# Preservers of eigenvalue inclusion sets

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**Abstract**

For a square matrix $A$, let $\mathcal{S}(A)$ be an eigenvalue inclusion set such as the Gershgorin region, the Brauer region in terms of Cassini ovals, and the Ostrowski region. Characterization is obtained for maps $\Phi$ on $n \times n$ matrices satisfying $\mathcal{S}(\Phi(A) - \Phi(B)) = \mathcal{S}(A - B)$ for all matrices $A$ and $B$. From these results, one can deduce the structure of additive or (real) linear maps satisfying $\mathcal{S}(A) = \mathcal{S}(\Phi(A))$ for every matrix $A$.

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## 1 Introduction

Motivated by pure and applied problems, researchers often need to understand the eigenvalues of matrices. For example, in numerical analysis or population dynamics, a square matrix $A$ satisfies $\lim_{m\to\infty} A^m = 0$ if and only if each eigenvalue has modulus less than 1; in stability theory of differential equations, the solution of the system of differential equations $x' = Ax$ is stable if and only if all the eigenvalues of $A$ lie in the left half plane; in the study of quadratic forms a Hermitian matrix is positive definite if and only if all the eigenvalues are positive real; see [2]. However, it is sometimes difficult to compute the eigenvalues efficiently and accurately, due to reasons such as the dimension of the matrix is too high, there are numerical or measuring errors in the entries, etc. So, researchers consider eigenvalue inclusion sets; see [2, 5]. For instance, the well known Gershgorin theorem asserts that the eigenvalues of a matrix lie in the union of circular disks centered at the diagonal entries with radii determined by the off diagonal entries (see Section 2). These eigenvalue inclusion sets give effective ways to estimate the location of the eigenvalues for matrices. To further improve the estimate, one may apply simple transformations such as diagonal similarities to a matrix to get better or easier estimates of the location of the eigenvalues of a given matrix.

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In this paper, we study maps on matrices leaving invariant the eigenvalue inclusion sets such as the Gershgorin region, the Brauer region in terms of Cassini ovals, and the Ostrowski region (see Sections 2, 3, and 4, respectively). The study of maps on matrix spaces leaving invariant some properties, functions or subsets is known as *preserver problems*. Early study on the subject focused on linear preservers; i.e., linear maps having the preserving properties; see [3] and its references. Recently, researchers work on more general preservers; see [4] and its references.

In our study, let $\mathcal{S}(A)$ be an eigenvalue inclusion set for the matrix $A$. We characterize maps $\Phi$ on square matrices satisfying $\mathcal{S}(\Phi(A)-\Phi(B)) = \mathcal{S}(A-B)$ for any two matrices $A$ and $B$. It is shown that such $\Phi$ are real affine maps. From our results or proofs, one can deduce the structure of additive or (real) linear maps satisfying the following:

$$\mathcal{S}(A) = \mathcal{S}(\Phi(A)). \tag{1.1}$$

Note that if one just assumes that a map $\Phi$ satisfies (1.1) for every matrix $A$, the structure of $\Phi$ can be quite arbitrary. For instance, one can partition the set of matrices into equivalence classes so that two matrices $A$ and $B$ belong to the same class if $\mathcal{S}(A) = \mathcal{S}(B)$. If $\Phi$ sends each of these classes back to itself, then $\Phi$ satisfies (1.1) for every matrix $A$.

We will always assume that $n \geq 2$ to avoid trivial consideration. The following notation and definitions will be used in our discussion.

$M_n$: the set of $n \times n$ complex matrices.

$\{E_{11}, E_{12}, \dots, E_{nn}\}$: the standard basis for $M_n$.

$\mathbf{P}_n$: the group of permutation matrices in $M_n$.

$\mathbf{DU}_n$: the group of diagonal unitary matrices in $M_n$.

$\mathbf{GP}_n = \{DP : D \in \mathbf{DU}_n,\ P \in \mathbf{P}_n\}$: the group of generalized permutation matrices in $M_n$.

## 2 Gershgorin Regions

In this section, we consider preservers of the Gershgorin region of $A = (a_{ij}) \in M_n$ defined by

$$G(A) = \cup_{k=1}^{n} G_k(A),$$

where for $k = 1, \dots, n$,

$$G_k(A) = \{\mu \in \mathbb{C} : |\mu - a_{kk}| \leq R_k\} \quad \text{with} \quad R_k = \sum_{j \neq k} |a_{kj}|.$$

It is known that $G(A)$ contains all the eigenvalues of $A$; one may see [2, Chapter 6] for general properties of $G(A)$.

**Theorem 2.1** *A map $\Phi : M_n \to M_n$ satisfies*

$$G(\Phi(A) - \Phi(B)) = G(A-B) \qquad \text{for all } A, B \in M_n \tag{2.1}$$

*if and only if $\Phi$ is a composition of maps of the following forms:*

(1) $A \mapsto \begin{pmatrix} A_1Q_1 \\ \vdots \\ A_nQ_n \end{pmatrix}$ *for* $A \in M_n$ *with rows* $A_1, \ldots, A_n$, *where* $Q_1, \ldots, Q_n \in \mathbf{GP}_n$ *are such that the* $(j,j)$ *entry of* $Q_j$ *is* 1 *for* $j = 1, \ldots, n$.

(2) $A \mapsto PAP^t + S$, *where* $P \in \mathbf{P}_n$ *and* $S \in M_n$.

(3) $A = (a_{rs}) \mapsto (\psi_{rs}(a_{rs}))$, *where* $\psi_{rs}$ *is either the complex conjugation map* $z \mapsto \overline{z}$ *or identity map* $z \mapsto z$, *and the latter always occurs if* $r = s$.

It is easy to verify that maps of the form (1), (2), and (3) in the theorem indeed satisfy (2.1). Hence the sufficiency of the theorem is clear. To prove the necessity part of the theorem, we need some lemmas.

**Lemma 2.2** *Let* $w = e^{i2\pi/n}$ *and*

$$\Gamma = \left\{ \frac{1 - w^k}{1 - w^j} : j \in \{1, \ldots, n-1\}, k \in \{1, \ldots, n\} \right\}. \tag{2.2}$$

*Suppose* $\mu \in \mathbb{C} \setminus \Gamma$. *Then* $R \in \mathbf{P}_n$ *is such that the set of entries of the vector* $\mu(w, w^2, \ldots, w^n) - (w, w^2, \ldots, w^n)R$ *equals* $\{(\mu - 1)w^j : 1 \le j \le n\}$ *if and only if* $R = I_n$.

*Proof.* The sufficiency is clear. To verify the necessity part, assume that the $k$th entry of the vector $\mu(w, w^2, \ldots, w^n) - (w, w^2, \ldots, w^n)R$ equals $\mu w^k - w^i$ with $k \neq i$. Then $\mu w^k - w^i = \mu w^j - w^j$ for some $j \in \{1, \ldots, n\}$ so that $\mu(w^j - w^k) = w^j - w^i$. If $j = i$, then $j \neq k$ and hence $\mu = 0 = (1 - w^n)/(1 - w)$; if $j \neq i$, then $\mu = (1 - w^{k-j})/(1 - w^{i-j})$. This contradicts the fact that $\mu \notin \Gamma$. ☐

The next lemma should be known to researchers on distance preserving maps. We include a proof for completeness.

**Lemma 2.3** *Let* $\mathbf{V} = \mathbb{C}^{1 \times m}$. *A map* $f : \mathbf{V} \to \mathbf{V}$ *satisfies* $\ell_1(f(x) - f(y)) = \ell_1(x - y)$ *for all* $x, y \in \mathbf{V}$ *if and only if there is* $z \in \mathbf{V}$ *and* $Q \in \mathbf{GP}_n$ *such that* $f$ *has the form*

$$(x_1, \ldots, x_m) \mapsto (f_1(x_1), \ldots, f_m(x_m))Q + z,$$

*where* $f_j$ *is either the identity map* $\mu \mapsto \mu$ *or the complex conjugation* $\mu \mapsto \bar{\mu}$.

*Proof.* The sufficiency is clear. We consider the converse. We divide the proof into two assertions.

**Assertion 1** Let $\mathcal{B} = \{x \in \mathbf{V} : \ell_1(x) \le 1\}$ and $\mathcal{E} = \{x \in \mathcal{B} : x \neq (u + v)/2 \text{ with different } u, v \in \mathcal{B}\}$ be the set of extreme points of $\mathcal{B}$. Then $x \in \mathcal{E}$ if and only if $x$ has only one nonzero entry with modulus 1.

*Proof.* Let $x = (x_1, \ldots, x_m) \in \mathcal{B}$. We consider three possible cases.

**Case 1** Suppose $x$ has a single nonzero entry at the $j$th position with $|x_j| = 1$. If $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in \mathcal{B}$ satisfy $x = (u+v)/2$, then $x_j = (u_j + v_j)/2$. Since $|x_j| = 1 \geq |u_j|, |v_j|$, we see that $u_j = v_j = x_j$. Because $u, v \in \mathcal{B}$, all other entries of $u$ and $v$ must be zero. Thus, $x = u = v$. Hence $x \in \mathcal{E}$.

**Case 2** Suppose $x$ has a single nonzero entry at the $j$th position such that $|x_j| < 1$. Then there exists a $\delta \neq 0$ such that $|x_j + \delta|, |x_j - \delta| \leq 1$. We can then let $u = (0, \ldots, 0, x_j + \delta, 0, \ldots, 0), v = (0, \ldots, 0, x_j - \delta, 0, \ldots, 0) \in \mathcal{B}$ such that $x = (u+v)/2$. Thus, $x \notin \mathcal{E}$.

**Case 3** Suppose $x \in \mathcal{B}$ has at least two nonzero entries, say, at the $j$th and $k$th positions such that $x_j = \rho_j e^{it_j}, x_k = \rho_k e^{it_k}$ and $\rho_j, \rho_k \in (0,1)$, where $\rho_j + \rho_k \leq 1$. Then there exists $u = (u_1, \ldots, u_m)$ with two nonzero entries, namely, $u_j = \delta e^{it_j}$ and $u_k = -\delta e^{it_k}$ with $\min\{\rho_j, \rho_k\} > \delta > 0$ so that $x + u, x - u \in \mathcal{B}$ are different vectors and $x = [(x+u) + (x-u)]/2 \notin \mathcal{E}$.

Combining these three cases we see that $x \in \mathcal{E}$ if and only if $x$ has only one nonzero entry with modulus 1.

**Assertion 2** The map $g(x) = f(x) - f(0)$ is real linear and satisfies $g(\mathcal{E}) = \mathcal{E}$, and $f$ has the asserted form with $f(0) = z$.

*Proof.* By the result in [1], we know that $g(x) = f(x) - f(0)$ is real linear. Note that $g(x) = 0$ if and only if $0 = \ell_1(g(x)) = \ell_1(x)$, i.e., $x = 0$. Hence $g$ is a real linear injective map, and therefore is bijective. As a result, $g(\mathcal{B}) = \mathcal{B}$. Clearly, $x = (u+v)/2$ for distinct $u, v \in \mathcal{B}$ if and only if $g(x) = (g(u) + g(v))/2$ with distinct $g(u), g(v) \in \mathcal{B}$. Thus, $g(\mathcal{E}) = \mathcal{E}$.

Let $\{e_1, e_2, \ldots, e_m\}$ be the standard basis for $\mathbb{C}^{1\times m}$. Because $g(\mathcal{E}) = \mathcal{E}$, we see that $g(e_j) = \mu_j e_{r_j}$ for some $r_j \in \{1, \ldots, m\}$ and $\mu_j \in \mathbb{C}$ with $|\mu_j| = 1$. Since $\ell_1(e_j + \gamma e_k) = \ell_1(g(e_j) + \gamma g(e_k))$ for all $\gamma \in \mathbf{R}$ and $j \neq k$, we see that $(r_1, \ldots, r_m)$ is a permutation of $(1, \ldots, m)$. Now, $g(ie_j) = \nu_j e_{s_j}$ for some $s_j \in \{1, \ldots, m\}$ and $\nu_j \in \mathbb{C}$ with $|\nu_j| = 1$. Since $\ell_1(e_j + \gamma i e_j) = \ell_1(g(e_j) + \gamma g(ie_j))$ for all $\gamma \in \mathbf{R}$, we see that $s_j = r_j$ and $\nu_j \in \{i\mu_j, -i\mu_j\}$. Thus, $g$ has the form $(x_1, \ldots, x_m) \mapsto (f_1(x_1), \ldots, f_m(x_m))Q$, where $f_1, \ldots, f_m$ and $Q$ satisfy the conclusion of the lemma. Thus, $f$ has the asserted form. □

**Proof of Theorem 2.1.** The sufficiency is clear, as remarked before. We consider the necessity. Replacing $\Phi$ by the map $X \mapsto \Phi(X) - \Phi(0)$, we may assume that $\Phi(0) = 0$ and $G(\Phi(X)) = \Phi(X)$ for all $X \in M_n$ in addition to the assumption that $G(\Phi(A) - \Phi(B)) = G(A - B)$ for all $A, B \in M_n$. Let $\Gamma$ be defined as in Lemma 2.2.

**Assertion 1** Let $D = \operatorname{diag}(w, w^2, \ldots, w^{n-1}, 1)$ for $w = e^{i2\pi/n}$. There is a permutation matrix $P$ such that $P\Phi(D)P^t = D$. Furthermore, if $\mu \in \mathbb{C} \setminus \Gamma$, then $P\Phi(\mu D)P^t = \mu D$.

To verify the assertion, note that $G(\Phi(\mu D)) = G(\mu D) = \{\mu w^j : 1 \leq j \leq n\}$. Hence, $\Phi(\mu D)$ has digonal entries $\mu w^j$ with $j = 1, \ldots, n$, and all off-diagonal entries equal to zero. Thus, there is a permutation matrix $P$ such that $P\Phi(D)P^t = D$. We may assume that $\Phi(D) = D$. Otherwise, replace $\Phi$ by the map $A \mapsto P^t\Phi(A)P$. Suppose $\mu \in \mathbb{C} \setminus \Gamma$. Then

$$G(\Phi(\mu D) - \Phi(D)) = G(\mu D - D) = \{(\mu - 1)w^j : j = 1, \ldots, n\}.$$

Thus, the vector of diagonal entries of $\Phi(\mu D) - \Phi(D)$ equals $\mu(w, w^2, \ldots, w^n) - (w, w^2, \ldots, w^n)R$ for some $R \in \mathbf{P}_n$, and the entries constitute the set $\{(\mu - 1)w^j : 1 \le j \le n\}$. By Lemma 2.2, the $(j, j)$ entry of $\Phi(\mu D)$ is $\mu w^j$ for each $j = 1, \ldots, n$, i.e., $\Phi(\mu D) = \mu D$.

**Assertion 2** Assume that $P$ in the conclusion in Assertion 1 is $I_n$. Then $\Phi(W_k) \subseteq W_k$ for $k = 1, \ldots, n$, where

$$W_k = \left\{ \sum_{j \neq k} a_j E_{kj} : a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n \in \mathbb{C} \right\}.$$

Moreover, define $\Phi_k : \mathbb{C}^{1\times(n-1)} \to \mathbb{C}^{1\times(n-1)}$ by

$$\Phi_k(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n) = (b_1, \ldots, b_{k-1}, b_{k+1}, \ldots, b_n)$$

if $\Phi(\sum_{j\neq k} a_j E_{kj}) = \sum_{j\neq k} b_j E_{kj}$. Then $\Phi_k$ satisfies the conclusion of Lemma 2.3 with $z = 0$.

To prove the assertion, let $A = \sum_{j\neq k} a_j E_{kj} \in W_k$ and $\Phi(A) = B = (b_{ij})$. Let $\mu \in (0, \infty) \setminus \Gamma$ satisfy $\mu|1 - w| > \sum_{j\neq k} |a_{kj}| = R_k$. Then $\Phi(\mu D) = \mu D$ by Assertion 1. Moreover, since $G(B) = G(A) = \{z \in \mathbb{C} : |z| \le R_k\}$,

$$\min\{\mu|w^i - w^j| : 1 \le i < j \le n\} = \mu|1 - w| > R_k \ge \max\{|b_{jj}| : 1 \le j \le n\}. \tag{2.3}$$

Thus,

$$G(\mu D - B) = G(\Phi(\mu D) - \Phi(A)) = G(\mu D - A) = \{\mu w^j : j \neq k\} \cup \{\gamma : |\gamma - \mu w^k| \le R_k\}.$$

For $i = 1, \ldots, n$, the $(i, i)$ entry of $\mu D - B$ equals $\mu w^i - b_{ii}$ is the center of one of the disks in $G(\mu D - A)$. Thus, $\mu w^i - b_{ii} = \mu w^j$ implies that $\mu|w^i - w^j| = |b_{ii}| < \mu|1 - w|$ by (2.3). It follows that $i = j$ and $b_{ii} = 0$. Thus, $\mu D - B$ has diagonal entries $\mu w, \mu w^2, \ldots, \mu w^{n-1}, \mu$. Since $G(\mu D - B) = G(\mu D - A)$, we see that only the $k$th row of $B$ can have nonzero off diagonal entries, and $\sum_{j\neq k} |b_{kj}| = R_k$.

Now, define $\Phi_k$ as in Assertion 2. Since $G(\Phi(A_1) - \Phi(A_2)) = G(A_1 - A_2)$ for any $A_1, A_2 \in W_k$ and $\Phi(0) = 0$, we see that $\ell_1(\Phi_k(u) - \Phi_k(v)) = \ell_1(u - v)$ for all $u, v \in \mathbb{C}^{1\times(n-1)}$. Thus, the last statement of Assertion 2 follows.

**Assertion 3** The map $\Phi$ has the asserted form.

By Assertion 2, we may compose the map $\Phi$ with maps of the form (1) – (3) described in the theorem and assume that

(I) $\Phi(\mu D) = \mu D$ whenever $\mu \in \mathbb{C} \setminus \Gamma$, and (II) $\Phi(X) = X$ whenever $X \in \cup_{k=1}^n W_k$.

Under conditions (I) and (II), we will show that $\Phi(A) = A$ for each $A \in M_n$. First, we consider the special case when $A = \mu D + B$ with $\mu \in \mathbb{C} \setminus \Gamma$ and $B \in W_k$ where $G(A)$ consists of $n$ disjoint disks with at most one of them having positive radius. Suppose $\Phi(A) = C = (c_{ij})$. Let $\nu > 0$ be such that $\nu, \nu\mu \notin \Gamma$ and $G(\nu\mu D - A) = G(\mu(\nu-1)D - B)$ consists of $n$ connected components. Then $\Phi(\nu\mu D) = \nu\mu D$ by (I), and

$$G(\nu\mu D - C) = G(\Phi(\nu\mu D) - \Phi(A)) = G(\nu\mu D - A) = G(\mu(\nu-1)D - B)$$

consists of $n$ circular disks with only one of them having positive radius. Considering the centers of the $n$ disks, we conclude that the diagonal entries of $\nu\mu D - C$ equal $\mu(\nu-1)w^j$ for $j = 1, \ldots, n$. Similarly, since $G(C) = G(A)$ consists of $n$ disks, we see that $C$ has diagonal entries lying in $\{\mu w^j : j = 1, \ldots, n\}$. Suppose the $c_{jj} = \mu w^k$ such that $k \neq j$. Then the vector of diagonal entries of $\Phi(\nu\mu D) - \Phi(A)$ equals $\nu\mu(w, w^2, \ldots, w^n) - \mu(w, w^2, \ldots, w^n)R$ for some $R \in \mathbf{P}_n$, and has entries which constitute the set $\{\mu(\nu-1)w^j : 1 \leq j \leq n\}$. It follows that the vector $\nu(w, w^2, \ldots, w^n) - (w, w^2, \ldots, w^n)R$ has entries in $\{(\nu-1)w^j : 1 \leq j \leq n\}$. By Lemma 2.2, we see that $c_{jj} = \mu w^j$. Since $G(C) = G(A)$, we see that only the $k$th row of $C$ can have off-diagonal entries. Since $G(\Phi(A) - \Phi(B)) = G(A - B) = \{\mu w^j : 1 \leq j \leq n\}$, we see that $\Phi(A) - \Phi(B)$ is a diagonal matrix, which is $\mu D$. It follows that $\Phi(A) = \mu D + \Phi(B) = \mu D + B$.

Next, we consider a general matrix $A = (a_{ij}) \in M_n$. Suppose $\Phi(A) = C = (c_{ij})$. Let $\mu \in (0, \infty) \setminus \Gamma$ be such that $G(\mu D - A)$ consists of $n$ disjoint disks. Moreover, we can choose $\mu > 0$ such that $\mu|1 - w| > \sum_{j=1}^n (|a_{jj}| + |c_{jj}|)$ so that $\mu w^j - c_{jj} \neq \mu w^k - a_{kk}$ for any $j \neq k$. Since $\Phi(\mu D) = \mu D$, $G(\mu D - C) = G(\Phi(\mu D) - \Phi(A)) = G(\mu D - A)$. By our choice of $\mu$, we see that $\mu w^j - c_{jj} = \mu w^j - a_{jj}$ so that $c_{jj} = a_{jj}$ for $j = 1, \ldots, n$. Furthermore, since

$$G\left(\mu D + \sum_{j\neq k} a_{kj}E_{kj} - C\right) = G\left(\Phi\left(\mu D + \sum_{j\neq k} a_{kj}E_{kj}\right) - \Phi(A)\right) = G\left(\mu D + \sum_{j\neq k} a_{kj}E_{kj} - A\right),$$

we see that

$$G_k\left(\mu D + \sum_{j\neq k} a_{kj}E_{kj} - C\right) = \left\{\mu w^k - c_{kk}\right\}.$$

Thus, $a_{kj} = c_{kj}$ for all $j \neq k$. Since $k$ is arbitrary, we see that $C = A$ as asserted. □

**Corollary 2.4** *Let $\Phi : M_n \to M_n$. The following conditions are equivalent.*

(a) *The map $\Phi$ is real linear and satisfies*

$$G(\Phi(A)) = G(A) \qquad \text{for all } A \in M_n.$$

(b) *The map $\Phi$ is additive and satisfies*

$$G(\Phi(A)) = G(A) \qquad \text{for all } A \in M_n.$$

(c) *The map $\Phi$ has the form in Theorem 2.1 with $S = 0$.*

*Proof.* The implications $(c) \Rightarrow (a) \Rightarrow (b)$ are clear. We focus on $(b) \Rightarrow (c)$. Clearly, $X \in M_n$ satisfies $G(X) = \{0\}$ if and only if $X = 0$ and thus $\Phi(X) = 0$ if and only if $X = 0$ since $G(\Phi(A)) = G(A)$ for all $A \in M_n$. For any $B \in M_n$, since $G(\Phi(B)+\Phi(-B)) = G(\Phi(B-B)) = \{0\}$, it follows that $\Phi(-B) = -\Phi(B)$. Therefore, $G(\Phi(A-B)) = G(\Phi(A)+\Phi(-B)) = G(\Phi(A)-\Phi(B))$ for all $A, B \in M_n$. Thus $\Phi$ has the asserted form in Theorem 2.1. Since $\Phi(0) = \{0\}$, we see that $S = 0$ in case condition (2) in Theorem 2.1 holds. The conclusion follows. ☐

**Corollary 2.5** *A (complex) linear map $\Phi : M_n \to M_n$ satisfies*

$$G(\Phi(A)) = G(A) \qquad \text{for all } A \in M_n$$

*if and only if it is a compositions of maps of the forms in (1) or (2) described in Theorem 2.1 with $S = 0$.*

It is clear that one can have the column vector version of Gershgorin regions. This is the same as considering $G(A^t)$, and we can prove analogous results on preservers.

# 3 Brauer Regions

In this section, we consider preservers of the Brauer region of a matrix $A = (a_{ij}) \in M_n$ defined as

$$C(A) = \cup_{1\le i<j\le n} C_{ij}(A),$$

with

$$C_{ij}(A) = \{\mu \in \mathbb{C} : |(\mu - a_{ii})(\mu - a_{jj})| \le R_i R_j\}$$

is a *Cassini oval*, with $R_i = \sum_{j\ne i} |a_{ij}|$ as defined in Section 2. The values $a_{ii}$ and $a_{jj}$ are the *foci* of the oval. One may see the discussion of the Cassini oval in standard references; for example, [2, Chapter 6] and Wikipedia. The following facts can be easily verified and will be used in our discussion.

**Lemma 3.1** *Let $A = (a_{ij}) \in M_n$.*

(a) *The set $C(A)$ consists of a collection of isolated points if and only if $A$ has at most one row with nonzero off diagonal entries. In such case, $C(A)$ coincides with the spectrum of $A$.*

(b) *If $|a_{ii} - a_{jj}|^2 > 4R_iR_j$, then $C_{ij}(A)$ consists of two closed convex regions, each of which contains a focus. Consequently, if $|a_{ii} - a_{jj}|^2 > 4\max\{|R_k|^2 : 1 \le k \le n\}$ whenever $i \ne j$, then $C(A)$ consists of $n$ disjoint connected regions, each of which contains a diagonal entry of $A$.*

(c) *Suppose $a_{kk}$ is an isolated point of $C(A)$. Then either $C(A)$ is a collection of isolated points so that (a) holds or the $k$th row of $A$ has only zero off diagonal entries.*

**Theorem 3.2** *A map $\Phi : M_n \to M_n$ satisfies*

$$C(\Phi(A) - \Phi(B)) = C(A - B) \qquad \text{for all } A, B \in M_n$$

*if and only if one of the following holds.*

(a) $n = 2$ *and* $\Phi$ *has the form*

$$A \mapsto P \begin{pmatrix} a_{11} & u\tau_1(a_{12}) \\ v\tau_2(a_{21}) & a_{22} \end{pmatrix} P^t + S \qquad or \qquad A \mapsto P \begin{pmatrix} a_{11} & u\tau_1(a_{21}) \\ v\tau_2(a_{12}) & a_{22} \end{pmatrix} P^t + S,$$

*where* $S \in M_2$, $P \in \mathbf{P}_2$, $u, v \in \mathbb{C}$ *satisfy* $|uv| = 1$, *and* $\tau_j$ *is the identity map or the conjugation map for* $j = 1, 2$.

(b) $\Phi$ *has the form described in Theorem 2.1.*

*Proof.* The sufficiency is clear. We consider the necessity. We may replace $\Phi$ by the map $A \mapsto \Phi(A) - \Phi(0)$ and assume that $C(A) = C(\Phi(A))$ for any $A \in M_n$.

**Case 1** Suppose $n = 2$. For any $\mu \in \mathbb{C}$, we have $C(\Phi(\mu E_{12})) = C(\mu E_{12}) = \{0\}$. It follows that $\Phi(\mu E_{12}) = f_1(\mu)E_{12} + f_2(\mu)E_{21}$ with $f_1(\mu)f_2(\mu) = 0$. Similarly, for any $\nu \in \mathbb{C}$, $C(\Phi(\nu E_{21})) = C(\nu E_{21}) = \{0\}$, we see that $\Phi(\nu E_{21}) = g_1(\nu)E_{12} + g_2(\nu)E_{21}$ with $g_1(\nu)g_2(\nu) = 0$. Since $C(\Phi(\mu E_{12}) - \Phi(\nu E_{21})) = C(\mu E_{12} - \nu E_{21})$ is a circular disk centered at the origin with radius $\sqrt{|\mu\nu|}$, $\Phi(\mu E_{12}) \neq 0$ and $\Phi(\nu E_{21}) \neq 0$ whenever $\mu\nu \neq 0$. Now for any $\mu_1, \mu_2 \in \mathbb{C}$,

$$C(\Phi(\mu_1 E_{12}) - \Phi(\mu_2 E_{12})) = C(\mu_1 E_{12} - \mu_2 E_{12}) = \{0\}.$$

We see that

(1) $\Phi(\mu E_{12}) = f(\mu)E_{12}$ for all $\mu \in \mathbb{C}$, or (2) $\Phi(\mu E_{12}) = f(\mu)E_{21}$ for all $\mu \in \mathbb{C}$.

We may assume that (1) holds. Otherwise, replace $\Phi$ by the map $A \mapsto \Phi(A)^t$. It will then follow that $\Phi(\nu E_{21}) = g(\nu)E_{21}$ for all $\nu \in \mathbb{C}$.

Note that $C(\Phi(\mu E_{12}) - \Phi(\nu E_{21})) = C(\mu E_{12} - \nu E_{21}) = \{z \in \mathbb{C} : |z^2| \leq |\mu\nu|\}$. Thus, we see that $|f(\mu)g(\nu)| = |\mu\nu|$. In particular, $f(\mu) = 0$ if and only if $\mu = 0$, and $g(\nu) = 0$ if and only if $\nu = 0$.

Since $C(\Phi(E_{11})) = C(E_{11}) = \{1, 0\}$, we see that $\Phi(E_{11})$ has diagonal entries $1, 0$ and at most one nonzero off diagonal entry. Since $C(\Phi(E_{11}) - \Phi(X)) = \Phi(E_{11} - X) = \{1, 0\}$ for $X \in \{E_{12}, E_{21}\}$, we see that $\Phi(E_{11}) = E_{11}$ or $\Phi(E_{11}) = E_{22}$. We may assume the former case holds. Otherwise, we may replace $\Phi$ by the map $A \mapsto P\Phi(A^t)P^t$ with $P = E_{12} + E_{21}$. Then we have $\Phi(E_{11}) = E_{11}$, $\Phi(\mu E_{12}) = f(\mu)E_{12}$ and $\Phi(\nu E_{21}) = g(\nu)E_{21}$. Now one may use the facts that $C(\Phi(\mu E_{11}) - \Phi(E_{11})) = \{\mu - 1, 0\}$, $C(\Phi(\mu E_{11}) - \Phi(E_{12})) = \{\mu, 0\}$, and $C(\Phi(\mu E_{11}) - \Phi(E_{21})) = \{\mu, 0\}$ to conclude that $\Phi(\mu E_{11}) = \mu E_{11}$. Similarly, we can argue that $\Phi(\mu E_{22}) = \mu E_{22}$.

Up to this point, we may assume that for $j \in \{1, 2\}$ and $\mu \in \mathbb{C}$, $\Phi(\mu E_{jj}) = \mu E_{jj}$ and there are two functions $f, g \in \mathbb{C} \mapsto \mathbb{C}$ such that $\Phi(\mu E_{12}) = f(\mu)E_{12}$ and $\Phi(\nu E_{21}) = g(\nu)E_{21}$, with $\nu \in \mathbb{C}$. Now suppose $\Phi(A) = (b_{ij})$ for $A = (a_{ij})$. Since $C(\Phi(A) - \Phi(a_{ij}E_{ij})) = C(A - a_{ij}E_{ij}) = \{a_{11}, a_{22}\}$ for $(i, j) \in \{(1, 2), (2, 1)\}$, we see that $b_{12} = f(a_{12})$ and $b_{21} = g(a_{21})$. For $i \in \{1, 2\}$, since $C(\Phi(A) - \Phi(\mu E_{ii})) = C(A - \mu E_{ii})$ for all $\mu \in \mathbb{C}$, we see that $b_{ii} = a_{ii}$. Thus,

$$\Phi(A) = \begin{pmatrix} a_{11} & f(a_{12}) \\ g(a_{21}) & a_{22} \end{pmatrix}$$

such that $|f(a_{12})g(a_{21})| = |a_{12}a_{21}|$.

Now, for any $\mu_1, \mu_2 \in \mathbb{C}$, since $C\left(\Phi(E_{21} + \mu_1 E_{12}) - \Phi(\mu_2 E_{12})\right) = C(E_{21} + \mu_1 E_{12} - \mu_2 E_{12})$, we see that $|uf(\mu_1) - uf(\mu_2)| = |\mu_1 - \mu_2|$ if $u = g(1)$. By Lemma 2.3, we see that $uf = \tau_1$ is the identity map or the conjugation map.

Similarly, using the fact that $C(\Phi(E_{12} + \nu_1 E_{21}) - \Phi(\nu_2 E_{21})) = C(E_{12} + \nu_1 E_{21} - \nu_2 E_{21})$, we can show that $vg = \tau_2$ is the identity map or the conjugation map, where $v = f(1)$.

Combining the above arguments, we get condition (a).

**Case 2** Suppose $n > 2$. We prove condition (b) holds by establishing several assertions. We will assume $\Gamma$ is defined as in Lemma 2.2.

**Assertion 1** Let $D = \text{diag}\,(w, w^2, \ldots, w^{n-1}, 1)$ for $w = e^{i2\pi/n}$. Then there is a permutation matrix $P$ such that $P\Phi(\mu D)P^t$ has $(j,j)$ entry equal to $\mu w^j$ for $j = 1, \ldots, n$, for any $\mu \in \mathbb{C} \setminus \Gamma$.

The verification to this assertion is identical to that of Assertion 1 in the proof of Theorem 2.1.

*In the following, we will always assume that D is defined as in Assertion 1, and the matrix P in the conclusion is the identity matrix.*

**Assertion 2** For $k \in \{1, \ldots, n\}$, if

$$W_k = \left\{ \sum_{j \neq k} a_j E_{kj} : a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n \in \mathbb{C} \right\},$$

then $\Phi(W_k) \subseteq W_k$.

For any $\mu > 0$ and $k \in \{2, \ldots, n\}$, since

$$C(\Phi(\mu D + E_{k1})) = C(\mu D + E_{k1}) = \{\mu w^j : 1 \leq j \leq n\},$$

we see that $\Phi(\mu D + E_{k1})$ has diagonal entries $\mu w, \ldots, \mu w^{n-1}, \mu$, and there is at most one row of $\Phi(\mu D + E_{k1})$ with nonzero off diagonal entries by Lemma 3.1(a). Note that

$$C(\Phi(\mu D + E_{k1}) - \Phi(D)) = C(\mu D + E_{k1} - D) = \{(\mu - 1)w^j : 1 \leq j \leq n\}.$$

Suppose $A = \sum_{j=2}^n a_j E_{1j} \neq 0$. Since $C(\Phi(A)) = C(A) = \{0\}$, we see that all diagonal entries of $\Phi(A)$ equal zero and there is at most one row of $\Phi(A)$ with nonzero off diagonal entries. Let $\mu > 0$ be sufficiently large so that the $(j,j)$ entry of $\Phi(\mu D + E_{k1})$ is $\mu w^j$ for sufficiently large $\mu > 0$. Then

$$C(\Phi(\mu D + E_{k1}) - \Phi(A)) = C(\mu D + E_{k1} - A)$$

is the union of $\{\mu w^j : 1 < j \leq n, j \neq k\}$ and the Cassini oval with two connected regions containing the foci $\mu w$ and $\mu w^k$, which is two disconnected regions. Thus, $\Phi(\mu D + E_{k1}) - \Phi(A)$ has nonzero off diagonal entries in row 1 and row $k$. Since this is true for all $k \in \{2, \ldots, n\}$, we see that the only nonzero row of $\Phi(A)$ is row 1. Furthermore, we see that for sufficiently large $\mu$, $\Phi(\mu D + E_{k1})$ has nonzero off diagonal entries at row $k$ for $k \in \{2, \ldots, n\}$.

Now, for $k \in \{2, \ldots, n\}$, let $A = \sum_{j \neq k} a_j E_{kj} \neq 0$. Since $C(\Phi(A)) = C(A) = \{0\}$, we see that $\Phi(A)$ has zero diagonal entries and has at most one nonzero row. Since

$$C(\Phi(\mu D + E_{m1}) - \Phi(A)) = C(\mu D + E_{m1} - A)$$

for $m \in \{1, \ldots, n\} \setminus \{k\}$ and sufficiently large $\mu$, we see that the $k$th row of $\Phi(A)$ is nonzero.

Combining the above arguments, we see that Assertion 2 holds.

**Assertion 3** Suppose $A = (a_{ij})$ and $\Phi(A) = (b_{ij})$. Then for $k \in \{1, \ldots, n\}$, we have

$$\text{(i) } b_{kk} = a_{kk} \quad \text{ and } \quad \text{(ii) } \sum_{j \neq k} b_{kj} E_{kj} = \Phi\left( \sum_{j \neq k} a_{kj} E_{kj} \right).$$

**Step 1** We prove (i). Since $C(\mu D - A) = C(\Phi(\mu D) - \Phi(A)) = C(\mu D - \Phi(A))$ for all sufficiently large $\mu > 0$, we see that $\Phi(A)$ has $(j, j)$ entry equal to $a_{jj}$. See Assertion 3 in the proof of Theorem 2.1.

**Step 2** We prove (ii) for the special case when $A = (a_{ij})$ has two rows with nonzero diagonal entries such that $C(A)$ consists of $n - 2$ distinct points and a Cassini oval consisting of two disconnected components. Without loss of generality, we assume that the first two rows of $A$ have nonzero off diagonal entries. Since $\Phi(A) = (b_{ij})$ satisfies $b_{jj} = a_{jj}$ for $j \in \{1, \ldots, n\}$, $C(A) = C(\Phi(A))$ and $C(\Phi(\mu D + E_{m1}) - \Phi(A)) = C(\mu D + E_{m1} - A)$ for $m \in \{1, \ldots, n\}$ and sufficiently large $\mu$, we see that only the first two rows of $\Phi(A)$ have nonzero off diagonal entries. Moreover, for $k = 1, 2$, $C(\Phi(A) - \Phi(\sum_{j \neq k} a_{kj} E_{kj})) = C(A - \sum_{j \neq k} a_{kj} E_{kj})$. Thus by Assertion 2, the $k$th row of $\Phi(A) - \Phi(\sum_{j \neq k} a_{kj} E_{kj})$ equals zero, i.e., $\sum_{j \neq k} b_{kj} E_{kj} = \Phi(\sum_{j \neq k} a_{kj} E_{kj})$.

**Step 3** We prove (ii) for the case when $A = (a_{ij}) \in M_n$ has a single row with nonzero off diagonal entries. Without loss of generality, we may assume the first row of $A$ has nonzero off diagonal entries. Let $X \in M_n$ have two nonzero rows such that $X = \mu D + \sum_{j \neq 1} a_{1j} E_{1j} + E_{kj}$, $\mu \in \mathbb{C} \backslash \Gamma$, $k \neq 1$, and $\Phi(X) = (y_{ij})$. By Steps 1 and 2, we see that $y_{ii} = x_{ii}$ for all $i \in \{1, \ldots, n\}$, only rows 1 and $k$ of $\Phi(X)$ have nonzero off diagonal entries, and that $\sum_{j \neq 1} y_{1j} E_{1j} = \Phi(\sum_{j \neq 1} a_{1j} E_{1j})$. Because

$$C\left(\Phi(A) - \Phi(X)\right) = C\left( (b_{ij}) - \left( \mu D + \sum_{j \neq 1} y_{1j} E_{1j} + \sum_{j \neq k} y_{kj} E_{kj} \right) \right) = C\left(A - X\right)$$

consists of $n$ distinct points for sufficiently large $\mu > 0$, we can conclude that $\Phi(A) - \Phi(X)$ has only 1 row with nonzero off diagonal entries. Since $C(\Phi(A)) = C(A)$ implies that $\Phi(A)$ has only 1 row with nonzero off diagonal entries, it follows that this row must be row 1 or row $k$. As this result holds for any $k \in \{2, \ldots, n\}$, we see that only row 1 of $\Phi(A)$ has nonzero off diagonal entries, and hence only row $k$ of $\Phi(A) - \Phi(X)$ has nonzero off diagonal entries. Therefore the first row of $\Phi(A) - \Phi(X)$ equals zero, i.e.,

$$\sum_{j \neq 1} b_{1j} E_{1j} = \sum_{j \neq 1} y_{1j} E_{1j} = \Phi\left( \sum_{j \neq 1} a_{1j} E_{1j} \right)$$

.

**Step 4** We prove (ii) for a diagonal matrix $A = (a_{ij})$. If $\Phi(A) = B = (b_{ij})$, then we can show $a_{jj} = b_{jj}$ for all $j \in \{1, \ldots, n\}$ as in the previous two cases. Since $C(B) = C(A)$ does not contain a disk with positive radius, we see that $B$ has at most one row with nonzero off diagonal entries. If $B$ has such a row, then we can find $X \in W_k$ so that $B - \Phi(X)$ has two rows with nonzero off diagonal entries. But then $C(A - X) = C(B - \Phi(X))$ contains a disk with positive radius, which is a contradiction. Hence, we see that $\Phi(A) = A$ if $A$ is a diagonal matrix.

**Step 5** We prove (ii) for a matrix $A = (a_{ij}) \in M_n$ with at least three rows having nonzero off diagonal entries. Let $\Phi(A) = B = (b_{ij})$ and $k \in \{1, \ldots, n\}$. We are going to show that $\sum_{j\neq k} b_{kj} = \Phi(\sum_{j\neq k} a_{kj}E_{kj})$. To this end, let $X = (x_{ij}) \in M_n$ be such that $X$ has exactly two nonzero rows indexed by $k$ and $k'$, where the off diagonal entries in these two rows are the same as those of $A = (a_{ij})$. Moreover, the diagonal entries $x_{11}, \ldots, x_{nn}$ are chosen so that

(1) $C(X)$ consists of $n - 2$ distinct points and a Cassini oval consisting of two connected components,

(2) $C(\Phi(X) - \Phi(A)) = C(X - A)$ consists of $n-2$ distinct points and a Cassini oval consisting of two connected components which include $\{x_{kk} - a_{kk}\}$ and $\{x_{k'k'} - a_{k'k'}\}$.

Since we have proved the result for the special cases covering the case for $X$, if $\Phi(X) = Y = (y_{ij})$, then

$$\sum_{j\neq k} y_{kj}E_{kj} = \Phi\left(\sum_{j\neq k} x_{kj}E_{kj}\right) = \Phi\left(\sum_{j\neq k} a_{kj}E_{kj}\right).$$

By (2), we see that

$$\sum_{j\neq k} y_{kj}E_{kj} = \sum_{j\neq k} b_{kj}E_{kj}.$$

It follows that

$$\sum_{j\neq k} b_{kj}E_{kj} = \Phi\left(\sum_{k\neq j} a_{kj}E_{kj}\right).$$

Since this is true for any $k \in \{1, \ldots, n\}$, the conclusion of the Assertion 3 holds.

**Assertion 4** Condition (b) in the theorem holds.

By Assertion 2, we may define $\Phi_k : \mathbb{C}^{1\times(n-1)} \to \mathbb{C}^{1\times(n-1)}$ by

$$\Phi_k(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n) = (b_1, \ldots, b_{k-1}, b_{k+1}, \ldots, b_n)$$

if $\Phi(\sum_{j\neq k} a_jE_{kj}) = \sum_{j\neq k} b_jE_{kj}$. We claim that $\ell_1(\Phi_1(a) - \Phi_1(a')) = \ell_1(a - a')$ for any $a = (a_2, \ldots, a_n), a' = (a'_2, \ldots, a'_n) \in \mathbb{C}^{1\times(n-1)}$. To see this, consider $A_k = \sum_{j=2}^n a_jE_{1j} + E_{k1}$ and $A' = \sum_{j=2}^n a'_jE_{1j}$. By Assertion 3 and the fact that $C(\Phi(A_k)) = C(A_k)$, $R_1(\Phi(A_k))R_k(\Phi(A_k)) =$

$R_1(A_k)R_k(A_k) = \ell_1(a)$. Also, since $C(\Phi(A_k) - \Phi(A')) = C(A_k - A')$, by Assertion 3 we have

$$\begin{aligned}
\ell_1(a - a') &= R_1(A_k - A')R_k(A_k) \\
&= R_1(\Phi(A_k) - \Phi(A'))R_k(\Phi(A_k)) \\
&= \ell_1(\Phi_1(a) - \Phi_1(a'))\ell_1(\Phi_k(1, 0, \ldots, 0))
\end{aligned}$$

By Assertion 3, $R_k(\Phi(A_k)) = R_k(\Phi(E_{k1}))$. Thus, for $k \in \{2, \ldots, n\}$,

$$\ell_1(\Phi_k(1, 0, \ldots, 0)) = R_k(\Phi(E_{k1})) = \ell_1(a - a')/\ell_1(\Phi_1(a) - \Phi_1(a')) = \nu > 0.$$

Since $C(\Phi(E_{21}) - \Phi(E_{k1})) = C(E_{21} - E_{k1}) = \{z \in \mathbb{C} : |z| \leq 1\}$, we have $1 = R_2(\Phi(E_{21}))R_k(\Phi(E_{k1})$ for $k \in \{3, \ldots, n\}$. (Here we use the fact that $n \geq 3$.) Thus, $\nu = 1$, and

$$\ell_1(a - a') = \ell_1(\Phi_1(a) - \Phi_1(a')).$$

So, $\Phi_1$ satisfies the conclusion of Lemma 2.3. Similarly, we can show that $\Phi_k$ satisfies the conclusion of Lemma 2.3 for $k \in \{2, \ldots, n\}$. Consequently, $\Phi$ has the asserted form. □

As with the Gershgorin discs, it is clear that one can have the column version of Cassini ovals by considering $C(A^t)$. However, there is no extension for

$$C_{ijk}(A) = \{\mu \in \mathbb{C} : |(\mu - a_{ii})(\mu - a_{jj})(\mu - a_{kk})| \leq R_i R_j R_k\}$$

or higher dimensions.

Note that we cannot deduce the structure of additive preservers of $C(A)$ using Theorem 3.2 as in Corollary 2.4 because $C(A) = \{0\}$ does not imply that $A = 0$. Nevertheless, we can apply a similar proof to characterize additive preservers of $C(A)$ and then deduce the results on (real or complex) linear preservers of $C(A)$.

## 4 Ostrowski Regions

In this section, we consider the Ostrowski region of $A \in M_n$ defined by

$$O_\xi(A) = \cup_{k=1}^n O_{\xi,k}(A),$$

for a given $\xi \in (0, 1)$, where

$$O_{\xi,k}(A) = \{\mu \in \mathbb{C} : |\mu - a_{kk}| \leq R_k(A)^\xi R_k(A^t)^{1-\xi}\}, \quad k \in \{1, \ldots, n\}.$$

One may see the discussion of the Ostrowski region in [2, Chapter 6].

**Theorem 4.1** *Let $\xi \in (0, 1)$. A map $\Phi : M_n \to M_n$ satisfies*

$$O_\xi(\Phi(A) - \Phi(B)) = O_\xi(A - B) \qquad \textit{for all } A, B \in M_n$$

*if and only if there exist $P \in \mathbf{P}_n$ and maps $\psi_{ij} : \mathbb{C} \to \mathbb{C}$, where $\psi_{jj}$ is the identity map and for $i \neq j$, $\psi_{ij}$ has the form $z \mapsto \nu_{ij} z$ or $z \mapsto \nu_{ij} \bar{z}$ for a norm one complex number $\nu_{ij}$, such that one of the following holds.*

(a) $(\xi, n) = (1/2, 2)$ *and the map $\Phi$ has the form in Theorem 3.2 (a).*

(b) *There is $S \in M_n$ such that the map $\Phi$ has the form $A = (a_{ij}) \mapsto P(\psi_{ij}(a_{ij}))P^t + S$.*

(c) $\xi = 1/2$ *and there is $S \in M_n$ such that $\Phi$ has the form $A = (a_{ij}) \mapsto P(\psi_{ij}(a_{ij}))^t P^t + S$.*

*Proof.* The sufficiency is clear. We consider the proof of the necessity. The proof for the case when $n = 2$ is similar to that of Theorem 3.2. So, we assume that $n > 2$ and $O_\xi(\Phi(A) - \Phi(B)) = O_\xi(A - B)$ for all $A, B \in M_n$. Replacing $\Phi$ by the map $X \mapsto \Phi(X) - \Phi(0)$, we may assume that $O_\xi(\Phi(A)) = O_\xi(A)$ for all $A \in M_n$. Let $D = \operatorname{diag}(w, w^2, \ldots, w^{n-1}, 1)$ with $w = e^{i2\pi/n}$. Furthermore, let $\Gamma$ be defined as in Lemma 2.2.

**Assertion 1** There is a permutation matrix $P$ such that one of the following holds for any $\mu \in \mathbb{C} \backslash \Gamma$.

(i) $P\Phi(E_{ij})P^t = u_{ij}E_{ij}$ and $P\Phi(\mu D + E_{ij})P^t = \mu D + v_{ij}E_{ij}$ with $u_{ij}, v_{ij} \in \mathbb{C}$ satisfying $|u_{ij}| = |v_{ij}| = 1$ whenever $1 \leq i, j \leq n$ and $i \neq j$.

(ii) $P\Phi(E_{ij})P^t = u_{ij}E_{ji}$ and $P\Phi(\mu D + E_{ij})P^t = \mu D + v_{ij}E_{ji}$ with $u_{ij}, v_{ij} \in \mathbb{C}$ satisfying $|u_{ij}| = |v_{ij}| = 1$ whenever $1 \leq i, j \leq n$ and $i \neq j$.

To prove the assertion, let $\mu \in \mathbb{C} \backslash \Gamma$. Suppose $\nu \in \mathbb{C}$ is such that $\nu, \nu\mu \notin \Gamma$. Since $O_\xi(\Phi(\nu\mu D)) = O_\xi(\nu\mu D) = \{\nu\mu w^j : 1 \leq j \leq n\}$, there is a permutation $P$ (depending on $\mu$ and $\nu$) such that $P\Phi(\nu\mu D)P^t = \nu\mu D + F$, where $F$ has zero diagonal entries and $R_j(F)R_j(F^t) = 0$ for all $j = 1, \ldots, n$. We will show that condition (i) or (ii) holds. Once this is done, we conclude that $P$ is independent of the choice of $\mu$ and $\nu$ by examining $\Phi(E_{ij})$ for $1 \leq i, j \leq n$.

For simplicity, we assume that $P = I_n$. Otherwise, replace $\Phi$ by the map $X \mapsto P^t\Phi(X)P$. For pairs $(i, j)$ with $i \neq j$ consider $V_{ij} = \Phi(\mu D + E_{ij})$. Since $O_\xi(V_{ij}) = O_\xi(\mu D + E_{ij})$, we see that $V_{ij}$ has diagonal entries $\mu w, \ldots, \mu w^{n-1}, \mu$. Since $\Phi(\nu\mu D)$ and $\nu\mu D$ have the same diagonal entries, and

$$O_\xi(\Phi(\nu\mu D) - V_{ij}) = O_\xi((\nu - 1)\mu D - E_{ij}) = \{(\nu - 1)\mu w^j : 1 \leq j \leq n\},$$

the vector of the diagonal of the matrix $(\nu\mu D - V_{ij})/\mu$ equals $\nu(w, w^2, \ldots, w^n) - (w, w^2, \ldots, w^n)R$ with $R \in \mathbf{P}_n$, and has entries in $\{(\nu - 1)w^j : 1 \leq j \leq n\}$. Since $\nu \in \mathbb{C} \setminus \Gamma$, it follows from Lemma 2.2 that $V_{ij} = \mu D + F_{ij}$ such that $F_{ij}$ has zero diagonal and $R_k(F_{ij})R_k(F_{ij}^t) = 0$ for $k = 1, \ldots, n$.

For pairs $(i, j)$ with $i \neq j$ let $U_{ij} = \Phi(E_{ij})$. Since $O_\xi(U_{ij}) = O_\xi(E_{ij}) = \{0\}$ we see that for $k = 1, \ldots, n$, $R_k(U_{ij})R_k(U_{ij}^t) = 0$ and $U_{ij}$ has zero diagonal entries. Moreover, $O_\xi(V_{ij} - U_{ji}) = O_\xi(\mu D + E_{ij} - E_{ji})$ contains non-degenerate circular disks centered at $\mu w^i$ and $\mu w^j$. Considering the disk with center $\mu w^i$, we see that either

- $R_i(U_{ij}) \neq 0 = R_i(U_{ij}^t)$ and $R_i(V_{ji}^t) \neq 0 = R_i(V_{ji})$ or
- $R_i(U_{ij}) = 0 \neq R_i(U_{ij}^t)$ and $R_i(V_{ji}^t) = 0 \neq R_i(V_{ji})$.

Suppose $R_1(U_{12}) \neq 0 = R_1(U_{12}^t)$. If $U_{12}$ has a nonzero $(k, j)$ entry with $j > 2$, then the $j$th row of $U_{12}$ is zero as $R_j(U_{12})R_j(U_{12}^t) = 0$. Since $O_\xi(V_{j1} - U_{1j}) = O_\xi(\mu D + E_{j1} - E_{1j})$ contains a unit disk centered at $\mu w^j$, either $R_j(V_{j1}^t)R_j(U_{1j}) \neq 0 = R_j(V_{j1}) = R_j(U_{1j}^t)$ or $R_j(V_{j1})R_j(U_{1j}^t) \neq 0 = R_j(V_{j1}^t) = R_j(U_{1j})$. In the former case, $O_\xi(E_{1j} - E_{12}) = O_\xi(U_{1j} - U_{12})$ contains a non-degenerate circular disk; in the latter case, $O_\xi(\mu D + E_{j1} - E_{12}) = O_\xi(V_{j1} - U_{12})$ contains a non-degenerate circular disk centered at $\mu w^j$. In both cases, we have a contradiction.

By the above paragraph, only the second column of $U_{12}$ can be nonzero. Now, suppose $U_{12}$ has a nonzero $(k, 2)$ entry with $k > 2$. Then $R_k(U_{12}) \neq 0 = R_k(U_{12}^t)$. Since $O_\xi(V_{k1} - U_{1k}) = O_\xi(\mu D + E_{k1} - E_{1k})$ contains a unit disk centered at $\mu w^k$, either $R_k(V_{k1}^t)R_k(U_{1k}) \neq 0 = R_k(V_{k1}) = R_k(U_{1k}^t)$ or $R_k(V_{k1})R_k(U_{1k}^t) \neq 0 = R_k(V_{k1}^t) = R_k(U_{1k})$. In the former case, $O_\xi(\mu D + E_{k1} - E_{12}) = O_\xi(V_{k1} - U_{12})$ contains a non-degenerate circular disk centered at $\mu w^k$; in the latter case, $O_\xi(E_{1k} - E_{12}) = O_\xi(U_{1k} - U_{12})$ contains a non-degenerate circular disk. In both cases, we have a contradiction. Thus, we conclude that $U_{12} = u_{12}E_{12}$ for some nonzero $u_{12}$.

If $R_1(U_{12}) = 0 \neq R_1(U_{12}^t)$, we can use a similar argument to show that $U_{12} = u_{12}E_{21}$ for some nonzero $u_{12}$.

Applying the above argument to $U_{ij}$, we see that $U_{ij} = u_{ij}E_{ij}$ or $u_{ij}E_{ji}$ for pairs $(i, j)$ with $i \neq j$.

Interchanging the roles of $U_{ij}$ and $V_{ij}$ in the above proof, we see that $V_{ij} = \mu D + v_{ij}E_{ij}$ or $\mu D + v_{ij}E_{ji}$ for pairs $(i, j)$ with $i \neq j$.

Now, suppose $U_{12} = u_{12}E_{12}$. Since $O_\xi(V_{j1} - U_{12}) = O_\xi(\mu D + E_{j1} - E_{12})$, we see that $V_{j1} = \mu D + v_{j1}E_{j1}$ with $|u_{12}|^\xi |v_{j1}|^{(1-\xi)} = 1$ for all $j = 2, \ldots, n$. Next, by the fact that $O_\xi(V_{j1} - U_{kj}) = O_\xi(\mu D + E_{j1} - E_{kj})$, we see that $U_{kj} = u_{kj}E_{jk}$ such that $|u_{kj}|^{(1-\xi)}|v_{j1}|^\xi = 1$ for all $(j, k)$ with $j \neq k$, $j > 1$ and $k \in \{1, \ldots, n\}$. In particular, there is $u_j$ such that $|u_{kj}| = |u_j|$ for all $k \neq j$. Since $O_\xi(U_{1j} - U_{j1}) = O_\xi(E_{1j} - E_{j1})$, we see that $U_{j1} = u_{j1}E_{j1}$ with $\max\{|u_{1j}|^\xi |u_{j1}|^{(1-\xi)}, |u_{1j}|^{(1-\xi)}|u_{j1}|^\xi\} = 1$ for $j = 2, \ldots, n$; since $O_\xi(V_{ij} - U_{ji}) = O_\xi(\mu D + E_{ij} - E_{ji})$, we see that $|v_{ij}|^\xi |u_{ji}|^{(1-\xi)} = 1 = |v_{ij}|^{(1-\xi)}|u_{ji}|^\xi$ for all pairs $(i, j)$ with $i \neq j$. It follows that $|u_{ij}| = |v_{ij}| = 1$ for pairs $(i, j)$ with $i \neq j$. Thus, condition (i) holds.

Suppose $U_{12} = u_{12}E_{21}$. We can use a similar argument to show that condition (ii) holds.

**Assertion 2** There exist functions $\psi_{ij}$ as described in the theorem such that the following holds.

(I) If conclusion (i) of Assertion 1 holds, then $P\Phi(\nu E_{ij})P^t = \psi_{ij}(\nu)E_{ij}$ for all $i \neq j$ and $\nu \in \mathbb{C}$.

(II) If conclusion (ii) of Assertion 1 holds, then $\xi = 1/2$ and $P\Phi(\nu E_{ij})P^t = \psi_{ij}(\nu)E_{ji}$ for all $i \neq j$ and $\nu \in \mathbb{C}$.

To prove the assertion, let $P$ satisfy the conclusion of Assertion 1. For simplicity, assume that $P = I_n$.

(I) Suppose condition (i) in Assertion 1 holds. Consider $\Phi(\nu E_{ij})$. Since

$$O_\xi(\Phi(\nu E_{ij}) - \mu D - v_{rs}E_{rs}) = O_\xi(\Phi(\nu E_{ij}) - \Phi(\mu D + E_{rs})) = O_\xi(\nu E_{ij} - \mu D - E_{rs})$$

for any pairs $(r, s)$ with $r \neq s$, we see that $\Phi(\nu E_{ij}) = \gamma E_{ij}$. Let $\psi_{ij} : \mathbb{C} \to \mathbb{C}$ be such that $\Phi(\nu E_{ij}) = \psi_{ij}(\nu)E_{ij}$.

We **claim** that $|\psi_{ij}(\nu_1) - \psi_{ij}(\nu_2)| = |\nu_1 - \nu_2|$ for any $\nu_1, \nu_2 \in \mathbb{C}$. Note that

$$O_\xi(\Phi(\nu_1 E_{ij} + E_{ji}) - (\mu D + v_{rs}E_{rs})) = O_\xi(\Phi(\nu_1 E_{ij} + E_{ji}) - \Phi(\mu D + E_{rs})) = O_\xi(\nu_1 E_{ij} + E_{ji} - \mu D - E_{rs})$$

for any pairs $(r,s)$ with $r \neq s$. It follows that $\Phi(\nu_1 E_{ij} + E_{ji}) = \gamma E_{ij} + v_{ji} E_{ji}$. By the fact that $O_\xi(\Phi(\nu_1 E_{ij} + E_{ji}) - \Phi(\nu_1 E_{ij})) = \{0\}$, we see that $\gamma = \psi_{ij}(\nu_1)$. Now,

$$O_\xi(\psi_{ij}(\nu_1)E_{ij} + v_{ji}E_{ji} - \psi_{ij}(\nu_2)E_{ij}) = O_\xi(\Phi(\nu_1 E_{ij} + E_{ji}) - \Phi(\nu_2 E_{ij})) = O_\xi((\nu_1 - \nu_2)E_{ij} + E_{ji}).$$

We get the desired conclusion. By Lemma 2.3, $\psi_{ij}$ has the asserted form.

(II) Suppose condition (ii) in Assertion 1 holds. We can use a similar argument to that in the proof of (I) to conclude that $\Phi(\nu E_{ij}) = \psi_{ij}(\nu)E_{ji}$ for all $(i,j)$ with $i \neq j$. Now,

$$O_\xi(\psi_{12}(2)E_{21} - \psi_{31}(1)E_{13})) = O_\xi(\Phi(2E_{12}) - \Phi(E_{31})) = O_\xi(2E_{12} - E_{31}).$$

Thus, $|2|^{(1-\xi)} = |2|^\xi$ and hence $\xi = 1/2$.

**Assertion 3** The map has the asserted form.

To prove the assertion, we may assume that condition (I) in Assertion 2 holds. Otherwise, replace $\Phi$ by the map $A \mapsto \Phi(A^t)$. For simplicity, we assume that $P = I_n$ and $\psi_{ij}$ is the identity map for all pairs $(i,j)$. We will show that $\Phi(A) = A$ for all $A \in M_n$. Note that if $\mu \in \mathbb{C} \setminus \Gamma$ with sufficiently large magnitude, we will have

$$\begin{aligned} & O_\xi(\Phi(A) - \mu D - v_{rs}E_{rs}) = O_\xi(\Phi(A) - \Phi(\mu D + E_{rs})) \\ = \quad & O_\xi(A - \mu D - E_{rs}) = O_\xi(\nu E_{ij} + E_{ki} - \mu D - E_{rs}) \end{aligned} \tag{4.1}$$

We consider 5 special cases before the general case.

**Case 1** Suppose $A = \sum_{j=1}^n d_j E_{jj}$ is a diagonal matrix.

Assume $\Phi(A)$ has a nonzero entry at the $(p,q)$ position for some $p \neq q$. Since $O_\xi(\Phi(A)) = O_\xi(A) = \{d_1, \ldots, d_n\}$, we see that the $p$th column of $\Phi(A)$ is zero. But then we can choose a suitable $\mu \in \mathbb{C} \setminus \Gamma$ so that $O_\xi(\Phi(A) - \mu D - v_{qp}E_{qp})$ contains a non-degenerate disk centered at the $p$ diagonal entry of $\Phi(A) - \mu D - v_{qp}E_{qp}$, where as $O_\xi(A - \mu D - E_{qp}) = \{d_j - \mu w^j : 1 \leq j \leq n\}$, which is a contradiction. Thus, $\Phi(A)$ is a diagonal matrix. Since $O_\xi(\phi(A) - \mu D - v_{12}E_{12}) = O_\xi(A - \mu D - E_{12}) = \{d_j - \mu w^j : 1 \leq j \leq n\}$ for any $\mu \in \mathbb{C} \setminus \Gamma$, we see that $\Phi(A) = A$.

**Case 2** Suppose $A = \nu E_{ij} + E_{ki}$ with $\nu \neq 0$ and $i \notin \{j,k\}$.

For notational simplicity, we assume that $(i,j) = (1,2)$ so that $A = \nu E_{12} + E_{21}$ or $(i,j,k) = (1,2,3)$ so that $A = \nu E_{12} + E_{31}$. It is easy to adapt the arguments to the general case. Assume $\Phi(A)$ has a nonzero $(p,q)$ entry with $p \neq q$ and $(p,q) \in \{1,\ldots,n\} \times \{3,\ldots,n\}$. Taking $(r,s) = (q,p)$ in (4.1), we see that $O_\xi(\Phi(A) - \mu D - v_{qp}E_{qp}) = O_\xi(A - \mu D - E_{qp})$ has $\mu w^q$ as an isolated point. Since $\Phi(A) - \mu D - v_{qp}E_{qp}$ has nonzero $(p,q)$ position, the $q$th column of $\Phi(A)$ has only one nonzero entry at the $(q,p)$ position equal to $v_{qp}$. But then $O_\xi(\mu D - \Phi(A))$ contains a non-degenerate circular disk centered at $\mu w^q$, whereas $\mu w^q$ is an isolated point of $O_\xi(\mu D - A)$, which is a contradiction. Similarly, if $\Phi(A)$ has a nonzero $(p,q)$ entry with $(p,q) \in \{k,\ldots,n\} \times \{1,2\}$ with $k = 3$ or $4$ depending on $A = \nu E_{12} + E_{21}$ or $A = \nu E_{12} + E_{31}$. Taking $(r,s) = (q,p)$ in (4.1), we see that $O_\xi(\Phi(A) - \mu D - v_{qp}E_{qp}) = O_\xi(A - \mu D - E_{qp})$ has $\mu w^p$ as an isolated point. Thus, the $p$th row of

$\Phi(A)$ has only one nonzero entry at the $(q,p)$ position equal to $v_{qp}$. But then using $\Phi(\mu D)=\mu D$, we see that $O_\xi(\mu D-\Phi(A))$ will contain a non-degenerate circular disk centered at $\mu w^p$, whereas $O_\xi(\mu D-A)$ does not, which is a contradiction. Thus, $\Phi(A)$ only has a nonzero entry at the $(p,q)$ positions with $(p,q)\in K$, where $K=\{(1,2),(2,1)\}$ or $K=\{(1,2),(2,1),(3,1),(3,2)\}$ depending on $A=\nu E_{12}+E_{21}$ or $A=\nu E_{13}+E_{31}$. If $A=\nu E_{12}+E_{21}$, then $O_\xi(\Phi(A)-\nu E_{12})=O_\xi(A-\nu E_{12})=\{0\}$ and $O_\xi(\Phi(A)-E_{21})=O_\xi(A-E_{21})=\{0\}$. We conclude that $\Phi(A)=\nu E_{12}+E_{21}$. Suppose $A=\nu E_{12}+E_{31}$. Since $O_\xi(\Phi(A)-X)=O_\xi(A-X)$ for $X\in\{0,\nu E_{12},E_{23},\mu D+E_{12}\}$ for a suitable $\mu\in\mathbb{C}\setminus\Gamma$, we see that $\Phi(A)=A$ as asserted.

**Case 3** Suppose $A=\mu D+R$, where $R$ has nonzero off diagonal entries in at most one row and $\mu$ satisfies $\mu|1-w|>2$ or $\mu=0$.

For simplicity, assume that the nonzero off diagonal entries of $A=\mu D+\sum_{j=2}^n a_{1j}E_{1j}$ lie in the first row.

For $\mu|1-w|>2$ since (I) holds, we see that $\Phi(\mu D+E_{ij})=\mu D+v_{ij}E_{ij}$ with $|v_{ij}|=1$ for pairs $(i,j)$ with $i\neq j$. Suppose $\Phi(A)=(y_{ij})$. Since $O_\xi(\Phi(A)-\Phi(\mu D+E_{ij}))=O_\xi(A-\mu D-E_{ij})$ for all pairs $(i,j)$ with $i\neq j$, we see that $y_{ij}=0$ for $i>1$ and $i\neq j$. Moreover, since $O_\xi(\Phi(A)-\Phi(\nu E_{1j}+E_{21}))=O_\xi(A-\nu E_{1j}-E_{21})$ for all $\nu\in\mathbb{C}$ by Case 2, we see that $y_{1j}=a_{1j}$ for $j=2,\dots,n$.

Now, suppose $\mu=0$. Since $O_\xi(\Phi(A)-\Phi(E_{ij}))=O_\xi(A-E_{ij})$ for all pairs $(i,j)$ with $i\neq j$, we see that $y_{ij}=0$ for $i>1$ and $i\neq j$. Moreover, since $O_\xi(\Phi(A)-\Phi(\nu E_{1j}+E_{21}))=O_\xi(A-\nu E_{1j}-E_{21})$ for all $\nu\in\mathbb{C}$ by Case 2, we see that $y_{1j}=a_{1j}$ for $j=2,\dots,n$.

**Case 4** Suppose $A=a_{ji}E_{ji}+\sum_{k\neq i}a_{ik}E_{ik}$ with $a_{ji}\neq 0$ and $R_i(A)\neq 0$.

We may assume without loss of generality that $(i,j)=(1,2)$, $a_{21}\neq 0$, and $R_1(A)\neq 0$. Let $\Phi(A)=Y=(y_{rs})$.

There is a choice of $\mu$ for which $O_\xi(Y-\mu D)=O_\xi(A-\mu D)$ yields $y_{kk}=a_{kk}=0$ for $k=1,\dots,n$. We must have $y_{rs}=0$ for $r>2$ because

$$O_\xi\left(Y-(\mu D+\nu E_{1r})\right)=O_\xi\left(A-(\mu D+\nu E_{1r})\right)$$

which shows that the latter set will have a degenerate circular disk at $-\mu w^r$ whereas, if $y_{rs}\neq 0$, the former will have a non-degenerate circular disk at $-\mu w^r$ for $\nu\neq y_{1r}$. Now if $y_{21}\neq 0$ then

$$O_\xi\left(Y-\left(\sum_{k\neq 1}a_{1k}E_{1k}\right)\right)=O_\xi\left(A-\left(\sum_{k\neq 1}a_{1k}E_{1k}\right)\right)=O_\xi\left(a_{21}E_{21}\right)=\{0\}$$

which implies $y_{1k}=a_{1k}$ for all $k\neq 1$. Similarly, if $y_{12}\neq 0$ then $y_{2k}=a_{2k}$ for all $k\neq 2$. Now if $y_{21}=0, y_{12}\neq 0$ then $y_{21}=a_{21}\neq 0$ will be a contradiction. Thus $y_{21}=0$ implies $y_{12}=0$, which implies $O_\xi(Y)=\{0\}\neq O_\xi(\Phi(A))$, again a contradiction. Hence we have $\Phi(A)=A$.

**Case 5** Suppose $A=(a_{ij})$ has exactly two indices $i$ and $j$ with $R_i(A)\neq 0$ and $R_j(A)\neq 0$.

For simplicity, assume that $i=1$ and $j=2$. Let $\Phi(A)=Y=(y_{ij})$. By an appropriate choice of $\mu$ and using $O_\xi(Y-\mu D)=O_\xi(A-\mu D)$, we get that $y_{kk}=a_{kk}$ for $k=1,\dots,n$. As before, if

$y_{rs} \neq 0$ for $r > 2$ and $r \neq s$, then $O_\xi(Y - \mu D) = O_\xi(A - \mu D)$ implies $R_r(Y^t) = 0$. Now

$$O_\xi\left(Y - (\mu D + E_{1r})\right) = O_\xi\left(A - (\mu D + E_{1r})\right).$$

The latter set has a degenerate circular disk at $a_{rr} - \mu w^r$ whereas the former set has a non-degenerate circular disk centered at that point. Thus we must have $y_{rs} = 0$ for $r > 2$ and $r \neq s$.

Now, using essentially the same arguments as in Case 4 yields $\Phi(A) = A$. Again, generalizing to any $i$ and $j$ with $i \neq j$, we have $\Phi(A) = A$ for all $A$ with both $R_i(A) \neq 0$ and $R_j(A) \neq 0$.

**General Case** We complete the proof by now taking $A = (a_{ij})$ to be arbitrary. Let $\Phi(A) = Y = (y_{ij})$. As before, we can get $y_{kk} = a_{kk}$ by using $\mu D$ with an appropriate choice of $\mu$.

If $R_k(Y^t) \neq 0$ then using

$$O_\xi\left(Y - \left(\mu D + \sum_{j \neq i} a_{kj} E_{kj}\right)\right) = O_\xi\left(A - \left(\mu D + \sum_{j \neq i} a_{kj} E_{kj}\right)\right)$$

we get that the $y_{kj} = a_{kj}$ for all $j \neq k$ since the latter set has a degenerate circular disk centered at $a_{kk} - \mu w^k$.

If $R_k(Y^t) = 0$ then using

$$O_\xi\left(Y - \left(\mu D + \nu E_{ik} + \sum_{j \neq i} a_{kj} E_{kj}\right)\right) = O_\xi\left(A - \left(\mu D + \nu E_{ik} + \sum_{j \neq i} a_{kj} E_{kj}\right)\right)$$

where $i \neq j$ and $\nu \neq a_{ik}$, we conclude that $y_{kj} = a_{kj}$ for all $j \neq k$ since the latter set has a degenerate circular disk centered at $a_{kk} - \mu w^k$. $\square$

One can use a similar proof to obtain the structure of additive preservers of $O_\xi(A)$, and then deduce the results on linear preservers.

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