r \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} w & 2r \\ 0 & w \end{pmatrix} \oplus \begin{pmatrix} w^2 & 2r \\ 0 & w^2 \end{pmatrix} \qquad \text{with } \ w = e^{2\pi i/3} = (-1 + i\sqrt{3})/2. \tag{3.1}$$

Not only does the study illustrate how to use our results in Section 2, it also shows that the higher rank numerical range may have some geometrical properties resembling convex sets with constant width.

3.1 Diameter and width

Theorem 3.1 *Let A be the matrix described in (3.1). Suppose $r \ge \sqrt{3}/2$. Then*

$$\lambda_2(\Re(e^{-it} A)) - \lambda_5(\Re(e^{-it} A)) = 2r \tag{3.2}$$

for all $0 \le t \le 2\pi$. In fact, the six eigenvalues of $\Re(e^{-it}A)$ can be computed explicitly and ordered according to t-intervals:

(i) If $t \in [0, \pi/3]$, then

$$\begin{aligned} r + \cos t \ge r + \frac{1}{2}(-\cos t + \sqrt{3}\sin t) \ge r + \frac{1}{2}(-\cos t - \sqrt{3}\sin t) \\ \ge -r + \cos t \ge -r + \frac{1}{2}(-\cos t + \sqrt{3}\sin t) \ge -r + \frac{1}{2}(-\cos t - \sqrt{3}\sin t), \end{aligned} \tag{3.3}$$

(ii) If $t \in [\pi/3, 2\pi/3]$, then

$$\begin{aligned} r + \frac{1}{2}(-\cos t + \sqrt{3}\sin t) \ge r + \cos t \ge r + \frac{1}{2}(-\cos t - \sqrt{3}\sin t) \ge -r + \frac{1}{2}(-\cos t + \sqrt{3}\sin t) \\ \ge -r + \cos t \ge -r + \frac{1}{2}(-\cos t - \sqrt{3}\sin t), \end{aligned} \tag{3.4}$$

(iii) If $t \in [2\pi/3, \pi]$, then

$$\begin{aligned} r + \frac{1}{2}(-\cos t + \sqrt{3}\sin t) \ge r + \frac{1}{2}(-\cos t - \sqrt{3}\sin t) \ge r + \cos t \ge -r + \frac{1}{2}(-\cos t + \sqrt{3}\sin t) \\ \ge -r + \frac{1}{2}(-\cos t - \sqrt{3}\sin t) \ge -r + \cos t. \end{aligned} \tag{3.5}$$

Proof. Note that for any $t \in [0, 2\pi)$, if $\alpha_t = \frac{1}{2}(-\cos t + \sqrt{3}\sin t)$ and $\beta_t = \frac{-1}{2}(\cos t + \sqrt{3}\sin t)$, then

$$\Re(e^{-it}A) = \cos t\Re(A) + \sin t\Im(A) = \begin{pmatrix} \cos t & re^{-it} \\ re^{it} & \cos t \end{pmatrix} \oplus \begin{pmatrix} \alpha_t & re^{-it} \\ re^{it} & \alpha_t \end{pmatrix} \oplus \begin{pmatrix} \beta_t & re^{-it} \\ re^{it} & \beta_t \end{pmatrix}$$

with eigenvalues

$$\cos t \pm r,\ \frac{1}{2}(-\cos t + \sqrt{3}\sin t) \pm r,\ \frac{1}{2}(-\cos t - \sqrt{3}\sin t) \pm r.$$

Since $\lambda_k(\Re(e^{-it}A))$ is $-\lambda_{n+1-k}(\Re(e^{-i(t+\pi)}A))$, we only need to consider $t \in [0, \pi]$.

The third inequality in (3.3), (3.4), (3.5) holds due to the fact that

$$2r - \sqrt{3}\Big(\frac{\sqrt{3}}{2}\cos t + \frac{1}{2}\sin t\Big) = 2r - \sqrt{3}\sin(t + \frac{\pi}{3}) \geq 0$$

for $t \in [0, \pi/3]$,

$$2r - \sqrt{3}\sin t \geq 0$$

for $t \in [\pi/3, 2\pi/3]$, and

$$2r - \sqrt{3}\Big(-\frac{3}{2}\cos t + \frac{1}{2}\sin t\Big) = 2r - \sqrt{3}\sin(t - \frac{\pi}{3}) \geq 0$$

for $t \in [2\pi/3, \pi]$ under the assumption that $r \geq \sqrt{3}/2$. The second and the fifth inequalities in (3.3), and the first and the fourth inequalities in (3.5) follow from $\sin t \geq 0$ for $t \in [0, \pi]$. The first and fourth inequalities in (3.3) hold for the reason that

$$\sqrt{3}\cos t - \sin t \geq 0$$

for $t \in [0, \pi/3]$. The first, second, fourth and the fifth inequalities in (3.4) follow from

$$\sin t \pm \sqrt{3}\cos t \geq 0$$

for $t \in [\pi/3, 2\pi/3]$. The second and the fifth inequalities in (3.5) hold owing to the fact that

$$-\sqrt{3}\cos t - \sin t \geq 0$$

for $t \in [2\pi/3, \pi]$. This completes the proof of the inequalities (3.3), (3.4), (3.5). Applying the five ordered eigenvalues of $\Re(e^{-it}A))$ in the respect t-intervals of $(i), (ii), (iii)$, we obtain that the difference $\lambda_2(\Re(e^{-it}A)) - \lambda_5(\Re(e^{-it}A))$ is $2r$. Now, the first assertion readily from (3.3) – (3.5). ■

Remark

1. If we ignore the distinction of the third and the fourth eigenvalues of $\Re(e^{-it}A)$, we may relax the assumption $r \geq \sqrt{3}/2$ by $r \geq 3/4$ in Theorem 3.1.

2. Given an $n \times n$ matrix B, if $\lambda_1(\Re(e^{-it}B)) - \lambda_n(\Re(e^{-it}B))$ is constant for all $t \in [0, 2\pi]$, then by [3, Theorem 2.1], $\Lambda_1(B)$ has constant width. By (3.2), $\lambda_2(\Re(e^{-it}B)) - \lambda_{n-2}(\Re(e^{-it}B))$ is constant for all $t \in [0, 2\pi]$. One may regard the boundary generating curve of $\Lambda_2(A(r))$ as a generalization of a curve enclosing a convex set of constant width. However, we may have $\max\{\Re(e^{-it}\mu) : \mu \in \Lambda_2(A(r))\} < \lambda_2(\Re(e^{-it}A(r)))$ for some t so that $\Lambda_2(A(r))$ is not of constant width.

3. Let $w = e^{2\pi i/5}$. Consider the 10×10 matrix $A = \begin{pmatrix} 1 & 2r \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} w & 2r \\ 0 & w \end{pmatrix} \oplus \begin{pmatrix} w^2 & 2r \\ 0 & w^2 \end{pmatrix} \oplus \begin{pmatrix} w^3 & 2r \\ 0 & w^3 \end{pmatrix} \oplus \begin{pmatrix} w^4 & 2r \\ 0 & w^4 \end{pmatrix}$. We may generalize the result of (3.2) in Theorem 3.1 to conclude that $\lambda_3(\Re(e^{-it}A)) - \lambda_8(\Re(e^{-it}A)) = 2r$ for all t which characterizes the boundary generating curve of $\Lambda_3(A)$.

Using the ordered eigenvalues of $\Re(e^{-it}A)$ in Theorem 3.1, we are ready to describe the boundary of $\Lambda_2(A)$.

Theorem 3.2 *Let A be the 6×6 matrix defined in (3.1). Suppose $r \geq \sqrt{3}/2$. Then, the boundary of $\Lambda_2(A)$ lies on the union of the 3 lines*

$$L_j = \{z \in \mathbb{C} : \Re(w^{j-1}z) = r - \frac{1}{2}\} \quad with \quad w = e^{i2\pi/3} \quad for\ j = 1, 2, 3,$$

and the 3 circular arcs

$$C_1 = \left\{1 + re^{it} : t \in [\frac{\pi}{3}, \frac{2\pi}{3}] \cup [\frac{4\pi}{3}, \frac{5\pi}{3}]\right\}, \quad C_2 = \left\{e^{-2i\pi/3} + re^{it} : t \in [0, \frac{\pi}{3}] \cup [\pi, \frac{4\pi}{3}]\right\},$$

$$C_3 = \left\{e^{-4i\pi/3} + re^{it} : t \in [\frac{2\pi}{3}, \pi] \cup [\frac{5\pi}{3}, 2\pi]\right\},$$

where these arcs have the normalized tangent ie^{it} for each parameter t.

Proof. By using the second and the fifth largest eigenvalue of $\Re(e^{-it}A)$ on subintervals $(0, \pi/3)$, $(\pi/3, 2\pi/3)$, $(2\pi/3, \pi)$, the boundary generating curve $z_2(t) = X(t) + iY(t)$ of $\Lambda_2(A)$ is computed as

$$(X(t), Y(t)) = \begin{cases} (1 + r\cos t, r\sin t), & \text{if } t \in (\pi/3, 2\pi/3) \cup (4\pi/3, 5\pi/3), \\ (-1/2 + r\cos t, \sqrt{3}/2 + r\sin t), & \text{if } t \in (0, \pi/3) \cup (\pi, 4\pi/3), \\ (-1/2 + r\cos t, -\sqrt{3}/2 + r\sin t, & \text{if } t \in (2\pi/3, \pi) \cup (5\pi/3, 2\pi). \end{cases}$$

for $r \geq \sqrt{3}/2$. To consider the relation between the curve $z_2(t)$ and the range $\Lambda_2(A)$, we consider the curve

$$\{(x(t), y(t)) = (-\frac{\cos t}{f_2(t)}, -\frac{\sin t}{f_2(t)}) : 0 \leq t \leq 2\pi\} \tag{3.6}$$

and its convex hull $D_2(A)$. As the result of Theorem 2.1, the second largest eigenvalue $f_2(t) = \lambda_2(\Re(e^{-it}A))$ of the Hermitian matrix $\Re(e^{-it}A)$

$$f_2(t) = \begin{cases} r + \cos t, & \text{if} t \in (\pi/3, 2\pi/3) \cup (4\pi/3, 5\pi/3), \\ r + \frac{1}{2}(-\cos t + \sqrt{3}\sin t) & \text{if } t \in (0, \pi/3) \cup (\pi, 4\pi/3), \\ r + \frac{1}{2}(-\cos t - \sqrt{3}\sin t) & \text{if } t \in (2\pi/3, \pi) \cup (5\pi/3, 2\pi), \end{cases}$$

for $r \geq \sqrt{3}/2$. The curve $(x(t), y(t))$ lies on the union of the 3 conic curves

$$\{(x, y) \in \mathbb{R}^2 : (r^2 - 1)x^2 - 2x + r^2y^2 - 1 = 0\},$$

$$\{(x, y) \in \mathbb{R}^2 : (4r^2 - 1)x^2 + 2\sqrt{3}xy + (4r^2 - 3)y^2 + 4x - 4\sqrt{3}y - 4 = 0\},$$

$$\{(x, y) \in \mathbb{R}^2 : (4r^2 - 1)x^2 - 2\sqrt{3}xy + (4r^2 - 3)y^2 + 4x + 4\sqrt{3}y - 4 = 0\}.$$

We consider the part of the curve $(x(t), y(t))$ for

$$t \in (-\delta, \delta) \cup (4\pi/3 - \delta, 4\pi/3] \cup (2\pi/3 - \delta, 2\pi/3 + \delta) \cup (4\pi/3 - \delta, 4\pi/3 + \delta)$$

for $0 < \delta < \pi/3$. For $t = 0$, the point $(x(0), y(0)) = (-1/(r - 1/2), 0)$. We also get

$$(x(4\pi/3), y(4\pi/3)) = (\frac{1}{2r-1}, \frac{\sqrt{3}}{2r-1}), \quad (x(2\pi/3), y(2\pi/3)) = (\frac{1}{2r-1}, -\frac{\sqrt{3}}{2r-1}).$$

At each point $(x(t), y(t))$ of this curve for $t = 0$, $t = 2\pi/3$ or $t = 4\pi/3$, this curve has 2 tangents. At the point $(x, y) = (-1/(r - 1/2), 0)$, the two tangents of the curve $(x(t), y(t))$ are

$$y = \pm\frac{2r-1}{\sqrt{3}}(x + \frac{2}{2r-1}), \tag{3.7}$$

which are also expressed as

$$(r - \frac{1}{2})x \pm (-\frac{\sqrt{3}}{2})y + 1 = 0.$$

This expression corresponds to the end points $(r - \frac{1}{2}) \pm (-\frac{\sqrt{3}}{2})i$ of one flat portion of the boundary of $\Lambda_2(A)$, where

$$(r - \frac{1}{2}) + \frac{\sqrt{3}}{2}i = z(0^+) = X(0^+) + iY(0^+)$$

and

$$(r - \frac{1}{2}) - \frac{\sqrt{3}}{2}i = z(0^-) = X(0^-) + iY(0^-). \qquad \blacksquare$$

The following result gives more details of Theorem 3.2 and determines the diameter and width of $\Lambda_2(A)$.

Theorem 3.3 *Let A be the 6×6 matrix defined in (3.1).*

(i) *If $\sqrt{3}/2 \leq r \leq 1$, the boundary of $\Lambda_2(A)$ is the regular triangle on the lines L_1, L_2, L_3 with vertices $-(2r - 1), (r - \frac{1}{2})(1 + \sqrt{3}i), (r - \frac{1}{2})(1 - \sqrt{3}i)$, and hence the diameter and the width of $\Lambda_2(A)$ are respectively given by $\sqrt{3}(2r - 1)$ and $\frac{3}{2}(2r - 1)$.*

(*ii*) *If $r>1$, the boundary of $\Lambda_2(A)$ has 6 circular arcs for the t-subintervals:*

$$\Sigma(r)=[-\frac{\pi}{3}+\delta(r),\frac{\pi}{3}-\delta(r)]\cup[\frac{\pi}{3}+\delta(r),\pi-\delta(r)]\cup[\pi+\delta(r),\frac{4\pi}{3}-\delta(r)],$$

where $0<\delta(r)<\pi/3$ is defined by $\sin(\delta(r))=\sqrt{3}/(2r)$. These circular arcs and the flat portions on $L_1\cup L_2\cup L_3$ form the boundary of $\Lambda_2(A)$.

(*iii*) *If $1<r\le\sqrt{3}$, the diameter and the width of $\Lambda_2(A)$ are respectively given by $\frac{\sqrt{3}}{2}(\sqrt{4r^2-3}+1)$ and $r+\frac{\sqrt{4r^2-3}}{2}$.*

(*iv*) *If $r>\sqrt{3}$, the angle $\delta(r)<\pi/6$ and the set $\Sigma(r)$ contains nonempty intervals*

$$\{t\in\Sigma(r):-\frac{\pi}{3}<t<\frac{2\pi}{3},t+\pi\in\Sigma(r)\}.$$

On these intervals the function $d(t)$ defined in (1.1) attains the value $2r$ which is the diameter of $\Lambda_2(A)$. In this case the width of $\Lambda_2(A)$ is $r+\frac{\sqrt{4r^2-3}}{2}$.

Proof. We adopt the notation in the proof of Theorem 3.2.

(*i*) Assume that $\sqrt{3}/2\le r\le 1$. Then the absolute values of the slopes of the tangents (3.7) are less than or equal to $1/\sqrt{3}=\tan(\pi/6)$ for $\sqrt{3}/2\le r\le 1$. The origin $(x,y)=(0,0)$ of the xy-plane belongs to the domain bounded by the hyperbolic arcs for $r<1$ and the parabolic arcs for $r=1$. It follows that the curve $(x(t),y(t))$ for $\sqrt{3}/2\le r\le 1$ is contained in the closed triangular domain with the 3 vertices $(x(0),y(0),(x(2\pi/3),y(2\pi/3)),(x(4\pi/3),y(4\pi/3))$ for $\sqrt{3}/2\le r\le 1$. The two lines

$$\frac{u}{2r-1}+\frac{\sqrt{3}v}{2r-1}+1=0,\ \frac{u}{2r-1}-\frac{\sqrt{3}v}{2r-1}+1=0$$

intersect at $(u,v)=(-(2r-1),0)$. By the duality of the convex set, the boundary of the range $\Lambda_2(A)$ for $\sqrt{3}/2\le r\le 1$ is the regular triangle with vertices $-(2r-1),(r-\frac{1}{2})(1+\sqrt{3}i),(r-\frac{1}{2})(1-\sqrt{3}i)$.

(*ii*) Assume that $r>1$. Then the above 3 points $(x(2k\pi/3),y(2k\pi/3))$, $k=0,1,2$, give the inequalities

$$\Re(z)\le r-\frac{1}{2},\quad \Re(e^{-2i\pi/3}z)\le r-\frac{1}{2},\quad \Re(e^{-4i\pi/3}z)\le r-\frac{1}{2}$$

for $z\in\Lambda_2(A)$. We use the duality method to determine the range $\Lambda_2(A)$. For this purpose, we consider the curve $(x(t),y(t))$ on a typical interval $2\pi/3\le t\le 4\pi/3$. In the interval $2\pi/3<t<4\pi/3$, the function $x(t)$ attains its maximum $x_0=\frac{\sqrt{4r^2-3}-1}{2(r^2-1)}$ twice. The two maximum points $2\pi/3<t_1=\pi-\delta(r)<t_2=\pi+\delta(r)<4\pi/3$ satisfy $y(t_2)=-y(t_1)>0$, where the angle $0<\delta(r)<\pi/3$ is given by $\delta(r)=\arcsin(\sqrt{3}/(2r))$, equivalently, $\sin\delta(r)=\frac{\sqrt{3}}{2r}$. The line segment joining two points $(x(t_1),y(t_1))$, $(x(t_2),y(t_2))$ passes through the point $(x_0,0)=(\frac{\sqrt{4r^2-3}-1}{2(r^2-1)},0)$. The point $(x(t),y(t))$ at $t=\pi$ is $(2/(2r+1),0)$. The assumption $r>1$ implies that $2/(2r+1)<x_0$. The gap is crucial for the fact that $\Lambda_2(A)$ is not of constant width. Since $(x_0,0)$ belongs to $D_2(A)$, it follows that for $z\in\Lambda_2(A)$,

$$\Re(z)\ge-\frac{1}{x_0}=-\frac{1+\sqrt{4r^2-3}}{2},$$

and thus

$$-\frac{1+\sqrt{4r^2-3}}{2} \leq \Re(z) \leq r - \frac{1}{2}$$

for $z \in \Lambda_2(A)$. The distance between the first term and the third term is

$$r - \frac{1}{2} + \frac{1+\sqrt{4r^2-3}}{2} = r + \frac{\sqrt{4r^2-3}}{2} < 2r.$$

For the angle $\pi - \delta(r) < t < \pi + \delta(r)$, we replace an interior point by the point

$$\Big(\frac{-\cos t}{\tilde{f}_2(t)}, \frac{-\sin t}{\tilde{f}_2(t)}\Big)$$

on the line segment joining $(x(t_1), y(t_1))$ and $(x(t_2), y(t_2))$ where the modified second eigenvalue $0 < \tilde{f}_2(t) < f_2(t)$ is given by

$$\tilde{f}_2(t) = \frac{1+\sqrt{4r^2-3}}{2} \cos(t-\pi).$$

(iv) Assume that $r > \sqrt{3}$. Then $\delta(r) < \pi/6$, thus $\frac{\pi}{6} + \epsilon \in \Sigma(r)$ and also $\frac{7\pi}{6} + \epsilon \in \Sigma(r)$ for sufficient small $|\epsilon|$. Hence for $t = \pi/6 + \epsilon$, $d(t) = 2r$ which is the diameter of $\Lambda_2(A)$. On the other hand if $-\delta(r) \leq t \leq \delta(r)$, then $d(-t) = d(t)$ and

$$d(t) = r + \cos(t - \frac{2\pi}{3}) + \frac{1+\sqrt{4r^2-3}}{2} \cos t$$

for $0 \leq t \leq \delta(r)$. On the interval $[0, \delta(r)]$, the function $d(t)$ is increasing. Its maximum is $d(\delta(r)) = 2r$ and its minimum is $d(0) = r + \frac{\sqrt{4r^2-3}}{2}$. This shows that the diameter of $\Lambda_2(A)$ is $2r$ and the width of $\Lambda_2(A)$ is

$$d(0) = r + \frac{\sqrt{4r^2-3}}{2} < 2r,$$

$\Lambda_2(A)$ is not of constant width.

(iii) Assume that $1 < r \leq \sqrt{3}$. In the case $r = \sqrt{3}$, by using the above arguments, we can show that the maximum of $d(t)$ on $0 \leq t \leq \pi$ is attained at $t = \pi/6$, $t = \pi/2$, $t = 5\pi/6$ and $d(\pi/2) = 2r = 2\sqrt{3}$, and the minimum of $d(t)$ on $0 \leq t < \pi$ is attained at $t = 0$, $t = \pi/3$, $t = 2\pi/3$ and

$$d(0) = r + \frac{\sqrt{4r^2-3}}{2} = \sqrt{3} + \frac{3}{2}.$$

We treat the case $1 < r < \sqrt{3}$. The function $d(t)$ on $0 \leq t \leq 2\pi$ satisfies $d(-t) = d(t)$, $d(t+2\pi/3) = d(t)$ and $d(t+\pi) = d(t)$. So, we determine the function $d(t)$ for $0 \leq t \leq \delta(r)(< \pi/3)$. If $\pi/3 - \delta(r) \leq t \leq \delta(r)$, then

$$\begin{aligned} d(t) &= \frac{1+\sqrt{4r^2-3}}{2} \cos(t - \frac{\pi}{3}) + \frac{1+\sqrt{4r^2-3}}{2} \cos t \\ &= \frac{\sqrt{3}}{2}(1 + \sqrt{4r^2-3}) \cos(t - \frac{\pi}{6}). \end{aligned}$$

Its maximum is $d(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}(\sqrt{4r^2-3}+1)$, and its minimum is $d(\delta(r)) = \frac{3(\sqrt{4r^2-3}+1)^2}{8r}$. If $0 \le t \le \pi/3 - \delta(r)$, then

$$d(t) = r + \cos(t - \frac{2\pi}{3}) + \frac{1+\sqrt{4r^2-3}}{2}\cos t.$$

This function is increasing on $[0, \pi/3 - \delta(r)]$ and

$$d(0) = r + \frac{\sqrt{4r^2-3}}{2}, \quad d(\frac{\pi}{3} - \delta(r)) = d(\delta(r)).$$

Therefore, the diameter and the width of $\Lambda_2(A)$ for $1 < r < \sqrt{3}$ are respectively given by $d(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}(\sqrt{4r^2-3}+1)$ and $d(0) = r + \frac{\sqrt{4r^2-3}}{2}$.

■

Fig. 1 displays the diameter and width of $\Lambda_2(A(r))$. The upper curve is the diameter and the lower one is the width for $r \ge \sqrt{3}/2 \approx 0.866$. Clearly, $\Lambda_2(A(r))$ is not of constant width.

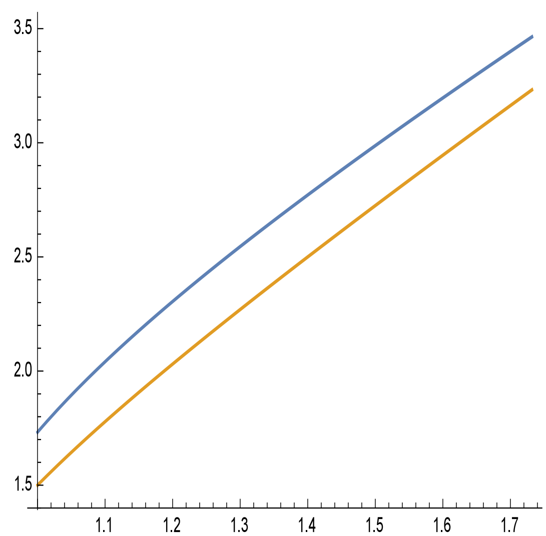

Fig. 1. Diameter and width

3.2 Boundary curve

Let A be the matrix defined in (3.1). In this section, we consider boundary generating curve of $\Lambda_2(A)$, and illustrate the results in Section 2 for the case $r = 3\sqrt{3}/2$. The six eigenvalue functions of $\Re(e^{-it}A(r))$ are given by

$$
\begin{aligned}
f_1(t) &= \cos t + r\\
f_2(t) &= \cos t - r,\\
f_3(t) &= \cos(t - \frac{2\pi}{3}) + r = \frac{1}{2}(-\cos t + \sqrt{3}\sin t) + r,\\
f_4(t) &= \cos(t - \frac{2\pi}{3}) - r,\\
f_5(t) &= \cos(t + \frac{2\pi}{3}) + r = \frac{1}{2}(-\cos t - \sqrt{3}\sin t) + r,\\
f_6 &= \cos(t + \frac{2\pi}{3}) - r.
\end{aligned}
$$

If $-2\pi/3 \le t \le -\pi/3$ or $\pi/3 \le t \le 2\pi/3$, the second largest eigenvalue of $\Re(e^{-it}A(r))$ is $f_1(t)$. On these two t-subintervals, the boundary generating curve of $\Lambda_2(A(r))$ lies on the circle $z(t) = 1 + re^{it}$. If $-\pi/3 \le t \le 0$ or $2\pi/3 \le t \le \pi$, the second largest eigenvalue of $\Re(e^{-it}A(r))$ is $f_5(t)$. On these two t-subintervals, the boundary generating curve of $\Lambda_2(A(r))$ lies on the circle $z(t) = e^{-2\pi i/3} + re^{it}$. If $0 \le t \le \pi/3$ or $\pi \le t \le 4\pi/3$, the second largest eigenvalues of $\Re(e^{-it}A(r))$ is $f_3(t)$. On these two t-subintervals, the boundary generating curve of $\Lambda_2(A(r))$ lies on the circle $z(t) = e^{2\pi i/3} + re^{it}$.

The boundary generating curve (2.1) of $\Lambda_2(A(r))$ is given by

$$\Re(z(t)) = f_j(t)\cos t - f_j(t)'\sin t, \quad \Im(z(t)) = f_j \sin t + f_j'(t)\cos t,$$

where $j \in \{1, 3, 5\}$ are determined depending on the above t-subintervals of $[-2\pi/3, 4\pi/3]$. Corresponding to the non-differentiable points $t = -2\pi/3$, $t = 0$, $t = 2\pi/3$ of the function $\lambda_2(\Re(e^{-it}A))$, the boundary of the range $\Lambda_2(A(r))$ has the three flat portions:

$$\ell_1 = \{\frac{3\sqrt{3}}{2} - \frac{1}{2} + is : -\frac{\sqrt{3}}{2} \le s \le \frac{\sqrt{3}}{2}\}$$

for $t = 0$, and

$$\ell_2 = e^{-2\pi i/3}\ell_1, \quad \ell_3 = e^{2\pi i/3}\ell_1$$

for $t = -2\pi/3$, $t = 2\pi/3$ respectively. We consider the 6 parts of the boundary generating curve:

$$
\begin{aligned}
C_1^{(1)} &= \{z(t) : -2\pi/3 \le t \le -\pi/3\}, & C_3^{(1)} &= \{z(t) : -\pi/3 \le t \le 0\},\\
C_2^{(1)} &= \{z(t) : 0 \le t \le \pi/3\}, & C_1^{(2)} &= \{z(t) : \pi/3 \le t \le 2\pi/3\},\\
C_3^{(2)} &= \{z(t) : 2\pi/3 \le t \le \pi\}, & C_2^{(2)} &= \{z(t) : \pi \le t \le 4\pi/3\}.
\end{aligned}
$$

The two arcs $C_3^{(2)}$ and $C_2^{(2)}$ intersect at $X_0 = -\sqrt{6} - \frac{1}{2} \sim -2.94949$. The two arcs $C_1^{(1)}$ and $C_3^{(1)}$ intersect at $e^{2\pi i/3}X_0 = \frac{1+2\sqrt{6}}{4} - i\frac{\sqrt{3}+6\sqrt{2}}{4}$, and the two arcs $C_2^{(1)}$ and $C_1^{(2)}$ intersect at $e^{-2\pi i/3}X_0 = \frac{1+2\sqrt{6}}{4} + i\frac{\sqrt{3}+6\sqrt{2}}{4}$. At these intersection points, the connected sets $C_1^{(1)} \cup C_3^{(1)}$,$C_2^{(1)} \cup C_1^{(2)}$ and $C_3^{(2)} \cup C_2^{(2)}$ are divided into two parts, one belongs to $\Lambda_2(A(r))$ and one does not belong to $\Lambda_2(A(r))$. We remove the parts which are not belonging to $\Lambda_2(A(r))$. Fig. 2 shows the graphic of the boundary generating curve of $\Lambda_2(A(r))$. Additional dashed arcs are added to recognize the relative position of the boundary generating curve on the three circles.

Set $0 < t_0 = \delta(3\sqrt{3}/2) = \arcsin(1/3) < \arcsin(1/2) = \pi/6$. At each angle t, the curve $z(t)$ has the normalized tangent vector $-\sin t + i \cos t = ie^{it}$. We conclude that the boundary of $\Lambda_2(A(r))$ is consisting of

$$\{z(t) : -2\pi/3 \leq t \leq -\pi/3 - t_0\} \cup \{z(t) : -\pi/3 + t_0 \leq t \leq \pi/3 - t_0\} \cup \{z(t) : \pi/3 + t_0 \leq t \leq \pi - t_0\}$$

and the 3 line segments ℓ_1, ℓ_2, ℓ_3, see Fig. 3. If $t \in [0, \pi]$ belongs to one of the intervals $[t_0, \pi/3 - t_0]$, $[\pi/3 + t_0, 2\pi/3 - t_0]$, $[2\pi/3 + t_0, \pi - t_0]$, then

$$\max\{\Re(e^{-it}\mu) : \mu \in \Lambda_2(A(r))\} = \max \Re(z(t)e^{-it})$$

and

$$\min\{\Re(e^{-it}\mu) : \mu \in \Lambda_2(A(r))\} = \min -\Re(z(t + \pi)e^{-i(t+\pi)})$$

and the difference $\max\{\Re(e^{-it}\mu) : \mu \in \Lambda_2(A(r))\} - \min\{\Re(e^{-it}\mu) : \mu \in \Lambda_2(A(r))\}$ is the constant $2r = 3\sqrt{3}$. The boundary $\partial\Lambda_2(A)$ on these circular arcs forms a generalization of curve of constant width.

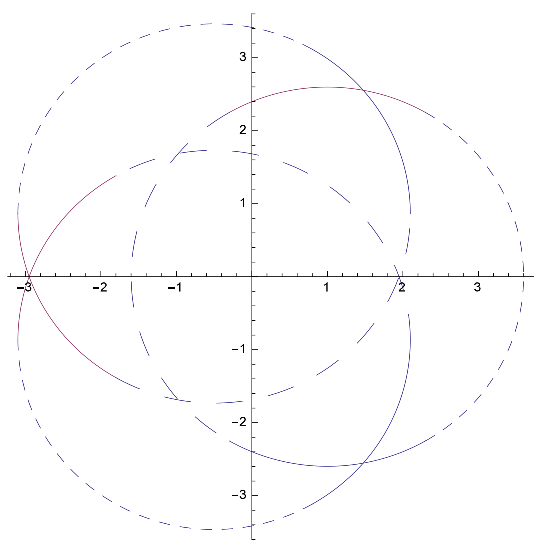

Fig. 2. Boundary generating curve of $\Lambda_2(A(r))$: Three intersected solid arcs

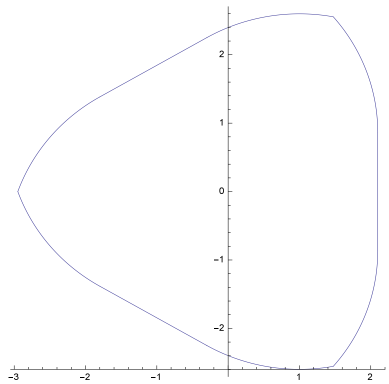

Fig. 3. Boundary of $\Lambda_2(A(r))$

The curve $(x(t), y(t))$ (3.6) plays an important role to construct the set $D_2(A)$. Figs. 4, 5, 6 display respectively the graphics of this curve for $r = 0.89$, $r = 1.7$ and $r = 3\sqrt{3}/2$.

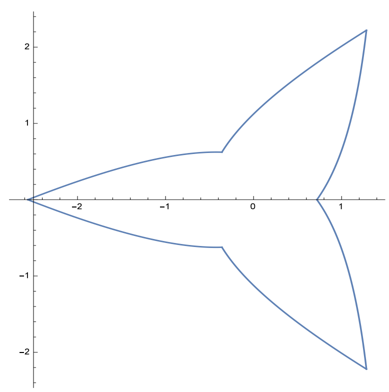

Fig. 4. The curve $(x(t), y(t))$ for $r = 0.89$

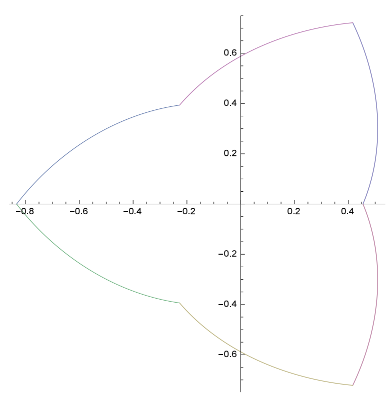

Fig. 5. The curve $(x(t), y(t))$ for $r = 1.7$

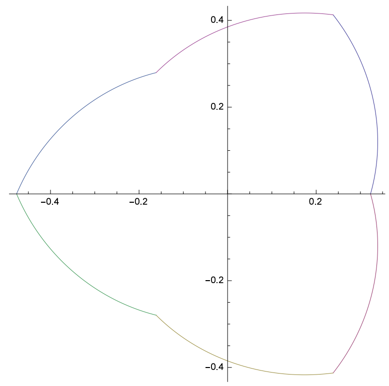

Fig. 6. The curve $(x(t), y(t))$ for $r = 3\sqrt{3}/2$

Remark By Theorems 3.1–3.3, we have $f_2(t) > \tilde{f}_2(t)$ for almost all $t \in [0, 2\pi)$ for the matrix $A \in M_6$ satisfying (3.1). For $k > 2$, we may let $A_k = A \oplus M \oplus \cdots \oplus M$ for $k-2$ copies of $M = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$ so that $W(A) \subseteq W(M)$. Then $\tilde{f}_k(t)$ of A_k equals $\tilde{f}_2(t)$ of A, and $f_k(t)$ of A_k equals $f_2(t)$ of A. So for A_k, $f_k(t) > \tilde{f}_k(t)$ for almost all $t \in [0, 2\pi)$.

Acknowledgment

The authors would like to thank the referee for carefully reading and valuable comments which helped to improve the quality of the paper.

Mao-Ting Chien is partially supported by Ministry of Science and Technology, Taiwan, under MOST 108-2115-M-031-001.

Chi-Kwong Li is an affiliate member of the Institute for Quantum Computing, University of Waterloo. His research was supported by USA NSF grant DMS 1331021 and Simons Foundation Grant 351047.

Hiroshi Nakazato thanks partial support from the Mathematics Research Promotion Center, Taiwan, for a visit to Soochow University on this work.

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