Eigenvalues of the sum of matrices from unitary similarity orbits

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Abstract

Let A and B be $n \times n$ complex matrices. Characterization is given for the set $\mathcal{E}(A, B)$ of eigenvalues of matrices of the form $U^*AU + V^*BV$ for some unitary matrices U and V. Consequences of the results are discussed and computer algorithms and programs are designed to generate the set $\mathcal{E}(A, B)$. The results refine those of Wielandt on normal matrices. Extensions of the results to the sum of matrices from three or more unitary similarity orbits are also considered.

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1 Introduction

Denote by M_n the set of $n \times n$ complex matrices. Let $A, B \in M_n$. There has been a great deal of interest in studying the eigenvalues of matrices of the form $U^*AU + V^*BV$ for some unitary matrices $U, V \in M_n$ because of motivations from theory as well as applications; see [2, 4, 7, 11, 17, 18]. The study has been very successful for Hermitian matrices. Klyachko [12] (see also [9, 11, 13] etc.) gave a necessary and sufficient conditions for the real numbers $c_1, \ldots, c_n$ to be the eigenvalues of the sum of two Hermitian matrices in M_n with eigenvalues $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$.

The problem for non-Hermitian matrices is more challenging. For two given matrices $A, B \in M_n$, let $\mathcal{E}(A, B)$ be the set of eigenvalues of matrices of the form $U^*AU + V^*BV$ for some unitary matrices U and V. Wielandt [19] (see also, [3] and [15]) determined the set $\mathcal{E}(A, B)$ for two normal matrices $A, B \in M_n$. There is not much information about the set $\mathcal{E}(A, B)$ for general matrices $A, B \in M_n$. The purpose of this paper is to address this problem.

In Section 2, we characterize $\mathcal{E}(A, B)$ for two given matrices $A, B \in M_n$. Additional results concerning normal matrices and essentially Hermitian matrices (normal matrices with collinear eigenvalues) are presented in Sections 3 and 4. In Section 5, we consider extension of our results to the sum of three of more matrices, and mention some related problems. In Section 6, we describe how to use our results to design computer algorithms and programs to generate the set $\mathcal{E}(A, B)$.

2 Main results

First, we characterize the matrix pair $(A, B) \in M_n \times M_n$ such that $0 \notin \mathcal{E}(A, B)$. We need the concept of Davis-Wielandt shell [5, 6] of $A \in M_n$ defined by

$$DW(A) = \{(x^*Ax, x^*A^*Ax) : x \in \mathbb{C}^n, x^*x = 1\} \subseteq \mathbb{C} \times \mathbb{R} \sim \mathbb{R}^3.$$

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Theorem 2.1 *Let $A, B \in M_n$. The following are equivalent.*

(a) $\det(U^*AU + V^*BV) \neq 0$ *for any unitary matrices $U, V \in M_n$.*

(b) $DW(A) \cap DW(-B) = \emptyset$.

(c) *There is $\xi \in \mathbb{C}$ such that the singular values of $A + \xi I_n$ and $B - \xi I_n$ lie in two disjoint closed intervals in* $[0, \infty)$.

Proof. If (c) holds, then $\|(A + \xi I_n)u\| > \|(B - \xi I_n)v\|$ for all unit vectors u, $v \in \mathbb{C}^n$, or $\|(A + \xi I_n)u\| < \|(B - \xi I_n)v\|$ for all unit vectors u, $v \in \mathbb{C}^n$. Thus, $(U^*AU + V^*BV)x \neq 0$ for all unitary matrices U, V and unit vector $x \in \mathbb{C}^n$. Thus, condition (a) holds.

Suppose (a) holds. Assume that $DW(A) \cap DW(-B)$ is non-empty. Then there are orthonormal pairs (u_1, u_2) and (v_1, v_2) such that

$$Au_1 = \mu u_1 + \nu u_2 \quad \text{and} \quad -Bv_1 = \mu v_1 + \nu v_2$$

with $(\mu, \mu^2 + \nu^2) \in DW(A) \cap DW(-B)$. Suppose U is unitary with u_1, u_2 as its first two columns, and V is unitary with v_1, v_2 as its first two columns. Then $U^*AU + V^*BV$ has zero first column, and hence has zero determinant, which is a contradiction. So, (b) holds.

Suppose (b) holds. Since $DW(A)$ and $DW(-B)$ are compact convex sets, by the separation theorem, there is a linear functional f such that $f(\alpha) > f(\beta)$ for all $(\alpha, \beta) \in DW(A) \times DW(-B)$. So, there is $\nu \in \mathbb{R}$ and $\mu \in \mathbb{C}$ such that

$$x^*(\nu A^*A + \mu A + \overline{\mu} A^*)x > y^*(\nu B^*B - \mu B - \overline{\mu} B^*)y$$

for any unit vectors $x, y \in \mathbb{C}^n$. We may perturb ν and assume that $\nu \neq 0$. Furthermore, we assume that $\nu > 0$; otherwise multiply -1 to the inequality. Then for $\xi = \bar{\mu}/\sqrt{\nu}$, we see that

$$x^*(A + \xi I_n)^*(A + \xi I_n)x > y^*(B - \xi I_n)^*(B - \xi I_n)y$$

for all unit vectors $x, y \in \mathbb{C}^n$. So, condition (c) holds. □

Note that $\mu \in \mathcal{E}(A, B)$ if and only if there exist unitary matrices $U, V \in M_n$ such that $\det(UAU^* + VBV^* - \mu I_n) = 0$. Using Theorem 2.1, we have the following.

Theorem 2.2 *Let $A, B \in M_n$ and $\mu \in \mathbb{C}$. The following are equivalent.*

(a) $\mu \notin \mathcal{E}(A, B)$.

(b) $DW(A) \cap DW(\mu I_n - B) = \emptyset$.

(c) *There is $\xi \in \mathbb{C}$ such that the singular values of $A + \xi I_n$ and $B - \mu I_n - \xi I_n$ lie in two disjoint closed intervals in* $[0, \infty)$.

3 Normal matrices

If $A, B \in M_n$ are normal, then $DW(A)$ and $DW(\mu I_n - B)$ are polytopes with at most n vertices in $\mathbb{C} \times \mathbb{R} \sim \mathbb{R}^3$. We have the following.

Theorem 3.1 *Suppose $A, B \in M_n$ are normal. Then the conditions* (a) – (c) *in Theorem 2.2 are equivalent to*

(d) *There is a circular disk containing all eigenvalues of one of the matrices A or $\mu I_n - B$, and excluding all the eigenvalues of the other matrix.*

Proof. Suppose A and B are normal. Then the singular values of A and $\mu I_n - B$ are the absolute values of the eigenvalues of the two matrices. One readily sees that Theorem 2.2 (c) is equivalent to condition (d). □

Theorem 3.1 has been proven by Wielandt [19, Theorem 1], where both lines and circles are used for the separation. As pointed out in [19], $\mathcal{E}(A,B)$ depends only on the spectra $\sigma(A)$, $\sigma(B)$ of A and B. Hence, for any nonempty finite subsets S, T of $\mathbb{C}$, we can define $\mathcal{E}(S,T) = \mathcal{E}(A,B)$, where A, B are any normal matrices of the same size such that $\sigma(A) = S$ and $\sigma(B) = T$.

If each of A and B has at most two distinct eigenvalues, then $\mathcal{E}(A,B)$ can be easily determined by Theorem 4.6 in Section 4. For other cases, we have the following theorem which is useful in constructing the set $\mathcal{E}(A,B)$ analytically or using computer programs; see Section 6.

Theorem 3.2 *Let $A, B \in M_n$ be normal matrices one of which has at least 3 distinct eigenvalues and the other has at least 2 distinct eigenvalues. Then conditions* (a)–(c) *in Theorem 2.2 are equivalent to*

(e) *For $(p,q) \in \{(2,3),(3,2)\}$, and any subset of p distinct distinct eigenvalues of A and q distinct eigenvalues of B, there is a circle containing all elements of one of the sets, and excluding all the elements of the other sets.*

Consequently, we have

$$\mathcal{E}(A,B) = \bigcup \{\mathcal{E}(S,T) : S \subseteq \sigma(A), T \subseteq \sigma(B) \quad \text{with } (|S|,|T|) \in \{(2,3),(3,2)\}\},$$

where $|S|$ and $|T|$ are the cardinalities of S and T, respectively.

Proof. Suppose A or B has at least 3 distinct eigenvalues and the other has at least 2 distinct eigenvalues. Then condition (d) fails to hold if and only if there are p distinct eigenvalues of A and q distinct eigenvalues of B with $(p,q) \in \{(3,2),(2,3)\}$ constituting an obstacle for the existence of the circle [14, Theorem 8.2]. Thus, Theorem 3.1 (d) is equivalent to (e). □

To construct $\mathcal{E}(A,B)$, one can further reduce the collection of subsets in the above theorem. To this end, we need the following the lemma showing that there is a one-one correspondence between the triangles on the boundary faces of the convex set $DW(B)$ and those on the boundary faces of $DW(\mu I - B)$ with $\mu = s + it$.

Lemma 3.3 *Suppose $s, t, a_j, b_j \in \mathbb{R}$, $1 \le j \le 5$. Let*

$$P_j = \left(a_j, b_j, a_j^2 + b_j^2\right) \quad \text{and } Q_j = \left(s - a_j, t - b_j, (s-a_j)^2 + (t-b_j)^2\right).$$

Suppose P_1, P_2, P_3 are not collinear. If P_4 and P_5 lie in the same open (or close) half space determined by P_1, P_2, P_3, then Q_4 and Q_5 lie in the same open (or close) half space determined by Q_1, Q_2, Q_3.

Proof. Suppose P_1, P_2, P_3 are not collinear. Then Q_1, Q_2, Q_3 are not collinear. Let Π_1 and Π_2 be the planes determined by P_1, P_2, P_3 and Q_1, Q_2, Q_3 respectively.

For $(a_{pq}) \in M_3$, denote by $\det((a_{pq})) = |a_{pq}|$. For $j = 4, 5$, we have

$$((P_2 - P_1) \times (P_3 - P_1)) \cdot (P_j - P_1) = \begin{vmatrix} a_2 - a_1 & b_2 - b_1 & a_2^2 + b_2^2 - a_1^2 - b_1^2 \\ a_3 - a_1 & b_3 - b_1 & a_3^2 + b_3^2 - a_1^2 - b_1^2 \\ a_j - a_1 & b_j - b_1 & a_j^2 + b_j^2 - a_1^2 - b_1^2 \end{vmatrix}$$

and

$$
\begin{aligned}
&((Q_2 - Q_1) \times (Q_3 - Q_1)) \cdot (Q_j - Q_1) \\
= &\begin{vmatrix} a_1 - a_2 & b_1 - b_2 & a_2^2 + b_2^2 - a_1^2 - b_1^2 + 2s(a_1 - a_2) + 2t(b_1 - b_2) \\ a_1 - a_3 & b_1 - b_3 & a_3^2 + b_3^2 - a_1^2 - b_1^2 + 2s(a_1 - a_3) + 2t(b_1 - b_3) \\ a_1 - a_j & b_1 - b_j & a_j^2 + b_j^2 - a_1^2 - b_1^2 + 2s(a_1 - a_j) + 2t(b_1 - b_j) \end{vmatrix} \\
= &\begin{vmatrix} a_1 - a_2 & b_1 - b_2 & a_2^2 + b_2^2 - a_1^2 - b_1^2 \\ a_1 - a_3 & b_1 - b_3 & a_3^2 + b_3^2 - a_1^2 - b_1^2 \\ a_1 - a_j & b_1 - b_j & a_j^2 + b_j^2 - a_1^2 - b_1^2 \end{vmatrix} \\
= &((P_2 - P_1) \times (P_3 - P_1)) \cdot (P_j - P_1).
\end{aligned}
$$

The result follows from the fact that P_4 and P_5 lie in the same open half space determined by Π_1 if and only if the triple products

$$((P_2 - P_1) \times (P_3 - P_1)) \cdot (P_4 - P_1) \quad \text{and} \quad ((P_2 - P_1) \times (P_3 - P_1)) \cdot (P_5 - P_1)$$

have the same sign and similar assertion for Q_j and Π_2. □

Theorem 3.4 *Let $A, B \in M_n$ be normal matrices with eigenvalues $a_1, \ldots, a_n$, and $b_1, \ldots, b_n$. Then $\mu \in \mathcal{E}(A, B)$ if and only if there is $X = \operatorname{diag}(w_1, w_2, w_3)$ and $Y = \operatorname{diag}(z_1, z_2)$ such that $DW(X) \cap DW(\mu I_2 - Y) \neq \emptyset$, where either*

(a) $w_1, w_2, w_3 \in \sigma(A)$ and $z_1, z_2 \in \sigma(B)$ so that $DW(\operatorname{diag}(w_1, w_2, w_3))$ lies on the boundary of $DW(A)$ and $DW(\operatorname{diag}(z_1, z_2))$ lies on the boundary of $DW(B)$, or

(b) $w_1, w_2, w_3 \in \sigma(B)$ and $z_1, z_2 \in \sigma(A)$ so that $DW(\operatorname{diag}(w_1, w_2, w_3))$ lies on the boundary of $DW(B)$ and $DW(\operatorname{diag}(z_1, z_2))$ lies on the boundary of $DW(A)$.

Proof. Note that for any $z_1, z_2, z_3 \in \sigma(B)$, $DW(\operatorname{diag}(\mu - z_1, \mu - z_2, \mu - z_3))$ lies on the boundary of $DW(\mu I_n - B)$ if and only if $DW(\operatorname{diag}(z_1, z_2, z_3))$ lies on the boundary of $DW(B)$. Now, $DW(A)$ and $DW(\mu I_n - B)$ are two convex polytopes in $\mathbb{C} \times \mathbb{R}$ with vertices in $\mathbf{P} = \{(z, |z|^2) : z \in \mathbb{C}\}$. So, $DW(A) \cap DW(\mu I_n - B) \neq \emptyset$ if and only if one of the polytopes intersects a boundary face of the other polytopes. Suppose $DW(\mu I_n - B)$ intersects a boundary face of $DW(A)$. Then there are three vertices, say, $(w_j, |w_j|^2)$ with $w_j \in \sigma(A)$ for $j = 1, 2, 3$, of the boundary face of $DW(A)$ intersecting $DW(\mu I_n - B)$. Note that the vertices of $DW(\mu I_n - B)$ belongs to $\mathbf{P}$. So, $DW(\operatorname{diag}(w_1, w_2, w_2))$ must intersect with some boundary face of $DW(\mu I_n - B)$. Consequently, there are three vertices on the boundary face of $DW(\mu I_n - B)$ whose convex hull intersect with $DW(\operatorname{diag}(w_1, w_2, w_3))$. Now, for two triangular laminas each having vertices in $\mathbf{P}$ to have nonempty intersection, there must be non-empty intersection of a triangular lamina with an edge of another triangular lamina. By Lemma 3.3, there is a one-one correspondence between the triangles on the boundary faces of $DW(\mu I_n - B)$ and those on the boundary faces of $DW(B)$. Thus, condition (a) or (b) holds. □

One can also consider the boundary $\partial\mathcal{E}(A, B)$ of $\mathcal{E}(A, B)$. By Theorem 4.6 in Section 4, if $A, B \in M_n$ are normal and each of them has at most two distinct eigenvalues, then $\mathcal{E}(A, B)$ has empty interior, i.e., $\partial\mathcal{E}(A, B) = \mathcal{E}(A, B)$. We will exclude these special cases. The following lemma is needed for further discussion.

Lemma 3.5 *Let $S = \{w_1, w_2, w_3\}$ and $T = \{z_1, z_2\}$ be subsets of $\mathbb{C}$. Then*

$$\partial\mathcal{E}(S, T) = \mathcal{E}(\{w_1, w_2\}, T) \cup \mathcal{E}(\{w_1, w_3\}, T) \cup \mathcal{E}(\{w_2, w_3\}, T).$$

Proof. Clearly, the result holds if S or T is a singleton. In the following, we may assume that $z_1 \neq z_2$. If $w_j = w_k$ for some $1 \le j < k \le 3$, then $\mathcal{E}(S,T) = \mathcal{E}(\{w_i, w_l\}, T)$, where $l \notin \{j,k\}$, which has no interior point.

Suppose $w_1, w_2, w_3 \in \mathbb{C}$ are distinct. Let $X = \text{diag}\,(w_1, w_2, w_3)$, $Y = \text{diag}\,(z_1, z_2)$, $X_{jk} = \text{diag}\,(w_j, w_k)$ for $1 \le j < k \le 3$. By Theorem 2.2, $\mu \in \mathcal{E}(S,T)$ if and only if $DW(X) \cap DW(\mu I_2 - Y) \neq \emptyset$. Note that $DW(\mu I_2 - Y)$ is a line segment with vertices in $\mathbf{P}$ while $DW(X)$ is a triangular lamina with three edges $DW(X_{12})$, $DW(X_{23})$ and $DW(X_{13})$. Thus, μ is a boundary point of $\mathcal{E}(S,T)$ if and only if the line segment $DW(\mu I_2 - Y)$ intersects the triangular lamina $DW(X)$ at its boundary, which is the union of line segments $DW(X_{12})$, $DW(X_{23})$ and $DW(X_{13})$. The result follows. ☐

By the above lemma and Theorem 3.2, we have

Theorem 3.6 *Suppose $A, B \in M_n$ are normal matrices, each having at least 2 distinct eigenvalues. Then*

$$\partial\mathcal{E}(A,B) \subseteq \bigcup \left\{\mathcal{E}(S,T) : S \subseteq \sigma(A), T \subseteq \sigma(B) \text{ with } |S| = |T| = 2\right\}.$$

4 Essentially Hermitian matrices

Recall that a normal matrix is essentially Hermitian if all of its eigenvalues lie on a straight line. Let us warm up our discussion with the following results and examples on Hermitian matrices.

Theorem 4.1 *Suppose $A, B \in M_n$ are Hermitian matrices with eigenvalues $a_1 \ge a_2 \ge \cdots \ge a_n$ and $b_1 \ge b_2 \ge \cdots \ge b_n$. Then*

$$\mathcal{E}(A,B) = [a_n + b_n,\ a_1 + b_1] \setminus \bigcup_{j=1}^{n-1} (a_{j+1} + b_1,\ a_j + b_n) \cup (b_{j+1} + a_1,\ b_j + a_n)\,,$$

where $(c,d) = \emptyset$ if $c \ge d$.

Proof. By Theorem 3.1 (d), $\mu \notin \mathcal{E}(A,B)$ if and only if $\{a_1, a_2, \cdots, a_n\}$ can be separated from $\{\mu - b_1, \mu - b_2, \cdots, \mu - b_n\}$ by a circle. For $\mu \in \mathbb{R}$, this happens if and only if one of the following conditions is satisfied:

1. $\mu - b_1 > a_1 \Leftrightarrow \mu > a_1 + b_1$.
2. $\mu - b_n < a_n \Leftrightarrow \mu < a_n + b_n$.
3. For some $1 \le j \le n-1$, $a_{j+1} < \mu - b_1 \le \mu - b_n < a_j \Leftrightarrow a_{j+1} + b_1 < \mu < a_j + b_n$.
4. For some $1 \le j \le n-1$, $\mu - b_j < a_n \le a_1 < \mu - b_{j+1} \Leftrightarrow b_{j+1} + a_1 < \mu < b_j + a_n$.

Hence, the result follows. ☐

We have the following corollary.

Corollary 4.2 *Suppose $A, B \in M_n$ satisfy the hypotheses of Theorem 4.1. If*

$$b_1 - b_n \ge \max_{1 \le j \le n-1} (a_j - a_{j+1}) \quad \text{and} \quad a_1 - a_n \ge \max_{1 \le j \le n-1} (b_j - b_{j+1}),$$

then $\mathcal{E}(A,B) = [a_n + b_n,\ a_1 + b_1]$.

Example 4.3 *Suppose $n \ge 2$, $A, B \in M_n$ are Hermitian with eigenvalues $a_1 = 5$, $a_n = 2$, $b_1 = 4$, and $b_n = 1$. Then $\mathcal{E}(A,B) = [3,9]$ is independent of the choices of a_i and b_j for $2 \le i,\ j \le n-1$.*

Example 4.4 *Suppose $A, B \in M_3$ are Hermitian with eigenvalues $a_1 = 5$, $a_3 = 1$, $b_1 = 4$, and $b_3 = 2$. If $a_2 = 3$, then $\mathcal{E}(A, B) = [3, 9]$; if $a_2 \neq 3$, then $\mathcal{E}(A, B) \subsetneq [3, 9]$.*

It is interesting to note that sometimes the set $\mathcal{E}(A, B)$ depends only on the extreme eigenvalues of A and B as shown in Example 4.3, but it is not always the case as shown in Example 4.4.

In perturbation theory, if $A, B \in M_n$ are Hermitian such that $\|B\|$ is larger than the smallest singular value of A, then it may happen that $A + B$ is singular. However, if we know more about the eigenvalues of A and B, one can get a better perturbation bound.

Example 4.5 *Suppose $A, B \in M_n$ are Hermitian such that $\sigma(A) \subseteq \mathbb{R} \backslash (-r, s)$ for some $r, s \in (0, \infty)$ and $\sigma(B) \subseteq [-u, v]$ for some $u, v \in [0, \infty)$ such that $-r + v < 0$ and $-u + s > 0$. Then $A + B$ is invertible.*

In [19, Theorem 2], Wielandt described a procedure to construct $\mathcal{E}(A, B)$ for a Hermitian matrix A and a skew-Hermitian matrix B with eigenvalues $a_1, \dots, a_n$ and $b_1, \dots, b_n$. In particular, it was shown that the set $\mathcal{E}(A, B)$ is the intersection of all hyperbolic regions containing the set $\{a_j + b_k : 1 \leq j, k \leq n\}$. However, details of the proof were not given. In the following, we extend the result of Wielandt to any pair of essentially Hermitian matrices A and B. A detailed proof is given for the result.

To present the result and proof, we need some basic facts in the co-ordinate geometry of $\mathbb{R}^2$ (identified with $\mathbb{C}$). Suppose $w_1, w_2, z_1, z_2 \in \mathbb{C}$ such that $P = \text{conv}\,\{w_r + z_s : r, s \in \{1, 2\}\}$ is a nondegenerate parallelogram. Then there is a unique rectangular hyperbola passing through the vertices of P. The hyperbola degenerate to a pair of perpendicular line if and only if the four sides of P have equal length. Otherwise, each branch of the hyperbola will pass through a pair of vertices of P corresponding to a side of P with shorter length, i.e., the two sides of P of longer lengths lie in the closed region lying between the two branches of the hyperbola. For a nondegenerate rectangular hyperbola, the connected closed region with the hyperbola as boundary is the *inner hyperbolic region*, the two disconnected closed regions with the hyperbola as boundary is the *outer hyperbolic region*. Of course, the complement of a closed hyperbolic region is an open hyperbolic region, and vice versa. In case the hyperbola degenerated to a pair of perpendicular lines, the inner (and outer) hyperbolic region becomes the union of two unbounded triangular regions connected at their vertices.

Suppose A and B are two essentially Hermitian matrices. If the line through $\sigma(A)$ and the line through $\sigma(B)$ are parallel, then there are $\alpha, \beta \in \mathbb{C}$ and $\phi \in \mathbb{R}$ such that $H = e^{-i\phi}(A - \alpha I)$ and $K = e^{-i\phi}(B - \beta I)$ are Hermitian. Then

$$\mathcal{E}(A, B) = e^{i\phi}\mathcal{E}(H, K) + (\alpha + \beta)$$

and the result follows from Theorem 4.1. For the other cases, we have the following result.

Theorem 4.6 *Suppose $A, B \in M_n$ are non-scalar essentially Hermitian matrices. Then there exist $\alpha, \beta \in \mathbb{C}$, $r_1 \geq r_2 \geq \cdots \geq r_n$ and $s_1 \geq s_2 \geq \cdots \geq s_n$, $\phi, \theta \in \mathbb{R}$ such that the eigenvalues of A and B are $a_j = \alpha + r_j e^{i\phi}$, $1 \leq j \leq n$ and $b_j = \beta + s_j e^{i\theta}$, $1 \leq j \leq n$ respectively. Let $\Gamma = [r_n, r_1] \times [s_n, s_1]$. Assume that $e^{i(\phi - \theta)} \notin \{1, -1\}$, i.e., the two sets of eigenvalues do not lie on two parallel lines.*

(i) *Let $S(a, b) = \{a_u + b_v : 1 \leq u, v \leq n\}$, and $1 \leq j < n$. If $a_j \neq a_{j+1}$, then $S(a, b)$ is a subset of the closed hyperbolic region*

$$H(a, j) = \{e^{i\phi}x + e^{i\theta}y + \alpha + \beta : (x, y) \in \mathbb{R}^2 \text{ with } (y - s_1)(y - s_n) \leq (x - r_j)(x - r_{j+1})\};$$

if $b_j \neq b_{j+1}$, then $S(a, b)$ is a subset of the closed hyperbolic region

$$H(b, j) = \{e^{i\phi}x + e^{i\theta}y + \alpha + \beta : (x, y) \in \mathbb{R}^2 \text{ with } (y - s_j)(y - s_{j+1}) \geq (x - r_1)(x - r_n)\}.$$

(ii) *The set* $\mathcal{E}(A,B)$ *is the intersection of* $P = \mathrm{conv}\,\{a_r + b_s : r,s \in \{1,n\}\}$ *and all closed hyperbolic regions in* (i).

(iii) *Each connected component of* $\mathcal{E}(A,B)$ *is simply connected with boundary consists of segments of hyperbolas given in* (i).

In particular, if each A *and* B *has exactly two distinct eigenvalues, say* $a_1 = \cdots = a_k \neq a_{k+1} = \cdots = a_n$ *and* $b_1 = \cdots = b_\ell \neq b_{\ell+1} = \cdots = b_n$*, then* $\mathcal{E}(A,B)$ *are two segments of a hyperbola equal to*

$$\begin{aligned}\mathcal{E}(A,B) &= P \cap H(a,k) \cap H(b,\ell)\\ &= \{e^{i\phi}x + e^{i\theta}y + \alpha + \beta : (x,y) \in \Gamma \text{ with } (y-s_1)(y-s_n) = (x-r_1)(x-r_n)\}.\end{aligned}$$

Our proof depends on the following lemma.

Lemma 4.7 *Suppose* $A,B \in M_n$ *satisfy the assumption in Theorem 4.6. Then* $\mu \notin \mathcal{E}(A,B)$ *if and only if one of the following holds.*

(a) *The line segment joining* a_1, a_n *and the line segment joining* $\mu - b_1, \mu - b_n$ *do not intersect.*

(b) *There exist* $t_1, t_2 \in [0,1]$ *and* $j \in \{1,\dots,n-1\}$ *such that*

$$\mu - (t_1 b_1 + (1-t_1)b_n) = t_2 a_j + (1-t_2)a_{j+1} \qquad \text{and}$$

$$t_1|\mu - b_1|^2 + (1-t_1)|\mu - b_n|^2 < t_2|a_j|^2 + (1-t_2)|a_{j+1}|^2.$$

(c) *There exist* $t_1, t_2 \in [0,1]$ *and* $j \in \{1,\dots,n-1\}$ *such that*

$$\mu - (t_1 b_j + (1-t_1)b_{j+1}) = t_2 a_1 + (1-t_2)a_n \qquad \text{and}$$

$$t_1|\mu - b_j|^2 + (1-t_1)|\mu - b_{j+1}|^2 > t_2|a_1|^2 + (1-t_2)|a_n|^2.$$

Proof. Under the given assumption, $DW(A)$ and $DW(\mu I_n - B)$ will be a vertical polygonal disks in $\mathbb{C} \times \mathbb{R}$ with vertices in $\{(z,|z|^2) : z \in \mathbb{C}\}$. The two disks have no intersection if and only if

(1) the projections of the two disks on $\mathbb{C}$ do not intersect, or

(2) the projections on $\mathbb{C}$ intersect but one disk is above the other disk.

Case (1) is equivalent to (a), and (2) is equivalent to (b) or (c). □

Proof of Theorem 4.6. Suppose $\mu \notin \mathcal{E}(A,B)$. Consider the three cases in Lemma 4.7:

(a) The line segment joining a_1, a_n and the line segment joining $\mu - b_1, \mu - b_n$ have no intersection if and only if for all $0 \le t_1,\ t_2 \le 1$,

$$\begin{aligned} t_2 a_1 + (1-t_2)a_n &\neq t_1(\mu - b_1) + (1-t_1)(\mu - b_n)\\ \mu &\neq t_1 b_1 + (1-t_1)b_n + t_2 a_1 + (1-t_2)a_n\\ \mu &\notin P = \mathrm{conv}\,\{a_r + b_s : r,s \in \{1,n\}\}\\ \mu &\notin \{e^{i\phi}x + e^{i\theta}y + \alpha + \beta : (x,y) \in \Gamma\}.\end{aligned}$$

(b) Suppose for some $t_1, t_2 \in [0,1]$ and $j \in \{1,\dots,n-1\}$ such that

$$\mu - (t_1 b_1 + (1-t_1)b_n) = t_2 a_j + (1-t_2)a_{j+1} \tag{4.1}$$

and

$$t_1|\mu - b_1|^2 + (1-t_1)|\mu - b_n|^2 < t_2|a_j|^2 + (1-t_2)|a_{j+1}|^2. \tag{4.2}$$

Let $\mu - \alpha - \beta = e^{i\phi}u + e^{i\theta}v$ with $u, v \in \mathbb{R}$. From (4.1) and $a_j = \alpha + e^{i\phi}r_j$ and $b_j = \beta + e^{i\theta}s_j$ for $1 \le j \le n$ with $e^{i(\phi-\theta)} \notin \{1, -1\}$, we have

$$u = t_2 r_j + (1-t_2)r_{j+1} \quad \text{and} \quad v = (t_1 s_1 + (1-t_1)s_n)$$

or equivalently,

$$t_1 = \frac{s_n - v}{s_n - s_1} \quad \text{and} \quad t_2 = \frac{r_{j+1} - u}{r_{j+1} - r_j}.$$

We have

$$\begin{aligned} & t_2|a_j|^2 + (1-t_2)|a_{j+1}|^2 \\ = \; & t_2|\alpha + e^{i\phi}r_j|^2 + (1-t_2)|\alpha + e^{i\phi}r_{j+1}|^2 \\ = \; & t_2(|\alpha|^2 + (\overline{\alpha}e^{i\phi} + \alpha e^{-i\phi})r_j + r_j^2) + (1-t_2)(|\alpha|^2 + (\overline{\alpha}e^{i\phi} + \alpha e^{-i\phi})r_{j+1} + r_{j+1}^2) \\ = \; & |\alpha|^2 + (\overline{\alpha}e^{i\phi} + \alpha e^{-i\phi})u + (r_j + r_{j+1})u - r_j r_{j+1} \end{aligned}$$

as $t_2 r_j + (1-t_2)r_{j+1} = u$ and $t_2 r_j^2 + (1-t_2)r_{j+1}^2 = (r_j + r_{j+1})u - r_j r_{j+1}$.

$$\begin{aligned} & t_1|\mu - b_1|^2 + (1-t_1)|\mu - b_n|^2 \\ = \; & t_1|\alpha + e^{i\phi}u + e^{i\theta}(v - s_1)|^2 + (1-t_1)|\alpha + e^{i\phi}u + e^{i\theta}(v - s_n)|^2 \\ = \; & t_1\Big[|\alpha|^2 + (\overline{\alpha}e^{i\phi} + \alpha e^{-i\phi})u + (\overline{\alpha}e^{i\theta} + \alpha e^{-i\theta})(v - s_1) \\ & +(e^{i(\theta-\phi)} + e^{-i(\theta-\phi)})u(v - s_1) + u^2 + (v - s_1)^2\Big] \\ & +(1-t_1)\Big[|\alpha|^2 + (\overline{\alpha}e^{i\phi} + \alpha e^{-i\phi})u + (\overline{\alpha}e^{i\theta} + \alpha e^{-i\theta})(v - s_n) \\ & +(e^{i(\theta-\phi)} + e^{-i(\theta-\phi)})u(v - s_n) + u^2 + (v - s_n)^2\Big] \\ = \; & |\alpha|^2 + (\overline{\alpha}e^{i\phi} + \alpha e^{-i\phi})u + u^2 - (v - s_1)(v - s_n) \end{aligned}$$

as $t_1(v - s_1) + (1-t_1)(v - s_n) = 0$ and $t_1(v - s_1)^2 + (1-t_1)(v - s_n)^2 = -(v - s_1)(v - s_n)$.

Putting these values into (4.2), we have

$$\begin{aligned} 0 \; < \; & t_2|a_j|^2 + (1-t_2)|a_{j+1}|^2 - t_1|\mu - b_1|^2 - (1-t_1)|\mu - b_n|^2 \\ = \; & (v - s_1)(v - s_n) - (u - r_j)(u - r_{j+1}). \end{aligned}$$

For any $z = e^{i\phi}x + e^{i\theta}y + \alpha + \beta$ with $x, y \in \mathbb{R}$, define

$$f(z) = (y - s_1)(y - s_n) - (x - r_j)(x - r_{j+1}).$$

With $a_k + b_m = e^{i\phi}r_k + e^{i\theta}s_m + \alpha + \beta$, we have

$$f(a_k + b_m) = (s_m - s_1)(s_m - s_n) - (r_k - r_j)(r_k - r_{j+1}) \le 0.$$

Thus, $H(a,j) = \{z : f(z) \le 0\}$ is a closed hyperbolic region satisfying (i).

Similarly, if condition (c) in Lemma 4.7 is satisfied, we have a closed hyperbolic region $H(b,j)$ satisfying (i).

By Lemma 4.7 and (i), we see that $\mathcal{E}(A,B)$ is a subset of the intersection of P and the hyperbolic regions described in (i), and no points in the complement of the intersection belongs to $\mathcal{E}(A,B)$. Thus, assertion (ii) of the theorem follows.

From the above discussion, we can see that the complement of $\mathcal{E}(A,B)$ is a union of open hyperbolic regions. So, if $z \in \mathbb{C} \setminus \mathcal{E}(A,B)$, then there exists a half line L containing z with $L \cap \mathcal{E}(A,B) = \emptyset$. Hence, every connected component of $\mathcal{E}(A,B)$ is simply connected.

Suppose the boundary of the parallelogram $P = \operatorname{conv}\{a_u + b_v : u, v \in \{1,n\}\}$ is graduated by the points $a_r + b_j$ and $a_j + b_r$ with $r \in \{1,n\}$ and $j \in \{1,\dots,n\}$. Then the intersection of the hyperbolas $H(a,j)$ (respectively, $H(b,j)$) with P will have end points $a_r + b_s$ with $r \in \{j, j+1\}$ and $s \in \{1,n\}$ (respectively, $r \in \{1,n\}$ and $s \in \{j,j+1\}$).

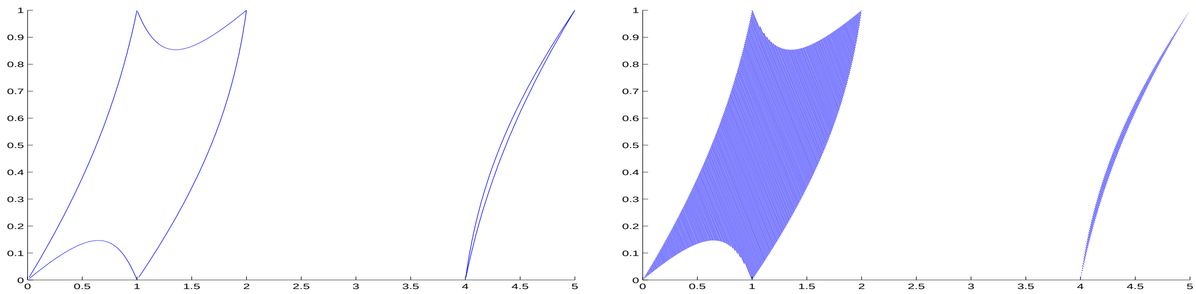
Combining the arguments in the last two paragraphs, we get condition (iii). $\square$

Remark 4.8 The above result gives a simple procedure to determine the region $\mathcal{E}(A,B)$ for A and B satisfying the conditions in Theorem 4.6:

Sketch the hyperbolas corresponding to the intersection of P and the closed hyperbolic regions $H(a,j)$ and $H(b,j)$ for $1 \le j < n$ (see Section 6.2). Then $\mathcal{E}(A,B)$ consists of the simply connected regions in P determined by these curves.

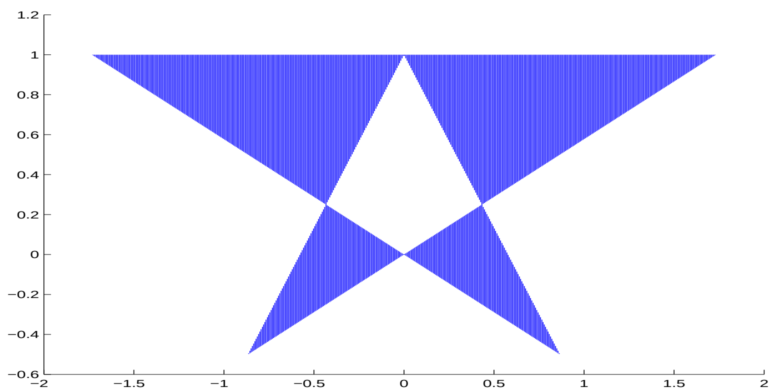
Remark 4.9 Notice that all 2×2 normal matrices are essentially Hermitian. Then for any 2×2 non-scalar normal matrices A and B, $\mathcal{E}(A,B)$ is either a union of line segments or a pair of hyperbola by Theorems 4.1 and 4.6. In both cases, $\mathcal{E}(A,B)$ has empty interior.

Example 4.10 *Consider $A = \operatorname{diag}(0,1,4)$ and $B = \operatorname{diag}(0, 1+i)$. The following pictures depict the segments of hyperbolas corresponding to $H(a,1)$, $H(a,2)$ and $H(b,1)$ and the set $\mathcal{E}(A,B)$.*



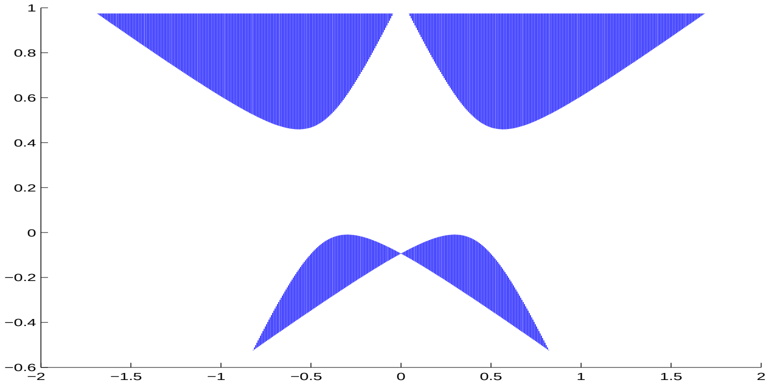
Suppose $A, B \in M_n$ are normal matrices. The connected components of $\mathcal{E}(A,B)$ may not be simply connected in general as shown in the following example.

Example 4.11 *Let $\omega = e^{i2\pi/3}$. Using the method described in Section 6, we can show that for $A = \operatorname{diag}(-i, -i\omega, -i\omega^2)$ and $B = \operatorname{diag}(-i\omega, -i\omega, -i\omega^2)$, $\mathcal{E}(A,B)$ is not simply connected.*



Although the conclusion of Theorem 4.6 does not hold for arbitrary normal matrices $A, B \in M_n$, one can see form Theorem 3.6 that the boundary of $\mathcal{E}(A,B)$ is a subset of the union of hyperbolas determined by eigenvalue pairs of A and eigenvalue pairs of B. We have the following example.

Example 4.12 *Let* $\omega = e^{i2\pi/3}$, $A = \operatorname{diag}(-i, -i\omega, -i\omega^2)$ *and* $B = 0.95\operatorname{diag}(-i\omega, -i\omega, -i\omega^2)$. *Then the boundary of* $\mathcal{E}(A,B)$ *are subsets of the union of hyperbolas.*



It is interesting to note that the matrices in Example 4.12 are obtained from those in Example 4.11 by shirking B by a factor of 0.95, and hence the two pictures of $\mathcal{E}(A,B)$ have some resemblance even though part of the boundary changes from straight line segments to curve segments. In general, it is not hard to show that $(A,B) \mapsto \mathcal{E}(A,B)$ is a continuous function, say, using the usual topology on $M_n \times M_n$ and the Hausdorff metric for compact sets in $\mathbb{C}$.

5 Extensions and open problems

One may ask whether the results can be extended to the sum of k matrices from k different unitary similarity orbits for $k > 2$. For Hermitian matrices $A_1, \ldots, A_k$, there is a complete description of the eigenvalues of the matrices in $\mathcal{U}(A_1) + \cdots + \mathcal{U}(A_k)$; see [8]. For non-Hermitian matrices $A_1, \ldots, A_k \in M_n$, we can extend the idea in Section 2 to determine the set of complex numbers μ, which is the eigenvalue of a matrix in $\mathcal{U}(A_1) + \cdots + \mathcal{U}(A_k)$. To this end, we need the concept of the modified Davis-Wielandt shell of $A \in M_n$ defined by

$$MDW(A) = \left\{ \left(x^*Ax, \sqrt{\|Ax\|^2 - |x^*Ax|^2}\, e^{it} \right) : x \in \mathbb{C}^n,\ x^*x = 1,\ t \in \mathbb{R} \right\} \subseteq \mathbb{C} \times \mathbb{C}.$$

Note that $(\mu_1, \mu_2) \in MDW(A)$ if and only if there is a unitary matrix U such that the first column of U^*AU equals $[\mu_1, \mu_2, 0, \ldots, 0]^t$.

Theorem 5.1 *Let* $A_1, \ldots, A_k \in M_n$ *and* $\mu \in \mathbb{C}$. *The following are equivalent.*

(a) *There are unitary* $U_1, \ldots, U_k \in M_n$ *such that* $\det(\sum_{j=1}^k U_j A_j U_j^* - \mu I_n) = 0$.

(b) $(\mu, 0) \in MDW(A_1) + \cdots + MDW(A_k)$.

(c) $[MDW(A_1) + \cdots + MDW(A_{k-1})] \cap MDW(\mu I_n - A_k) \neq \emptyset$.

Proof. We may assume that $k \geq 3$. The implications $(c) \iff (b) \Rightarrow (a)$ are clear. Suppose (a) holds. Then there are unitary matrices $U_1, \ldots, U_k$ such that the first column of $\sum_{j=1}^k U_j^* A_j U_j$ equals $[\mu, 0, \ldots, 0]^t$. Let v_j be obtained from the first column of $U_j^* A_j U_j$ by removing its first entry μ_j. Then $\sum_{j=1}^k v_j = 0$. Relabel A_j so that $\|v_1\| \geq \cdots \geq \|v_k\|$. Then $\|v_1\| \leq \|v_2\| + \cdots + \|v_k\|$. Thus, there exist $t_1, \ldots, t_k \in \mathbb{R}$ such that $\sum_{j=1}^k \|v_j\| e^{it_j} = 0$. It follows that $(\mu_j, \|v_j\| e^{it_j}) \in MDW(A_j)$ for $j = 1, \ldots, k$ such that $(\mu, 0) = \sum_{j=1}^k (\mu_j, \|v_j\| e^{it_j})$. Thus, condition (b) holds. □

Besides the unitary similarity orbits, one may consider orbits of matrices under other group actions and consider the eigenvalues of the sum of matrices from different orbits.

For example, we can consider the usual similarity orbit of $A \in M_n$

$$\mathcal{S}(A) = \{SAS^{-1} : S \in M_n \text{ is invertible}\};$$

the unitary equivalence orbit of $A \in M_n$

$$\mathcal{V}(A) = \{UAV : U, V \in M_n \text{ are unitary}\};$$

the unitary congruence orbit of $A \in M_n$

$$\mathcal{U}^t(A) = \{UAU^t : U \in M_n \text{ is unitary}\}.$$

For example, if $A, B \in M_n$ are not scalar, then any $\mu \in \mathbb{C}$ can be an eigenvalues of $SAS^{-1} + B$. Can we prove this for complex orthogonal similarity?

One may also consider the eigenvalues of usual product, Lie product, and Jordan product of matrices from different orbits; e.g., see [10, 16]. Of course, one may ask similar problems for matrices over reals or arbitrary fields or rings.

For example, our results in Section 2.1 hold for real eigenvalues for real matrices $UAU^t + VBV^t$, where U, V are real orthogonal matrices.

6 Computer algorithms and programs

Using the result in Section 2, we can use positive semi-definite programming package to test whether $\mu \in \mathcal{E}(A, B)$ as follows. For every $(\xi, |\xi|^2) \in DW(\mu I - B)$, we check whether $(\xi, |\xi|^2) \in DW(A)$, equivalently, we check whether there is a real linear combination of of the three Hermitian matrices:

$$\mathrm{Re}(A - \xi I), \mathrm{Im}(A - \xi I), A^*A - |\xi|^2 I$$

is positive definite. (This can be done by positive semi-definite programming package.) If there is no such combination, then $(\xi, |\xi|^2) \in DW(A)$.

Of course, the above test is inefficient and hard to implement. The situation will improve significantly for normal matrices. One can use standard linear programming package to check whether the two convex polytopes $DW(A)$ and $DW(\mu I - B)$ have nonempty intersection.

The situation further improves if we use Theorem 3.4 and focus on $DW(X) \cap DW(\mu I_2 - Y)$ for normal matrices $X \in M_3$ and $Y \in M_2$. For convenience, we use $\mathcal{E}(X, Y)$ to denote the set of $\mu \in \mathbb{C}$ such that $DW(X) \cap DW(\mu I_2 - Y) \neq \emptyset$, even X and Y may not have the same size. Then the set $\mathcal{E}(A, B)$ is the union of $\mathcal{E}(X, Y)$, where $X = \mathrm{diag}\,(w_1, w_2, w_3) \in M_3$ and $Y = \mathrm{diag}\,(z_1, z_2) \in M_2$ described in Theorem 3.4. Furthermore, if both A and B have only two distinct eigenvalues, respectively, say w_1, w_2 and z_1, z_2, then $\mathcal{E}(A, B) = \mathcal{E}(X, Y)$ with $X = \mathrm{diag}\,(w_1, w_2)$ and $Y = \mathrm{diag}\,(z_1, z_2)$.

In the following, we will focus on $\mathcal{E}(X, Y)$ so that either $(X, Y) \in M_2 \times M_2$ or $(X, Y) \in M_3 \times M_2$ with distinct eigenvalues. Also as $\mathcal{E}(X, Y)$ depends only on the eigenvalues of X and Y, we may assume that X and Y are diagonal in our discussion.

We describe an easy point-wise test for $x + iy \in \mathcal{E}(X, Y)$ in the following.

6.1 A point-wise test

The two-two case

We begin with the simple case when $X = \text{diag}\,(w_1, w_2), Y = \text{diag}\,(z_1, z_2) \in M_2$, and determine whether a given point $x+iy \in \mathcal{E}(X,Y)$, for four given complex numbers $w_1 = a_1+ib_1$, $w_2 = a_2+ib_2$, $z_1 = c_1 + id_1$ $z_2 = c_2 + id_2$ so that w_1, w_2 are distinct, and z_1, z_2 are distinct.

Let $P_j = \left(a_j, b_j, a_j^2 + b_j^2\right)$ and $Q_j = \left(x - c_j, y - d_j, (x - c_j)^2 + (y - d_j)^2\right)$ for $j = 1, 2$. Then $x + iy \in \mathcal{E}$ if and only if

$$\overline{P_1P_2} \cap \overline{Q_1Q_2} \neq \emptyset. \tag{6.1}$$

Since all the 4 points P_1, P_2, Q_1, Q_2 lie on the boundary of the convex set $\{(x, y, z) : x^2 + y^2 \leq z\} \subseteq \mathbb{R}^3$, (6.1) holds if and only if the 4 points lie on the same plane and P_1 and P_2 lie on opposite closed half plane determined by the line through Q_1 and Q_2.

Let $\mathbf{u} = \overrightarrow{Q_1Q_2}$, $\mathbf{v} = \overrightarrow{Q_1P_2}$ and $\mathbf{r} = \mathbf{u} \times \mathbf{v} = (r_1, r_2, r_3)$. Define

$$\Delta_0 = \left| \begin{array}{ccc} c_1 - c_2 & a_1 + c_1 - x & a_2 - a_1 \\ d_1 - d_2 & b_1 + d_1 - y & b_2 - b_1 \\ \begin{array}{c}(x - c_2)^2 + (y - d_2)^2 \\ -(x - c_1)^2 - (y - d_1)^2\end{array} & a_1^2 + b_1^2 - (x - c_1)^2 - (y - d_1)^2 & a_2^2 + b_2^2 - a_1^2 - b_1^2 \end{array} \right|,$$

.

$$\Delta_1 = \left| \begin{array}{ccc} c_1 - c_2 & a_1 + c_1 - x & r_1 \\ d_1 - d_2 & b_1 + d_1 - y & r_2 \\ \begin{array}{c}(x - c_2)^2 + (y - d_2)^2 \\ -(x - c_1)^2 - (y - d_1)^2\end{array} & a_1^2 + b_1^2 - (x - c_1)^2 - (y - d_1)^2 & r_3 \end{array} \right|$$

Then P_1, P_2, Q_1, Q_2 all lie on the same plane if and only if $\Delta_0 = 0$. Suppose $\Delta_0 = 0$. Then P_1 and P_2 lie on opposite closed half plane determined by the line through Q_1 and Q_2 if and only if $\Delta_1 \leq 0$.

Assertion 6.1 *For normal matrices $X, Y \in M_2$ with eigenvalues described above, $x+iy \in \mathcal{E}(X,Y)$ if and only if $\Delta_0 = 0$ and $\Delta_1 \leq 0$.*

The three-two case

Next, we describe the test to determine whether a given point

$$x + iy \in \mathcal{E}\left(\text{diag}\,(w_1, w_2, w_3), \text{diag}\,(z_1, z_2)\right)$$

for any given complex numbers w_1, w_2, w_3, z_1, z_2 so that w_1, w_2, w_3 are distinct and z_1, z_2 are distinct. Let $w_j = a_j + ib_j$ for $j = 1, 2, 3$, and $z_k = c_k + id_k$ for $k = 1, 2$. Then $x + iy \in \mathcal{E}(X, Y)$ if and only if there exist $0 \leq t_1 \leq 1$, $0 \leq t_1,\ t_2$, and $t_1 + t_2 \leq 1$ such that

$$\begin{aligned} &(1 - t_1) \begin{pmatrix} x - c_1 \\ y - d_1 \\ (x - c_1)^2 + (y - d_1)^2 \end{pmatrix} + t_1 \begin{pmatrix} x - c_2 \\ y - d_2 \\ (x - c_2)^2 + (y - d_2)^2 \end{pmatrix} \\ = \ &(1 - t_2 - t_3) \begin{pmatrix} a_1 \\ b_1 \\ a_1^2 + b_1^2 \end{pmatrix} + t_2 \begin{pmatrix} a_2 \\ b_2 \\ a_2^2 + b_2^2 \end{pmatrix} + t_3 \begin{pmatrix} a_3 \\ b_3 \\ a_3^2 + b_3^2 \end{pmatrix}, \end{aligned}$$

or equivalently,

$$\left(\begin{array}{ccc} c_2 - c_1 & a_2 - a_1 & a_3 - a_1 \\ d_2 - d_1 & b_2 - b_1 & b_3 - b_1 \\ \begin{array}{c}(x - c_1)^2 + (y - d_1)^2 \\ -(x - c_2)^2 - (y - d_2)^2\end{array} & a_2^2 + b_2^2 - a_1^2 - b_1^2 & a_3^2 + b_3^2 - a_1^2 - b_1^2 \end{array} \right) \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

$$= \begin{pmatrix} x-c_1-a_1 \\ y-d_1-b_1 \\ (x-c_1)^2+(y-d_1)^2 \\ -\left(a_1^2+b_1^2\right) \end{pmatrix}.$$

Let

$$\Delta_0 = \begin{vmatrix} c_2-c_1 & a_2-a_1 & a_3-a_1 \\ d_2-d_1 & b_2-b_1 & b_3-b_1 \\ (x-c_1)^2+(y-d_1)^2 & & \\ -(x-c_2)^2-(y-d_2)^2 & a_2^2+b_2^2-a_1^2-b_1^2 & a_3^2+b_3^2-a_1^2-b_1^2 \end{vmatrix},$$

$$\Delta_1 = \begin{vmatrix} x-c_1-a_1 & a_2-a_1 & a_3-a_1 \\ y-d_1-b_1 & b_2-b_1 & b_3-b_1 \\ (x-c_1)^2+(y-d_1)^2 & & \\ -\left(a_1^2+b_1^2\right) & a_2^2+b_2^2-a_1^2-b_1^2 & a_3^2+b_3^2-a_1^2-b_1^2 \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} c_2-c_1 & x-c_1-a_1 & a_3-a_1 \\ d_2-d_1 & y-d_1-b_1 & b_3-b_1 \\ (x-c_1)^2+(y-d_1)^2 & (x-c_1)^2+(y-d_1)^2 & \\ -(x-c_2)^2-(y-d_2)^2 & -\left(a_1^2+b_1^2\right) & a_3^2+b_3^2-a_1^2-b_1^2 \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} c_2-c_1 & a_2-a_1 & x-c_1-a_1 \\ d_2-d_1 & b_2-b_1 & y-d_1-b_1 \\ (x-c_1)^2+(y-d_1)^2 & & (x-c_1)^2+(y-d_1)^2 \\ -(x-c_2)^2-(y-d_2)^2 & a_2^2+b_2^2-a_1^2-b_1^2 & -\left(a_1^2+b_1^2\right) \end{vmatrix}.$$

By the above discussion, we have the following.

Assertion 6.2 *Suppose $X \in M_3$ and $Y \in M_2$ are normal with eigenvalues described as above. Assume that $\Delta_0 \neq 0$. Then $x+iy \in \mathcal{E}(A,B)$ if and only if*

$$(\Delta_1, \Delta_2, \Delta_3, \Delta_0-\Delta_1, \Delta_0-\Delta_2-\Delta_3)/\Delta_0$$

has nonnegative entries.

Suppose $\Delta_0 = 0$. Let

$$P_j = \left(a_j, b_j, a_j^2+b_j^2\right) \quad \text{for } j=1,2,3, \quad \text{and}$$

$$Q_k = \left(x-c_k, y-d_k, (x-c_k)^2+(y-d_k)^2\right) \quad \text{for } k=1,2.$$

Then the line L through Q_1 and Q_2 is parallel to the plane Π determine by P_1, P_2, and P_3. Since all the 5 points P_1, P_2, P_3, Q_1, Q_2 lie on the boundary of the convex set $\{(x,y,z) : x^2+y^2 \leq z\} \in \mathbb{R}^3$, $x+iy \in \mathcal{E}(X,Y)$ if and only if L lies on Π and each of the closed half space determined by L contains some P_i. Hence, L lies on Π if and only if $\Delta_0 = \Delta_1 = 0$. In such a case, let

$$\mathbf{u} = \overrightarrow{Q_1Q_2} = \left(c_1-c_2, d_1-d_2, (x-c_2)^2-(x-c_1)^2+(y-d_2)^2-(y-d_1)^2\right).$$

For $1 \leq j \leq 3$, let

$$\mathbf{v_j} = \overrightarrow{Q_1P_j} = \left(a_j-x+c_1, b_j-y+d_1, a_j^2+b_j^2-(x-c_1)^2-(y-d_1)^2\right).$$

If P_j and P_k lie on different half planes determined by L, then the cross products $\mathbf{u}\times\mathbf{v_j}$ and $\mathbf{u}\times\mathbf{v_k}$ are normals to Π, pointing in opposite directions. For $1 \le j \le 3$, let $\mathbf{r_j} = \mathbf{u}\times\mathbf{v_j} = (r_{1j}, r_{2j}, r_{3j})$ and

$$\Delta_j' = \begin{vmatrix} a_2 - a_1 & a_3 - a_1 & r_{1j} \\ b_2 - b_1 & b_3 - b_1 & r_{2j} \\ a_2^2 + b_2^2 - a_1^2 - b_1^2 & a_3^2 + b_3^2 - a_1^2 - b_1^2 & r_{3j} \end{vmatrix}.$$

We can now describe the remaining case in the following.

Assertion 6.3 *Suppose $X \in M_3$ and $Y \in M_2$ are normal with eigenvalues described as above. Assume that $\Delta_0 = 0$. Then $x + iy \in \mathcal{E}(X, Y)$ if and only if $\Delta_1 = 0$ and $\Delta_j' \le 0 \le \Delta_k'$ for some $1 \le j, k \le 3$.*

Based on Assertions 6.1 – 6.3 with Theorem 3.4, we have written the Matlab program PT.m (see `http://www.math.wm.edu/~ckli/program/PT.m`) to test whether a point $x + iy$ lies in $\mathcal{E}(A, B)$.

Also, if $A, B \in M_n$ are normal matrices, then $\mathcal{E}(A, B)$ is a subset of the set

$$\mathrm{conv}\,(\sigma(A) + \sigma(B)) = \mathrm{conv}\,\{a + b : a \in \sigma(A), b \in \sigma(B)\}.$$

One can then consider a grid in $\mathrm{conv}\,(\sigma(A)+\sigma(B))$ and apply the pointwise test to the grid points to plot $\mathcal{E}(A, B)$. The Matlab program PPT.m (see `http://www.math.wm.edu/~ckli/program/PPT.m`) is written based on this idea. An example of $\mathcal{E}(A, B)$ generated by the program will be given in Section 6.4.

6.2 Parametrization of $\mathcal{E}(A, B)$ for normal matrices

In this subsection, we give a parametrization of $\mathcal{E}(A, B)$. We start with the three-two case.

The three-two case

Consider the case when $X = \mathrm{diag}\,(w_1, w_2, w_3) \in M_3$ and $Y = \mathrm{diag}\,(z_1, z_2) \in M_2$. Write $w_j = a_j + ib_j$ for $j = 1, 2, 3$ and $z_k = c_k + id_k$ for $k = 1, 2$. Let $P_j = \left(a_j, b_j, a_j^2 + b_j^2\right)$ for $j = 1, 2, 3$ and $Q_k = \left(x - c_k, y - d_k, (x - c_k)^2 + (y - d_k)^2\right)$ for $k = 1, 2$. As $\mu \in \mathcal{E}(X, Y)$ if and only if $\mu + w_1 + z_1 \in \mathcal{E}(X - w_1 I_3, Y - z_1 I_2)$. We may assume that $w_1 = z_1 = 0$, i.e., $a_1 = b_1 = c_1 = d_1 = 0$. Notice that $\mathcal{E}(X, Y)$ is the set of $x + iy \in \mathbb{C}$ such that $\Delta(P_1P_2P_3) \cap \overline{Q_1Q_2} \ne \emptyset$ and this holds if and only if there exist $0 \le t \le 1$ such that $\overline{P_1P_4} \cap \overline{Q_1Q_2} \ne \emptyset$, where

$$P_4 = (a_4, b_4, r_4) = \left(ta_2 + (1 - t)a_3,\ tb_2 + (1 - t)b_3,\ t(a_2^2 + b_2^2) + (1 - t)(a_3^2 + b_3^2)\right). \qquad (6.2)$$

By the convexity of the function $(x, y) \mapsto x^2 + y^2$, we have $r_4 \ge a_4^2 + b_4^2$. Thus, there is $0 \le t_1, t_2 \le 1$ such that

$$(1 - t_1)\begin{pmatrix} x \\ y \\ x^2 + y^2 \end{pmatrix} + t_1 \begin{pmatrix} x - c_2 \\ y - d_2 \\ (x - c_2)^2 + (y - d_2)^2 \end{pmatrix} = (1 - t_2)\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} a_4 \\ b_4 \\ r_4 \end{pmatrix}$$

or equivalently,

$$x = c_2 t_1 + a_4 t_2, \qquad y = d_2 t_1 + b_4 t_2, \qquad (6.3)$$

and

$$t_1(x^2 + y^2 - (x - c_2)^2 - (y - d_2)^2) + r_4 t_2 = x^2 + y^2. \qquad (6.4)$$

Substituting (6.3) into (6.4) we get

$$(c_2^2+d_2^2)t_1(t_1-1)-(a_4^2+b_4^2)t_2^2+r_4t_2=0$$

which is a hyperbolic equation of t_1 and t_2 on $[0,1]$.

Suppose $(c_2^2+d_2^2)\geq r_4/(a_4^2+b_4^2)$. Then

$$t_1=\frac{1}{2}\pm\sqrt{\frac{1}{4}+\left(\frac{a_4^2+b_4^2}{c_2^2+d_2^2}\right)t_2^2-\left(\frac{r_4}{c_2^2+d_2^2}\right)t_2}, \tag{6.5}$$

and it is easy to check that $0\leq t_1\leq 1$ whenever $t_2\in[0,1]$.

Suppose $(c_2^2+d_2^2)<r_4/(a_4^2+b_4^2)$. Then

$$t_2=\frac{r_4}{2(a_4^2+b_4^2)}+\sqrt{\frac{r_4^2}{4(a_4^2+b_4^2)^2}+\left(\frac{c_2^2+d_2^2}{a_4^2+b_4^2}\right)t_1(t_1-1)} \tag{6.6}$$

or

$$t_2=\frac{r_4}{2(a_4^2+b_4^2)}-\sqrt{\frac{r_4^2}{4(a_4^2+b_4^2)^2}+\left(\frac{c_2^2+d_2^2}{a_4^2+b_4^2}\right)t_1(t_1-1)}. \tag{6.7}$$

Note that for t_2 defined in (6.6), $0\leq t_2\leq 1$ whenever $t_1\in[0,1]$ and for t_2 defined in (6.7), $0\leq t_2\leq 1$ whenever $t_1\in\left[\frac{1}{2}-\sqrt{\frac{1}{4}+\frac{a_4^2+b_4^2-r_4}{c_2^2+d_2^2}},\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{a_4^2+b_4^2-r_4}{c_2^2+d_2^2}}\right]$, provided that the expression in the square roots is nonnegative.

Assertion 6.4 *Suppose $X\in M_3$ and $Y\in M_2$ are normal with eigenvalues described as above. For each $t\in[0,1]$, determine t_1 and t_2 using the equations* (6.2), (6.5) - (6.7). *Then $x+iy\in\mathcal{E}(X,Y)$ if and only if it is given by the parametric equation* (6.3) *in terms of t_1 and t_2.*

The two-two case

Next, we consider the two by two case. By a similar argument of the three by two case with $(a_4,b_4,r_4)=(a_2,b_2,a_2^2+b_2^2)$, we have

$$x=c_2t_1+a_2t_2,\qquad y=d_2t_1+b_2t_2, \tag{6.8}$$

and

$$(c_2^2+d_2^2)t_1(t_1-1)-(a_2^2+b_2^2)t_2(t_2-1)=0$$

which is a hyperbolic equation of t_1 and t_2 on $[0,1]$. Then

$$t_1=\frac{1}{2}\pm\sqrt{\frac{1}{4}+\left(\frac{a_2^2+b_2^2}{c_2^2+d_2^2}\right)t_2(t_2-1)}, \tag{6.9}$$

lies in $[0,1]$ whenever $t_2\in[0,1]$ if $c_2^2+d_2^2\geq a_2^2+b_2^2$, or

$$t_2=\frac{1}{2}\pm\sqrt{\frac{1}{4}+\left(\frac{c_2^2+d_2^2}{a_2^2+b_2^2}\right)t_1(t_1-1)} \tag{6.10}$$

lies in $[0,1]$ whenever $t_1\in[0,1]$ if $c_2^2+d_2^2<a_2^2+b_2^2$.

Assertion 6.5 *Suppose* $X = \operatorname{diag}(0, w_2) \in M_2$ *and* $Y = \operatorname{diag}(0, z_2) \in M_2$. *Then* $x + iy \in \mathcal{E}(X, Y)$ *if and only if it is given by the parametric equation* (6.8) *in terms of* t_1 *and* t_2 *determined by equations* (6.9) *and* (6.10).

Based on Assertions 6.4 and 6.5 and Theorem 3.4, we have written the matlab program HPT.m (see `http://www.math.wm.edu/~ckli/program/HPT.m`) to generate $\mathcal{E}(X, Y)$. An example of $\mathcal{E}(A, B)$ generated by the program will be given in Section 6.4.

Using Theorem 3.6 and Assertion 6.5, we have written the matlab program BD32.m (see `http://www.math.wm.edu/~ckli/program/BD32.m`) to generate $\partial\mathcal{E}(X, Y)$, the boundary of $\mathcal{E}(X, Y)$ for normal $X \in M_3$ and $Y \in M_2$.

6.3 A different algorithm

To use the parametric approach in the previous subsection, one has to consider grid points for $t_2 \in [0, 1]$. For each choice of t_2 one has to determine intervals for t_3, then determine the value t_1, and draw two curves for t_3 in the two intervals. Here, we introduce a different algorithms to generate $\mathcal{E} = \mathcal{E}(X, Y)$ with $X = \operatorname{diag}(w_1, w_2, w_3)$ and $Y = \operatorname{diag}(z_1, z_2)$. To generate the points $x + iy \in \mathcal{E}(X, Y)$, we first determine the range for x. Then for each x in the range, we determine the range of y. Here we consider the three by two case only.

Let $w_j = a_j + ib_j$ and $z_k = c_k + id_k$ for $j = 1, 2, 3$ and $k = 1, 2$. Since $\mathcal{E}(\mu X, \mu Y) = \mu\mathcal{E}(X, Y)$, we may assume that $d_1 = d_2$. Also by a suitable relabeling, we can always assume $c_1 > c_2$ and $b_1 \geq b_2 \geq b_3$. Evidently, if $x + iy \in \mathcal{E}(X, Y)$, then

$$\min\{a_1, a_2, a_3\} + c_2 \leq x \leq \max\{a_1, a_2, a_3\} + c_1.$$

Now we choose an x satisfying the above inequalities and determine y so that $x + iy$ lies in $\mathcal{E}(X, Y)$.

In fact, except for the case when $b_1 = b_2 = b_3$, we may further assume that $b_1 < b_2 \leq b_3$. In the exceptional case, $X - b_1 I_3$ and $Y - d_1 I_2$ are Hermitian matrices. Then the result follows from Theorem 4.1 In detail, we have

Assertion 6.6 *Suppose* $a_1 < a_2 < a_3$, $b_1 = b_2 = b_3$, $c_1 < c_2$ *and* $d_1 = d_2$. *Then* $x + iy \in \mathcal{E}(X, Y)$ *if and only if* $y = b_1 + d_1$ *and*

$$x \in [a_1 + c_1, a_3 + c_2] \backslash (a_1 + c_2, a_2 + c_1) \cup (a_2 + c_2, a_3 + c_1) \cup (a_3 + c_1, a_1 + c_2).$$

From now, we suppose that $b_1 < b_2 \leq b_3$. As $\mathcal{E}(X - \mu I_3, Y + \mu I_2) = \mathcal{E}(X, Y)$, we can also assume $|w_1| = |w_2| \neq |w_3|$ if w_1, w_2, w_3 are collinear and $|w_1| = |w_2| = |w_3|$ otherwise.

Note that as $d_1 = d_2$ and $|w_1| = |w_2|$, the determinants Δ_i defined in Section 6.1 become

$$\begin{aligned}
\Delta_0 &= \begin{vmatrix} c_2 - c_1 & a_2 - a_1 & a_3 - a_1 \\ 0 & b_2 - b_1 & b_3 - b_1 \\ (x - c_1)^2 - (x - c_2)^2 & 0 & a_3^2 + b_3^2 - a_1^2 - b_1^2 \end{vmatrix}, \\
\Delta_1 &= \begin{vmatrix} x - c_1 - a_1 & a_2 - a_1 & a_3 - a_1 \\ y - d_1 - b_1 & b_2 - b_1 & b_3 - b_1 \\ (x - c_1)^2 + (y - d_1)^2 - (a_1^2 + b_1^2) & 0 & a_3^2 + b_3^2 - a_1^2 - b_1^2 \end{vmatrix}, \\
\Delta_2 &= \begin{vmatrix} c_2 - c_1 & x - c_1 - a_1 & a_3 - a_1 \\ 0 & y - d_1 - b_1 & b_3 - b_1 \\ (x - c_1)^2 - (x - c_2)^2 & (x - c_1)^2 + (y - d_1)^2 - (a_1^2 + b_1^2) & a_3^2 + b_3^2 - a_1^2 - b_1^2 \end{vmatrix}, \\
\Delta_3 &= \begin{vmatrix} c_2 - c_1 & a_2 - a_1 & x - c_1 - a_1 \\ 0 & b_2 - b_1 & y - d_1 - b_1 \\ (x - c_1)^2 - (x - c_2)^2 & 0 & (x - c_1)^2 + (y - d_1)^2 - (a_1^2 + b_1^2) \end{vmatrix},
\end{aligned}$$

where the last three determinants can been expressed in the form

$$\Delta_i = \Delta_{i2}(y-d_1)^2 + \Delta_{i1}(y-d_1) + \Delta_{i0} \quad i = 1,2,3,$$

with

$$\begin{aligned}
\Delta_{12} &= \begin{vmatrix} a_2-a_1 & a_3-a_1 \\ b_2-b_1 & b_3-b_1 \end{vmatrix}, \\
\Delta_{11} &= -\begin{vmatrix} a_2-a_1 & a_3-a_1 \\ 0 & a_3^2+b_3^2-a_1^2-b_1^2 \end{vmatrix}, \\
\Delta_{10} &= \begin{vmatrix} x-c_1-a_1 & a_2-a_1 & a_3-a_1 \\ -b_1 & b_2-b_1 & b_3-b_1 \\ (x-c_1)^2-(a_1^2+b_1^2) & 0 & a_3^2+b_3^2-a_1^2-b_1^2 \end{vmatrix}, \\
\Delta_{22} &= -\begin{vmatrix} c_2-c_1 & a_3-a_1 \\ 0 & b_3-b_1 \end{vmatrix}, \\
\Delta_{21} &= \begin{vmatrix} c_2-c_1 & a_3-a_1 \\ (x-c_1)^2-(x-c_2)^2 & a_3^2+b_3^2-a_1^2-b_1^2 \end{vmatrix}, \\
\Delta_{20} &= \begin{vmatrix} c_2-c_1 & x-c_1-a_1 & a_3-a_1 \\ 0 & -b_1 & b_3-b_1 \\ (x-c_1)^2-(x-c_2)^2 & (x-c_1)^2-(a_1^2+b_1^2) & a_3^2+b_3^2-a_1^2-b_1^2 \end{vmatrix}, \\
\Delta_{32} &= \begin{vmatrix} c_2-c_1 & a_2-a_1 \\ 0 & b_2-b_1 \end{vmatrix}, \\
\Delta_{31} &= -\begin{vmatrix} c_2-c_1 & a_2-a_1 \\ (x-c_1)^2-(x-c_2)^2 & 0 \end{vmatrix}, \\
\Delta_{30} &= \begin{vmatrix} c_2-c_1 & a_2-a_1 & x-c_1-a_1 \\ 0 & b_2-b_1 & -b_1 \\ (x-c_1)^2-(x-c_2)^2 & 0 & (x-c_1)^2-(a_1^2+b_1^2) \end{vmatrix}.
\end{aligned}$$

Note that

$$\Delta_0 = (c_2-c_1)\begin{vmatrix} b_2-b_1 & b_3-b_1 \\ 0 & a_3^2+b_3^2-a_1^2-b_1^2 \end{vmatrix} + ((x-c_1)^2-(x-c_2)^2)\begin{vmatrix} a_2-a_1 & a_3-a_1 \\ b_2-b_1 & b_3-b_1 \end{vmatrix}.$$

Therefore, $\Delta_0 = 0$ if and only if

$$w_1, w_2, w_3 \text{ are not collinear and } x = (c_1+c_2)/2. \tag{6.11}$$

Suppose (6.11) holds. Then $\Delta_0 = 0$ and by Assertion 6.3, $x+iy \in \mathcal{E}(X,Y)$ only if $\Delta_1 = 0$, in which the equality holds when

$$y = d_1 \pm \sqrt{(x-c_1)^2-(a_1^2+b_1^2)}.$$

Now we can check whether the point $x+iy$ in $\mathcal{E}(X,Y)$ by considering the values of Δ_i' defined in Assertion 6.3.

Exclude the above case. Then $\Delta_0 \neq 0$. By Assertion 6.2, $x+iy \in \mathcal{E}(X,Y)$ if and only if

$$\Delta_1/\Delta_0 \geq 0, \quad \Delta_2/\Delta_0 \geq 0, \quad \Delta_3/\Delta_0 \geq 0, \quad (\Delta_0-\Delta_1)/\Delta_0 \geq 0 \quad \text{and} \quad (\Delta_0-\Delta_2-\Delta_3)/\Delta_0 \geq 0.$$

In the following, we determine the possible range of y that satisfies the above inequalities.

Suppose $\alpha_1 \leq \beta_1, \ldots, \alpha_5 \leq \beta_5$ are the real solutions, if exist, of the following quadratic equations

$$\Delta_1 = \Delta_{12}(y-d_1)^2 + \Delta_{11}(y-d_1) + \Delta_{10} = 0, \tag{6.12}$$

$$\Delta_2 = \Delta_{22}(y-d_1)^2 + \Delta_{21}(y-d_1) + \Delta_{20} = 0, \tag{6.13}$$

$$\Delta_3 = \Delta_{32}(y-d_1)^2 + \Delta_{31}(y-d_1) + \Delta_{30} = 0, \tag{6.14}$$

$$\Delta_0 - \Delta_1 = -\Delta_{12}(y-d_1)^2 - \Delta_{11}(y-d_1) - \Delta_{10} + \Delta_0 = 0, \tag{6.15}$$

$$\Delta_0 - \Delta_2 - \Delta_3 = -(\Delta_{22}+\Delta_{32})(y-d_1)^2 - (\Delta_{21}+\Delta_{31})(y-d_1) - (\Delta_{20}+\Delta_{30}) + \Delta_0 = 0. \tag{6.16}$$

Also we keep to use α_i to denote the corresponding real solution if the quadratic equation is linear.

As $b_1 < b_2 \le b_3$,

$$\Delta_{22} = -(c_2-c_1)(b_3-b_1) < 0 \quad \text{and} \quad \Delta_{32} = (c_2-c_1)(b_2-b_1) > 0.$$

Thus, the inequalities $\Delta_2/\Delta_0 \ge 0$ and $\Delta_3/\Delta_0 \ge 0$ are satisfied if and only if y lies in the interval specified in the following

Table 1

Eq. (6.13)	Eq. (6.14)	$\Delta_0 > 0$	$\Delta_0 < 0$
Y	Y	$[\alpha_2,\beta_2]\backslash(\alpha_3,\beta_3)$	$[\alpha_3,\beta_3]\backslash(\alpha_2,\beta_2)$
Y	N	$[\alpha_2,\beta_2]$	No solution
N	Y	No solution	$[\alpha_3,\beta_3]$
N	N	No solution	No solution

where "Y" denotes the corresponding equation having real solution(s) and "N" otherwise.

Now we turn to equation (6.16). Note that

$$\Delta_{22} + \Delta_{32} = \begin{vmatrix} c_2-c_1 & a_2-a_3 \\ 0 & b_2-b_3 \end{vmatrix} \le 0$$

So the equation is linear, equivalently $\Delta_{22}+\Delta_{32} = 0$, if and only if $b_2 = b_3$, which can hold only if w_1, w_2, w_3 is not collinear. In this case, $a_3^2 + b_3^2 - a_1^2 - b_1^2 = |w_3|^2 - |w_1|^2 = 0$ and so

$$\Delta_{21} + \Delta_{31} = \begin{vmatrix} c_2-c_1 & a_3-a_2 \\ (x-c_1)^2-(x-c_2)^2 & a_3^2+b_3^2-a_1^2-b_1^2 \end{vmatrix} \ne 0.$$

Therefore the inequality $(\Delta_0 - \Delta_2 - \Delta_3)/\Delta_0 \ge 0$ is satisfied if and only if y lies in the intervals specified in the following

Table 2

	$b_2 \ne b_3 \quad (\Delta_{22}+\Delta_{32} \ne 0)$		$b_2 = b_3 \quad (\Delta_{22}+\Delta_{32} = 0)$	
Eq. (6.16)	$\Delta_0 > 0$	$\Delta_0 < 0$	$(\Delta_{21}+\Delta_{31})/\Delta_0 > 0$	$(\Delta_{21}+\Delta_{31})/\Delta_0 < 0$
Y	$(-\infty,\alpha_5] \cup [\beta_5,\infty)$	$[\alpha_5,\beta_5]$	$(-\infty,\alpha_5]$	$[\alpha_5,\infty)$
N	$(-\infty,\infty)$	No solution	/	/

Finally we consider the equations (6.12) and (6.15). Clearly, the equations are linear, i.e., $\Delta_{12} = 0$, if and only if w_1, w_2, w_3 are collinear. In addition, the equations are constant functions, i.e., $\Delta_{12} = 0$ and $\Delta_{11} = 0$, if and only if $a_1 = a_2 = a_3$. In case of being constant function,

$$\Delta_0 = (c_2-c_1)(b_2-b_1)(a_3^2+b_3^2-a_1^2-b_1^2) \quad \text{and} \quad \Delta_1 = (x-c_1-a_1)(b_2-b_1)(a_3^2+b_3^2-a_1^2-b_1^2).$$

Thus, the inequalities $\Delta_1/\Delta_0 \geq 0$ and $(\Delta_0 - \Delta_1)/\Delta_0 \geq 0$ are satisfied if and only if $c_1 \leq x - a_1 \leq c_2$, which always hold by our assumption on x.

Combining with the quadratic and linear cases, the inequalities $\Delta_1/\Delta_0 \geq 0$ and $(\Delta_0 - \Delta_1)/\Delta_0 \geq 0$ are satisfied if and only if y lies in the intervals specified in the following

Table 3

		Non-collinear $(\Delta_{12} \neq 0)$		Collinear $(\Delta_{12} = 0)$		
Eq. (6.12)	Eq. (6.15)	$\Delta_{12}/\Delta_0 > 0$	$\Delta_{12}/\Delta_0 < 0$	$\Delta_{11}/\Delta_0 > 0$	$\Delta_{11}/\Delta_0 = 0$	$\Delta_{11}/\Delta_0 < 0$
Y	Y	$[\alpha_4, \beta_4] \backslash (\alpha_1, \beta_1)$	$[\alpha_1, \beta_1] \backslash (\alpha_4, \beta_4)$	$[\alpha_1, \alpha_4]$	$(-\infty, \infty)$	$[\alpha_4, \alpha_1]$
Y	N	No solution	$[\alpha_1, \beta_1]$	/	/	/
N	Y	$[\alpha_4, \beta_4]$	No solution	/	/	/
N	N	No solution	No solution	/	/	/

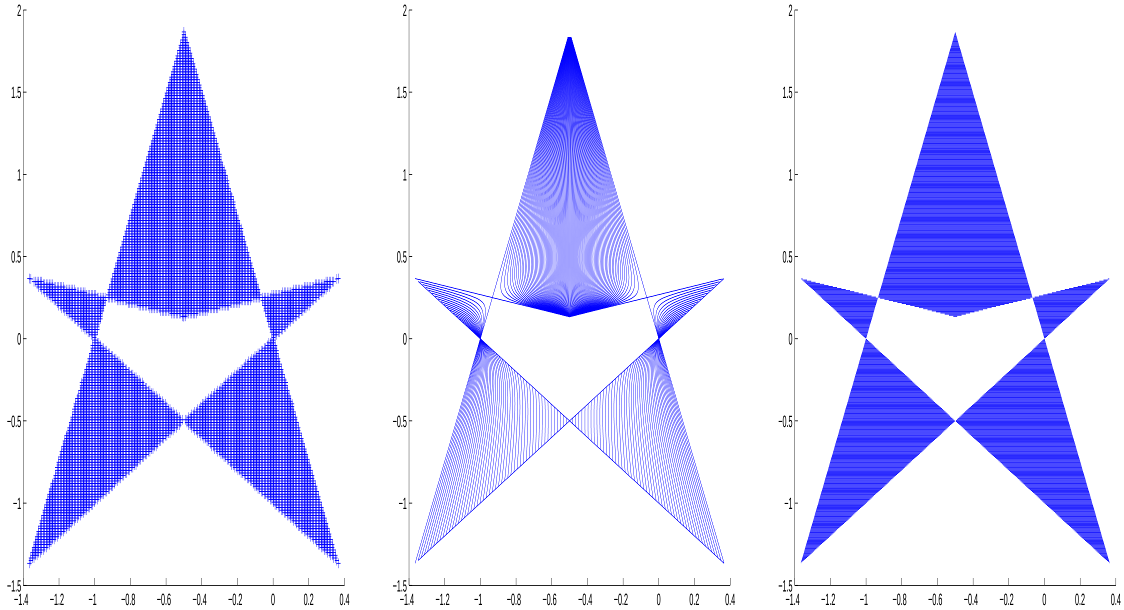
In summary, we have the following

Assertion 6.7 *Suppose $b_1 < b_2 \leq b_3$, $c_1 < c_2$, $d_1 = d_2$. Assume* (i) $|w_1| = |w_2| \neq |w_3|$ *if w_1, w_2, w_3 are collinear, and* (ii) $|w_1| = |w_2| = |w_3|$ *otherwise. Except for the case (6.11), for any $x \in [a_{\min} + c_1, a_{\max} + c_2]$, where $a_{\min} = \min\{a_1, a_2, a_3\}$ and $a_{\max} = \max\{a_1, a_2, a_3\}$, $x + iy \in \mathcal{E}(X, Y)$ if and only if y lies in the intersection the intervals specified in Tables 1, 2 and 3.*

Based on Assertions 6.6 – 6.7, we have written another Matlab program IPT.m (see `http://www.math.wm.edu/~ckli/program/IPT.m`) to generate $\mathcal{E}(A, B)$ for normal matrices A and B. An example of $\mathcal{E}(A, B)$ generated by the program will be given in Section 6.4.

6.4 An example of $\mathcal{E}(A, B)$ generated by the three approaches

Example 6.8 *Let $A = \operatorname{diag}(i, i\omega, i\omega^2)$ and $B = \operatorname{diag}(\omega, \omega^2)$ with $\omega = e^{i2\pi/3}$. The region of $\mathcal{E}(A, B)$ is plotted using Matlab programs based on the three different algorithms in Sections 6.1–6.3.*



$\mathcal{E}(A, B)$ plotted by PPT.m $\quad$ $\mathcal{E}(A, B)$ plotted by HPT.m $\quad$ $\mathcal{E}(A, B)$ plotted by IPT.m

In the above example, we see that the first program took the longest computer time and a lot of memory to determine and store $\mathcal{E}(A, B)$. The second program took less computer time and less memory, but it is not effective in approximating the straight line boundary of $\mathcal{E}(A,B)$ (using hyperbolas). Finally, the third program used to least among of computer time and memory to produce and store $\mathcal{E}(A,B)$.

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