Decomposable Numerical Ranges on Orthonormal Tensors

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Abstract

Let $1 \leq m \leq n$, and let $\chi : H \to \mathbb{C}$ be a degree 1 character on a subgroup H of the symmetric group of degree m. The generalized matrix function on an $m \times m$ matrix $B = (b_{ij})$ associated with χ is defined by $d_\chi(B) = \sum_{\sigma \in H} \chi(\sigma) \prod_{j=1}^m b_{j,\sigma(j)}$, and the decomposable numerical range of an $n \times n$ matrix A on orthonormal tensors associated with χ is defined by

$$W_\chi^\perp(A) = \{d_\chi(X^*AX) : X \text{ is an } n \times m \text{ matrix such that } X^*X = I_m\}.$$

We study relations between the geometrical properties of $W_\chi^\perp(A)$ and the algebraic properties of A, and determine the structure of those linear operators L on $n \times n$ complex matrices that satisfy $W_\chi^\perp(L(A)) = W_\chi^\perp(A)$ for all $n \times n$ matrices A. These results extend those of other researchers who treat the special cases of χ such as the principal or alternate character.

Keywords: Linear operators, decomposable numerical range.
AMS Classification: 15A04, 15A60, 47B49.

1 Introduction

Let M_n be the algebra of $n \times n$ complex matrices. Suppose $1 \leq m \leq n$ and $\chi : H \to \mathbb{C}$ is a degree 1 character on a subgroup H of the symmetric group S_m of degree m. The generalized matrix function associated with χ is defined, for $B = (b_{ij}) \in M_m$, by

$$d_\chi(B) = \sum_{\sigma \in H} \chi(\sigma) \prod_{j=1}^{m} b_{j,\sigma(j)}.$$

For instance, if χ is the alternate character on $H = S_m$ then $d_\chi(B) = \det(B)$; if χ is the principal character on $H = S_m$ then $d_\chi(B) = \text{per}(B)$, the permanent of B.

Define the decomposable numerical range of $A \in M_n$ on orthonormal tensors associated with χ by

$$W_\chi^\perp(A) = \{d_\chi(X^*AX) : X \text{ is an } n \times m \text{ matrix such that } X^*X = I_m\}.$$

When $m = 1$, it reduces to the classical numerical range of A, denoted by $W(A)$. The decomposable numerical range can be viewed as the image of the quadratic form $x^* \mapsto (K(A)x^*, x^*)$, defined by the induced matrix $K(A)$ associated with χ, on the decomposable unit tensors $x^* = x_1 * \cdots * x_m$ such that $\{x_1, \ldots, x_m\}$ is an orthonormal set in $\mathbb{C}^n$. We refer the readers to [8, 11] for general background.

The classical numerical range $W(A)$ is a useful tool for studying matrices and operators (see e.g. [5]). Likewise, the decomposable numerical range is a useful tool for studying induced matrices acting on symmetry classes of tensors. There has been considerable interest in studying the geometric properties of $W_\chi^\perp(A)$ and their relations to the algebraic properties of A, see [1, 6, 7] and their references. Another problem of interest is to determine the structure of the linear preservers of $W_\chi^\perp$, i.e., those linear operators L on M_n satisfying $W_\chi^\perp(L(A)) = W_\chi^\perp(A)$ for all $A \in M_n$, see for instance [13]. The purpose of this paper is to further the research in these directions.

In Section 2, we give a brief survey of some existing results related to our study. In Section 3, we investigate the interplay between the geometrical properties of $W_\chi^\perp(A)$ and the algebraic properties of the matrix A. In Section 4, we determine the structure of the linear preservers of $W_\chi^\perp$ for the unsolved cases.

We reserve the symbol ε to denote the alternate character on $H = S_m$. In such case, we write $W_\varepsilon^\perp(A)$ instead of $W_\chi^\perp(A)$.

Since $W_\chi^\perp(A)$ can be viewed as the image of the compact connected set

$$\{X : X \text{ is } n \times m \text{ with } X^*X = I_m\}$$

under the continuous map $X \mapsto d_\chi(X^*AX)$, the set $W_\chi^\perp(A)$ is compact and connected as well. Furthermore, we shall use the following well-known facts about $W_\chi^\perp(A)$ and the induced matrix $K(A)$ in our discussion.

Proposition 1.1 [8, Chapter 2] *If $A \in M_n$ is Hermitian, positive (semi-)definite, unitary, or normal, then $K(A)$ has the corresponding property.*

Proposition 1.2 *Suppose $H < S_n$ and χ are given. Let $\tau \in S_n$, $\tilde{H} = \{\tau^{-1}\sigma\tau : \sigma \in H\}$, and $\tilde{\chi}(\tau^{-1}\sigma\tau) = \chi(\sigma)$ for all $\sigma \in H$. Then $W_\chi^\perp(A) = W_{\tilde{\chi}}^\perp(A)$ for any $A \in M_n$.*

Proposition 1.3 *Suppose $1 \le m \le n$ and χ is a character on $H < S_m$. Let $A \in M_n$.*
(a) $W_\chi^\perp(\mu A) = \mu^m W_\chi^\perp(A)$ for any $\mu \in \mathbb{C}$.
(b) $W_\chi^\perp(U^*AU) = W_\chi^\perp(A) = W_\chi^\perp(A^t)$ for any unitary $U \in M_n$.
(c) For any character χ we have $W_\varepsilon^\perp(A) \subseteq W_\chi^\perp(A) \subseteq W(K(A))$.

Proof. (a) and (b) can be easily verified. For (c), note that if $z = \det(X^*AX) \in W_\varepsilon^\perp(A)$, there exists a unitary $V \in M_m$ so that V^*X^*AXV is in upper triangular form. Then $z = \det(V^*X^*AXV) = d_\chi(V^*X^*AXV) \in W_\chi^\perp(A)$. □

Since the study for the classical numerical range is quite complete, we always assume that $m \geq 2$ in Sections 3 and 4, unless otherwise stated.

2 Existing Results

We first mention some existing results relating the geometrical properties of $W_\chi^\perp(A)$ and the algebraic properties of A. When $m = 1$, the following is well known (see e.g. [5]).

Proposition 2.1 *Let $A \in M_n$.*

(a) *$W(A) = \{\lambda\}$ if and only if $A = \lambda I$.*

(b) *$W(A) \subseteq \mathbb{R}$ if and only if A is Hermitian.*

(c) *$W(A) \subseteq (0, \infty)$ if and only if A is positive definite.*

(d) *$W(A)$ has no interior point if and only if A is a normal matrix with eigenvalues lying on a straight line.*

For the principal character χ, we have the following results [2, 5, 6, 7].

Proposition 2.2 *Suppose $1 \leq m \leq n = 2$ and χ is the principal character on $H < S_m$. Let A be unitarily similar to $\begin{pmatrix} \lambda_1 & c \\ 0 & \lambda_2 \end{pmatrix}$.*

(a) *If $m = 1$, then $W_\chi^\perp(A) = W(A)$ is an elliptical disk with foci λ_1 and λ_2, and major axis $\{|c|^2 + |\lambda_1 - \lambda_2|^2\}^{1/2}$.*

(b) *If $H = S_2$, then $W_\chi^\perp(A)$ is an elliptical disk with foci $\lambda_1\lambda_2$ and $(\lambda_1^2 + \lambda_2^2)/2$, and major axis $|c|^2 + |\lambda_1 - \lambda_2|^2/2$.*

(c) *If $H = \{e\} < S_2$, then $W_\chi^\perp(A)$ is an elliptical disk with foci $\lambda_1\lambda_2$ and $(\lambda_1 + \lambda_2)^2/4$, and major axis $|c|^2/2 + |\lambda_1 - \lambda_2|^2/4$.*

Consequently, $W_\chi^\perp(A) = \{\lambda\}$ if and only if $A = \xi I$ with $\xi^m = \lambda$; $W_\chi^\perp(A)$ is a nondegenerate line segment if and only if A is a non-scalar normal matrix.

Proposition 2.3 *Suppose $1 < m \leq n$ and χ is the principal character on $H < S_m$. Let $A \in M_n$. Then $W_\chi^\perp(A) = \{\lambda\}$ if and only if $A = \xi I$ with $\xi^m = \lambda$. Moreover, if $(m, n) \neq (2, 2)$, the following conditions are equivalent.*

(a) *$W_\chi^\perp(A)$ is a subset of a straight line.*

(b) *$W_\chi^\perp(A)$ is a subset of a straight line passing through the origin.*

(c) *ξA is Hermitian for some nonzero $\xi \in \mathbb{C}$.*

Next, we turn to the alternate character ε on $H = S_m$. To avoid trivial consideration, we assume that $m < n$; otherwise, we have $W_\varepsilon^\perp(A) = \{\det(A)\}$. We have the following results, see [16] and [1, §4].

Proposition 2.4 *Suppose $1 < m < n$ and ε is the alternate character on $H = S_m$. Let $A \in M_n$. Then $W_\varepsilon^\perp(A) = \{\xi\}$ if and only if one of the following conditions holds.*
(a) *$\xi = 0$ and A has rank less than m.*
(b) *$A = \lambda I$ so that $\lambda^m = \xi$.*

Proposition 2.5 *Suppose $1 < m < n$ and ε is the alternate character on $H = S_m$. Let $A \in M_n$. Then $W_\varepsilon^\perp(A)$ is a non-degenerate line segment if and only if* $\operatorname{rank}(A) \geq m$ *and one of the following conditions holds.*
(a) *A is unitarily similar to $A_1 \oplus 0_{n-m}$ such that* $\det(A_1) \neq 0$.
(b) *ξA is Hermitian for some $\xi \in \mathbb{C}$ with $|\xi| = 1$.*
(c) *$1 < m < n-1$ and there exists $\xi \in \mathbb{C}$ with $|\xi| = 1$ such that ξA has eigenvalues $\lambda_1, \ldots, \lambda_n$, with $\lambda_1 = \cdots = \lambda_{n-1}$.*
(d) *$m = n - 1$ and A is invertible so that ξA^{-1} is hermitian for some $\xi \in \mathbb{C}$ with $|\xi| = 1$.*

In Section 3, we shall obtain analogous results for $W_\chi^\perp(A)$, for other χ, and give examples to show that generalizations are impossible in some cases.

Next, we turn to the existing results on linear preservers of $W_\chi^\perp(A)$. When $m = 1$, we have the following result of Pellegrini [12].

Proposition 2.6 *A linear preserver of the classical numerical range on M_n must be of the form $A \mapsto U^*AU$ or $A \mapsto U^*A^tU$ for some unitary U.*

If $m = n$, then $W_\varepsilon^\perp(A) = \{\det(A)\}$, and we have the following result of Frobenius [3].

Proposition 2.7 *If $m = n$, a linear preserver of the determinant on M_n must be of the form $A \mapsto MAN$ or $A \mapsto MA^tN$ for some $M, N \in M_n$ with* $\det(MN) = 1$.

For $m < n$ and $\chi = \varepsilon$ on S_m, we have the following result of Marcus and Filippenko [9].

Proposition 2.8 *Let $\chi = \varepsilon$ be the alternate character on $H = S_m$ with $m < n$. A linear preserver of $W_\varepsilon^\perp$ on M_n must be of the form $A \mapsto \xi U^*AU$ or $A \mapsto \xi U^*A^tU$ for some unitary $U \in M_n$ and $\xi \in \mathbb{C}$ with $\xi^m = 1$.*

When χ is the principal character on $H = S_m$ with $m \leq n$, we have the following result of Hu and Tam [6, 7]. (Note that there is a misprint in [6, Theorem 6]; one may see [4, Theorem 5.3] for the accurate statement.)

Proposition 2.9 *Let χ be the principal character on $H = S_m$, $m \leq n$. A linear preserver of $W_\chi^\perp$ on M_n must be of the form described in Proposition 2.8, or, when $m = n = 2$ and $H = S_2$, of the form*

$$A \mapsto \pm[V^*AV + (\pm i - 1)(\operatorname{tr} A)I/2] \quad \text{or} \quad A \mapsto \pm[V^*A^tV + (\pm i - 1)(\operatorname{tr} A)I/2]$$

for some unitary matrix $V \in M_2$.

In Section 4, we shall determine the structure of those linear preservers of $W_\chi^\perp$ for the remaining cases.

3 Geometric Properties

In this Section, we study some geometric properties of $W^\perp_\chi(A)$. First of all, by Propositions 2.4 and 2.5, we see that it is impossible to extend the result in Proposition 2.3 to other character in general. In fact, if χ is not the principal character on $H < S_m$ with $m > 1$, then $W^\perp_\chi(A) = \{0\}$ for any rank one matrix A (see [2]). Thus, one sees that low rank matrices are obstacles for extending the result in Proposition 2.3. We shall overcome this by imposing suitable restriction on the ranks of the matrices. We first establish the following lemma.

Lemma 3.1 *Suppose* $1 < m \le n$ *and* χ *is a character on* $H < S_m$. *Let* $A = (a_{pq}) \in M_n$ *be an upper triangular matrix,* $B = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$ *and* $\tilde H = \{\sigma \in H : \sigma(j) = j$ *for all* $j \ge 3\}$. *Denote by* $\tilde\chi$ *be the restriction of* χ *to* $\tilde H$. *Identify, in the natural way,* $\tilde H$ *with a subgroup of* S_2 *and* $\tilde\chi$ *with a character on it. Then* $(\prod_{j=3}^m a_{jj})W^\perp_{\tilde\chi}(B) \subseteq W^\perp_\chi(A)$.

Proof. If $V \in M_2$ is unitary and $V^*BV = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then the $n \times m$ matrix $\tilde V$, obtained from $V \oplus I_{n-2}$ by deleting the last $n-m$ columns, satisfies $\tilde V^*\tilde V = I_m$ and we have $d_\chi(\tilde V^* A \tilde V) = \gamma \prod_{j=3}^m a_{jj}$, where $\gamma = d_{\tilde\chi}(V^*BV) \in W^\perp_{\tilde\chi}(B)$. □

Theorem 3.2 *Suppose* $1 < m \le n$ *and* χ *is a character on* $H < S_m$ *such that* $d_\chi(\cdot) \ne \det(\cdot)$ *when* $m = n$. *Let* $A \in M_n$ *be such that* $\operatorname{rank}(A) \ge \min\{m+1, n\}$. *If* $W^\perp_\chi(A)$ *has no interior point, then* A *is normal.*

Proof. Suppose $m < n$. If $W^\perp_\chi(A)$ has no interior point, then $W^\perp_\varepsilon(A)$ has no interior point, by Proposition 1.3 (c). By [1, Theorem 3.1], A is normal.

Suppose $m = n$. Then $\det(A) \ne 0$ and $\chi \ne \varepsilon$. Thus, there is a transposition (p, q) in S_n such that

$$\text{(i) } (p,q) \notin H, \quad \text{or} \quad \text{(ii) } (p,q) \in H \text{ and } \chi(p,q) = 1.$$

Furthermore, replacing H by $\sigma H \sigma^{-1}$ for a suitable $\sigma \in S_n$, we may assume that $(p, q) = (1, 2)$.

Suppose A is not normal. By Lemma 1 in [10], A is unitarily similar to an upper triangular matrix (a_{pq}) with $a_{12} \ne 0$. By Proposition 1.3 (b), we may assume that $A = (a_{pq})$. If $B = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$, then, using the notations and result in Lemma 3.1, we have $\gamma \prod_{j=3}^m a_{jj} \in W^\perp_\chi(A)$ for any $\gamma \in W^\perp_{\tilde\chi}(B)$. Since $a_{12} \ne 0$, it follows from Proposition 2.2 (c) or (b), depending on whether (i) or (ii) holds, that γ can be any point in a circular disk centered at $a_{11}a_{22}$ with radius r, for a sufficiently small $r > 0$. Thus, $W^\perp_\chi(A)$ contains a circular disk centered at $\prod_{j=1}^n a_{jj}$ with radius $|r \prod_{j=3}^n a_{jj}| > 0$, contradicting the assumption that $W^\perp_\chi(A)$ has no interior point. □

We are now ready to characterize those A such that $W^\perp_\chi(A)$ is a singleton.

Theorem 3.3 *Suppose $1 < m \le n$ and χ is a character on $H < S_m$ such that $d_\chi(\cdot) \neq \det(\cdot)$ when $m = n$. Let $A \in M_n$ be such that $\operatorname{rank}(A) \ge \min\{m+1, n\}$. Then $W_\chi^\perp(A) = \{\mu\}$ if and only if $A = \lambda I$ with $\lambda^m = \mu$.*

Proof. The ($\Leftarrow$) part is clear. To prove the converse, we can apply Theorem 3.2 to conclude that A is unitarily similar to $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$. We have to show that all eigenvalues of A are equal.

It is well known (see [1] or the proof of Proposition 1.3) that if z is a product of m eigenvalues of A, then $z \in W_\varepsilon^\perp(A) \subseteq W_\chi^\perp(A)$. Thus, if $m < n$ and not all eigenvalues of A are equal, then $W_\chi^\perp(A)$ is not a singleton, which is a contradiction.

Suppose now $m = n$. Then $\det(A) \neq 0$ and $\chi \neq \varepsilon$. Using the argument in the proof of Theorem 3.2, we may assume that

$$\text{(i)}\ \ (1,2) \notin H, \quad \text{or} \quad \text{(ii)}\ \ (1,2) \in H \text{ and } \chi(1,2) = 1.$$

If not all eigenvalues of A are equal, then we may assume that $\lambda_1 \neq \lambda_2$. Then $W_\chi^\perp(A)$ contains all the elements of the form

$$d_\chi(B \oplus \operatorname{diag}(\lambda_3, \dots, \lambda_n)) = z \prod_{j=3}^{n} \lambda_j,$$

where B is unitarily similar to $\operatorname{diag}(\lambda_1, \lambda_2)$ and z is lying in a non-degenerate line segment with $\lambda_1\lambda_2$ as one of the endpoints by Proposition 2.3 (c) or (b), depending on whether (i) or (ii) holds. This again contradicts the fact that $W_\chi^\perp(A)$ is a singleton. □

A consequence of Theorem 3.3 is the following corollary.

Corollary 3.4 *Suppose $1 < m \le n$ and χ is a character on $H < S_m$ such that $d_\chi(\cdot) \neq \det(\cdot)$ when $m = n$. Let $A \in M_n$. The following conditions are equivalent.*
(a) *A is a scalar.*
(b) *There is $\mu \in \mathbb{C}$, which is not an eigenvalue of A, such that $W_\chi^\perp(A - \mu I)$ is a singleton.*
(c) *There is $\mu \in \mathbb{C}$ such that $W_\chi^\perp(A - \mu I) = \{\nu\}$ with $\nu \neq 0$.*
(d) *There exist distinct $\mu_1, \dots, \mu_k \in \mathbb{C}$ with*

$$k > \begin{cases} n/(n-m) & \text{if } m < n, \\ n & \text{if } m = n, \end{cases}$$

such that $W_\chi^\perp(A - \mu_j I)$ is a singleton.

Proof. The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (a) $\Rightarrow$ (d) follows easily from Theorem 3.3 and Proposition 1.3. Now, assume that (d) holds. When $m = n$, one of $\mu_1, \dots, \mu_k \in \mathbb{C}$ is not an eigenvalue of A, hence (b) holds. Suppose that $m < n$. If $\operatorname{rank}(A - \mu_j I) \le m$ for all $1 \le j \le k$, then μ_j is an eigenvalue of A, with algebraic multiplicity at least equal to $n - m$. It follows that the characteristic polynomial of A has degree $\ge k(n-m) > n$, a contradiction.

Thus, rank $(A - \mu_j I) \geq m+1$ for some $1 \leq j \leq k$. Now, we can apply Theorem 3.3 to the matrix $A - \mu_j I$ to get condition (b). $\square$

Note that conditions (c) and (d) are useful when we do not know the rank or the spectrum of A.

Next, we consider the situation when $W_\chi^\perp(A)$ is a line segment.

Theorem 3.5 *Suppose* (I) $1 < m < n$ *and* χ *is a character on* $H < S_m$ *such that* $\chi \neq \varepsilon$ *when* $H = S_m$, *or* (II) $m = n$ *and there exist* $1 \leq i < j < k \leq n$ *such that the restriction of* χ *on* $\tilde{H} = \{\sigma \in H : \sigma(r) = r$ *whenever* $r \neq i, j, k\}$ *is the principal character. Let* $A \in M_n$ *be such that* rank $(A) \geq \min\{m+1, n\}$. *The following conditions are equivalent.*
(a) $W_\chi^\perp(A)$ *is a subset of a straight line.*
(b) $W_\chi^\perp(A)$ *is a subset of a straight line that passes through the origin.*
(c) μA *is a nonzero Hermitian matrix for some* $\mu \in \mathbb{C}$.

Proof. The implications (c) $\Rightarrow$ (b) $\Rightarrow$ (a) are clear.

Suppose (a) holds. By Theorem 3.2, we may assume that $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. We consider two cases.

First, suppose $m < n$. Since rank $(A) > m$, we may suppose that $\lambda_1, \ldots, \lambda_{m+1}$ are nonzero. As in the proof of Theorem 3.2, we may assume that

$$\text{(i) } (1,2) \notin H, \quad \text{or} \quad \text{(ii) } (1,2) \in H \text{ and } \chi(1,2) = 1.$$

Assume that there are three eigenvalues of A, say $\lambda_1, \lambda_2, \lambda_3$, not lying on a line passing through the origin. Then $W_\chi^\perp(A)$ contains all the points of the form

$$d_\chi(B \oplus \operatorname{diag}(\lambda_4, \ldots, \lambda_{m+1})) = z \prod_{j=4}^{m+1} \lambda_j,$$

where $B = X^*(\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3))X$ for some 3×2 matrix X satisfying $X^*X = I_2$. Let $B = (b_{pq})$. If (i) holds, then $z = b_{11}b_{22}$ lies in $W_{\hat{\chi}}^\perp(\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3))$, where $\hat{\chi}$ is the principal character on $\hat{H} = \{e\} < S_2$. If (ii) holds, then $z = b_{11}b_{22} + b_{12}b_{21}$ lies in $W_{\hat{\chi}}^\perp(\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3))$, where $\hat{\chi}$ is the principal character on $\hat{H} = S_2$. By Proposition 2.3, such a collection of z cannot be a subset of a straight line, which is a contradiction. As a result, any three (and hence all) eigenvalues of A will lie on a straight line passing through the origin.

Next, suppose $m = n$ and there exist $1 \leq i < j < k \leq n$ satisfying the hypothesis. By Proposition 1.2, we may assume that $(i, j, k) = (1, 2, 3)$. Denote by $\tilde{\chi}$ the restriction of χ on $\tilde{H}$. Then $W_\chi^\perp(A)$ contains all the points of the form

$$d_\chi(B \oplus \operatorname{diag}(\lambda_4, \ldots, \lambda_n)) = z \prod_{j=4}^{m+1} \lambda_j,$$

where $B = X^*(\mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3))X$ for some unitary $X \in M_3$ and $z = d_{\tilde{\chi}}(B)$. By Proposition 2.3, we see that the three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ lie on a straight line passing through the origin. We can permute the diagonal entries and apply the same arguments to the resulting matrix. As a result, we see that any three (and hence all) eigenvalues of A lie on a straight line passing through the origin, i.e., ξA is hermitian for some nonzero $\xi \in \mathbb{C}$. $\square$

A consequence of Theorem 3.5 is the following corollary.

Corollary 3.6 *Suppose m, n, and χ satisfy the hypotheses of Theorem 3.5. Then $A \in M_n$ is Hermitian if and only if $W_\chi^\perp(A + \mu I) \subseteq \mathbb{R}$ for all (sufficiently large) $\mu \in \mathbb{R}$.*

Proof. If A is Hermitian and $\mu \in \mathbb{R}$, then $A + \mu I$ and the induced matrix $K(A + \mu I)$ are also Hermitian by Proposition 1.1. Proposition 1.3 (c) implies that

$$W_\chi^\perp(A + \mu I) \subseteq W(K(A + \mu I)) \subseteq \mathbb{R}.$$

Conversely, suppose $W_\chi^\perp(A + \mu I) \subseteq \mathbb{R}$. We can choose $\mu \in \mathbb{R}$ sufficiently large so that $A + \mu I$ is invertible, and apply Theorem 3.5 to conclude that $\xi(A + \mu I)$ is Hermitian, i.e., the eigenvalues of $A + \mu I$ lie on a line passing through the origin. Since this is true for all sufficiently large $\mu \in \mathbb{R}$, we see that the eigenvalues of A must be real numbers, hence A is hermitian. $\square$

In [14], it was shown that if $A \in M_n$ satisfies $d_\chi(X^*AX) > 0$ for all $n \times m$ matrices X with $d_\chi(X^*X) = 1$, then there exists $\xi \in \mathbb{C}$ with $\xi^m = 1$ such that ξA is positive definite. By Theorem 3.5, we have the following.

Corollary 3.7 *Suppose m, n, and χ satisfy the hypotheses of Theorem 3.5. Let $A \in M_n$.*

(a) *There exists $\xi \in \mathbb{C}$ with $\xi^m = 1$ such that ξA is positive definite if and only if $W_\chi^\perp(A) \subseteq (0, \infty)$.*

(b) *A is positive definite if and only if $W_\chi^\perp(A + \mu I) \subseteq (0, \infty)$ for all $\mu \in [0, \infty)$.*

(c) *A is positive semi-definite if and only if $W_\chi^\perp(A + \mu I) \subseteq (0, \infty)$ for all $\mu \in (0, \infty)$.*

Proof. (a) If there exists $\xi \in \mathbb{C}$ with $\xi^m = 1$ such that ξA is positive definite, then $K(A) = K(\xi A)$ is positive definite by Proposition 1.1. Proposition 1.3 (c) implies that $W_\chi^\perp(A) \subseteq W(K(A)) \subseteq (0, \infty)$. Conversely, suppose $W_\chi^\perp(A) \subseteq (0, \infty)$. Then $0 \notin W_\varepsilon^\perp(A) \subseteq W_\chi^\perp(A)$. It follows that (see e.g. [1]) A is invertible. By Theorem 3.5, ξA is Hermitian for some $\xi \in \mathbb{C}$ with $|\xi| = 1$. We may assume that ξA has a positive eigenvalue λ_1; otherwise, replace ξ by $-\xi$. We claim that:

$$(1) \text{ all other eigenvalues of } \xi A \text{ are positive,} \quad \text{and} \quad (2)\ \xi^m = 1.$$

Suppose (1) does not hold, i.e., ξA has a negative eigenvalue λ_2. Assume that rest of the eigenvalues are $\lambda_3, \ldots, \lambda_n$. If $m < n$, then $\lambda_1 \prod_{j=3}^{m+1} \lambda_j$ and $\lambda_2 \prod_{j=3}^{m+1} \lambda_j$ are elements in $W_\chi^\perp(\xi A)$ with different signs. By the connectedness of $W_\chi^\perp(\xi A)$, we see that $0 \in W_\chi^\perp(\xi A) = \xi^m W_\chi^\perp(A)$.

Hence $0 \in W_\chi^\perp(A)$, which is a contradiction. Next, suppose $m = n$. By the assumption on χ, we may assume that the restriction of χ on $\tilde{H} = \{\sigma \in H : \sigma(r) = r \text{ whenever } r \neq 1,2,3\}$ is the principal character. As a result, either

$$\text{(i)} \quad (1,2) \notin H \qquad \text{or} \qquad \text{(ii)} \quad (1,2) \in H \text{ with } \chi(1,2) = 1.$$

If (i) holds, then $\lambda_1\lambda_2 \prod_{j=3}^n \lambda_j$ and $4^{-1}(\lambda_1+\lambda_2)^2 \prod_{j=3}^n \lambda_j$ are elements in $W_\chi^\perp(\xi A)$ with different signs, by Lemma 3.1 and Proposition 2.2 (c). If (ii) holds, then $\lambda_1\lambda_2 \prod_{j=3}^n \lambda_j$ and $2^{-1}(\lambda_1^2 + \lambda_2^2) \prod_{j=3}^n \lambda_j$ are elements in $W_\chi^\perp(\xi A)$ with different signs, by Lemma 3.1 and Proposition 2.2 (b). In both cases, we have $0 \in W_\chi^\perp(\xi A) = \xi^m W_\chi^\perp(A)$, which is the desired contradiction. As a result, we see that condition (1) holds.

Now, all the eigenvalues of ξA are positive, and thus $W_\chi^\perp(\xi A) = \xi^m W_\chi^\perp(A)$ contains a positive real number. Since $W_\chi^\perp(A) \subseteq (0,\infty)$, we see that $\xi^m = 1$, i.e., condition (2) holds.

(b) If A is positive definite and $\mu \geq 0$, then $A + \mu I$ and the induced matrix $K(A+\mu I)$ are also positive definite by Proposition 1.1. Proposition 1.3 (c) implies that $W_\chi^\perp(A+\mu I) \subseteq W(K(A+\mu I)) \subseteq (0,\infty)$. Conversely, suppose $W_\chi^\perp(A+\mu I) \subseteq (0,\infty)$ for all $\mu \in [0,\infty)$. By (a), we see that A is Hermitian. If A has an eigenvalue $\lambda \leq 0$, then 0 is an eigenvalue of $A + \mu I$ with $\mu = -\lambda \geq 0$. Thus, $0 \in W_\varepsilon^\perp(A+\mu I) \subseteq W_\chi^\perp(A+\mu I) \not\subseteq (0,\infty)$. Hence, all eigenvalues of A are positive, and the result follows.

(c) The proof can be done by a similar argument as in (b), or by applying a continuity argument to (b). $\square$

The following example shows that condition (II) in Theorem 3.5 and Corollary 3.7 is necessary when $m = n$.

Example 3.8 Suppose that $m = n = 3$ and let $H < S_3$ be the group generated by the transposition $(2,3)$. Let χ be such that $\chi(2,3) = -1$ and suppose that $D = \operatorname{diag}(1, w, \bar{w}) \in M_3$, where $w = e^{i\pi/3}$. Let $U \in M_3$ be unitary such that the first column of U is $(u_1,u_2,u_3)^t$. If $U^*DU = (a_{pq})$, then $d_\chi(U^*DU) = a_{11}(a_{22}a_{33} - a_{23}a_{32})$. Note that

$$a_{11} = |u_1|^2 + |u_2|^2 w + |u_3|^2 \bar{w}$$

and

$$a_{22}a_{33} - a_{23}a_{32} = |u_1|^2 + |u_2|^2 \bar{w} + |u_3|^2 w$$

as $(a_{22}a_{33} - a_{23}a_{32})/\det(D)$ is just the $(1,1)$ entry of

$$(U^*DU)^{-1} = U^*D^{-1}U = U^*\operatorname{diag}(1, \bar{w}, w)U.$$

Hence

$$\begin{aligned} W_\chi^\perp(D) &= \{(t_1 + t_2 w + t_3\bar{w})(t_1 + t_2\bar{w} + t_3 w) : t_1,t_2,t_3 \geq 0,\ t_1+t_2+t_3 = 1\} \\ &= \{|t_1 + t_2 w + t_3 \bar{w}|^2 : t_1,t_2,t_3 \geq 0,\ t_1+t_2+t_3 = 1\} = [1/4, 1]. \end{aligned}$$

Inspired by the above example, we have the following result.

Theorem 3.9 *Suppose $m = n \geq 3$ and χ is a character on $H < S_m$ such that $\chi \neq \varepsilon$ when $H = S_m$. Furthermore, assume that condition* (II) *in Theorem 3.5 does not hold. If $A \in M_n$ is invertible and if $W_\chi^\perp(A)$ is a subset of a line passing through the origin, then A is a multiple of a Hermitian matrix or a multiple of a unitary matrix.*

Proof. Suppose m, n and χ satisfy the hypotheses of the theorem. As in the proof of Theorem 3.2, we may assume that

$$\text{(i)}\ \ (1,2) \notin H, \quad \text{or} \quad \text{(ii)}\ \ (1,2) \in H \text{ and } \chi(1,2) = 1.$$

However, if (ii) holds, then the subgroup $\tilde{H} = \{\sigma : \sigma(r) = r \text{ whenever } r \neq 1, 2, 3\}$ of H can only be of order 2 or order 6. In either case, the restriction of χ on $\tilde{H}$ must be the principal character as $\chi(1,2) = 1$. Since we assume that condition (II) of Theorem 3.5 does not hold, condition (ii) is ruled out.

Suppose $W_\chi^\perp(A)$ is a subset of a line passing through the origin. If $W_\chi^\perp(A)$ is a singleton, then A is a scalar matrix by Theorem 3.3, and the result follows. Thus, we assume that $W_\chi^\perp(A)$ is a nonzero line segment. By Theorem 3.2, A is unitarily similar to $B = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Since $W_\chi^\perp(A)$ is not a singleton, not all eigenvalues of A are equal. Replacing B by μB for a suitable $\mu \in \mathbb{C}$ with $|\mu| = 1$, we may assume that $W_\chi^\perp(B) \subseteq \mathbb{R}$. Now, for any $\lambda_j \neq \lambda_1$, we may permute the diagonal entries of B so that λ_1, λ_j are moved to the first two positions. Applying Lemma 3.1 to the resulting matrix, we see that $W_\chi^\perp(B)$ contains the line segment with endpoints $\det(B)$ and $\det(B)(\lambda_1^2 + \lambda_j^2)/(2\lambda_1\lambda_j)$, which is part of the line

$$\{\ \det(B)\,(1 + t\,(\lambda_j/\lambda_1 - 2 + \lambda_1/\lambda_j))\ :\ t \in \mathbb{R}\ \}.$$

Since $W_\chi^\perp(B) \subseteq \mathbb{R}$, we see that $\det(B) \in \mathbb{R}$ and thus $\lambda_j/\lambda_1 + \lambda_1/\lambda_j \in \mathbb{R}$. As a result, either $\lambda_j/\lambda_1 \in \mathbb{R}$ or $\lambda_j/\lambda_1 \notin \mathbb{R}$ with $|\lambda_j/\lambda_1| = 1$. Now, this is true for all λ_j with $\lambda_j \neq \lambda_1$. We see that B is permutationally similar to $\lambda_1(B_1 \oplus B_2)$ so that all the diagonal entries of B_1 are real and all the diagonal entries of B_2 are not real with moduli one. If B_2 does not exist then B is a multiple of a Hermitian, matrix, and so is A. If B_2 is non-trivial then all diagonal entries of B_1 must have moduli one. If it is not true, then B has an eigenvalue of the form $\lambda_k = r\lambda_1$ with $r \in \mathbb{R} \setminus \{\pm 1\}$. Repeating the previous argument with λ_k replacing λ_1, we see that B has some nonreal eigenvalue λ_j such that $|\lambda_k/\lambda_j| \neq 1$, which is a contradiction. Hence, if B_2 is non-trivial, then B_1 is unitary as well. Thus, B is a multiple of a unitary matrix, and so is A. □

By Theorem 3.9, we have the following corollary.

Corollary 3.10 *Suppose m, n and χ satisfy the hypotheses of Theorem 3.9. Let $A \in M_n$ be such that $W_\chi^\perp(A + \mu I) \subseteq \mathbb{R}$ for all $\mu \in \mathbb{R}$. Then either A is Hermitian or $A = \lambda I_k \oplus \bar{\lambda} I_{n-k}$ for some non-real complex number λ, where $1 \leq k < n$.*

Proof. Suppose m, n, χ and A satisfy the hypotheses. Then for each μ not equal to an eigenvalue of A, either $A + \mu I$ is a multiple of a Hermitian matrix or $A + \mu I$ is a multiple of a unitary matrix, by Theorem 3.9. Since this is true for infinitely many μ, we see that either all the eigenvalues of A are real, i.e., A is Hermitian, or A has two non-real distinct eigenvalues λ and $\bar{\lambda}$. □

4 Linear Preservers

In this Section, we prove the following result on linear preservers of $W_\chi^\perp$; it covers all the cases previously not treated.

Theorem 4.1 *Suppose $n \geq 3$ and $d_\chi(\cdot) \neq \det(\cdot)$ when $m = n$. A linear operator L on M_n satisfies $W_\chi^\perp(L(A)) = W_\chi^\perp(A)$ for all $A \in M_n$ if and only if it is of the form*

$$A \mapsto \xi U^* A U \quad \text{or} \quad A \mapsto \xi U^* A^t U$$

for some unitary U and $\xi \in \mathbb{C}$ with $\xi^m = 1$.

Proof. The ($\Leftarrow$) part is clear. For ($\Rightarrow$), we start by showing that linear preservers of $W_\chi^\perp$ on M_n are invertible.

Suppose L is a linear preserver of $W_\chi^\perp$ on M_n. Let $A \in M_n$ be nonzero such that $L(A) = 0$. Suppose A has singular value decomposition $U\mathrm{diag}\,(a_1, \ldots, a_n)V$, where U and V are unitary, and $a_1 \geq \cdots \geq a_n$. Set $B = UV$. Then B and $K(B)$ are unitary by Proposition 1.1. By Proposition 1.3 (c) and the fact that the numerical range of a normal matrix is just the convex hull of its spectrum, we have

$$W_\chi^\perp(B) \subseteq W(K(B)) \subseteq D_1 = \{\ z \in \mathbb{C}\ :\ |z| \leq 1\ \}$$

On the other hand, we have

$$|\det(A + B)| = \prod_{j=1}^{n} (1 + a_j) > 1.$$

Thus, there exist m eigenvalues of $A + B$ with product equal to μ such that $|\mu| > 1$. It follows that $\mu \in W_\varepsilon^\perp(A + B) \setminus D_1$, in contradiction to

$$W_\varepsilon^\perp(A+B) \subseteq W_\chi^\perp(A+B) = W_\chi^\perp(L(A+B)) = W_\chi^\perp(L(B)) = W_\chi^\perp(B) \subseteq D_1$$

Let L be a linear preserver of $W_\chi^\perp$. Since $W_\chi^\perp(L(I)) = W_\chi^\perp(I) = \{1\}$, by Theorem 3.3 we see that $L(I) = \xi I$ for some $\xi \in \mathbb{C}$ with $\xi^m = 1$. Replacing L by L/ξ, we may assume that $L(I) = I$.

Next, we show that L maps the set of positive semi-definite matrices into itself. Since every positive semi-definite matrix is a nonnegative combination of rank one orthogonal projections, it suffices to consider the images of rank one orthogonal projections under L. To this end, let A be a rank one orthogonal projection. By Propositions 1.1, 1.3 and 2.1, we have

$$W_\chi^\perp(A+tI) \subseteq W(K(A+tI)) \subseteq \mathbb{R} \quad \text{for all} \quad t \in \mathbb{R},$$

and

$$W_\chi^\perp(A+tI) \subseteq W(K(A+tI)) \subseteq (0,\infty) \quad \text{for all} \quad t > 0.$$

Suppose m, n and χ satisfy the hypotheses of Theorem 3.5. Since $W_\chi^\perp(L(A)+tI) = W_\chi^\perp(L(A+tI)) = W_\chi^\perp(A+tI) \subseteq [0,\infty)$ for all $t \geq 0$, it follows from Corollary 3.7 that $L(A)$ is positive definite.

Suppose m, n and χ satisfy the hypotheses of Theorem 3.9. Since $W_\chi^\perp(L(A)+tI) = W_\chi^\perp(L(A+tI)) = W_\chi^\perp(A+tI) \subseteq \mathbb{R}$ for all $t \in \mathbb{R}$, it follows from Corollary 3.10 that either $L(A)$ is a Hermitian matrix, or $L(A)$ is unitarily similar to $\mu I_k \oplus \bar{\mu} I_{n-k}$ for some non-real $\mu \in \mathbb{C}$, where $1 \leq k < n$. Since χ is not the principal character, we see that the latter case cannot happen; indeed, we have (see e.g. [2]) $0 \neq \det(L(A)) \in W_\chi^\perp(L(A))$ and since $\operatorname{rank}(A) = 1$, we have $\{0\} = W_\chi^\perp(A)$. So, $L(A)$ is Hermitian. If $L(A)$ has an eigenvalue $\lambda < 0$, then $0 = \det(L(A-\lambda I)) \in W_\chi^\perp(L(A-\lambda I)) = W_\chi^\perp(A-\lambda I) \subseteq (0,\infty)$, which is impossible. Thus $L(A)$ is positive semi-definite.

Since L is invertible, one can apply the previous arguments to L^{-1} to conclude that L^{-1} maps the set of positive semi-definite matrices into itself. Thus, L maps the set of positive semi-definite matrices *onto* itself. By a result of Schneider [15], we see that L is of the form

$$A \mapsto U^*AU \qquad \text{or} \qquad A \mapsto U^*A^tU$$

for some invertible $U \in M_n$. Using Theorem 3.3 for $\{1\} = W_\chi^\perp(I) = W_\chi^\perp(L(I)) = W_\chi^\perp(U^*U)$, we conclude that U is unitary. The result follows. □

One can get an analogous result for real linear preservers of $W_\chi^\perp$ on Hermitian matrices by a similar (and simpler) argument.

Acknowledgement

The first author was supported by an NSF grant of USA, and this research was done while he was visiting the University of Toronto supported by a faculty research grant of the College of William and Mary in the academic year 1998-1999. He would like to thank Professor M.D. Choi for making the visit possible. The second author was supported by a grant of Professors E. Bierstone, A. Khovanskii, P. Milman and M. Spivakovsky of the University of Toronto. Both authors would like to thank the staff of the University of Toronto for their warm hospitality.

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