Distances from a Hermitian pair to diagonalizable and non-diagonalizable Hermitian pairs *

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Abstract

Denote by $W(T)$, $r(T)$ and $\|T\|$ the numerical range, the numerical radius and the spectral norm of a complex matrix T. Let (A, B) be a pair of Hermitian matrices. It is shown that if $0 \in W(A + iB)$ then

$$d(A, B) = \inf\{|\mu| : \mu \notin W(A + iB)\}$$

is an upper bound for

$$\inf\{r(E + iF) : (A + E) + i(B + F) \text{ is diagonalizable by congruence}\};$$

if $0 \notin W(A + iB)$ then the Crawford number

$$c(A, B) = \min\{|\mu| : \mu \in W(A + iB)\}$$

is equal to

$$\min\{r(E + iF) : (A + E) + i(B + F) \text{ is not diagonalizable by congruence}\},$$

which in turn is equal to

$$\inf\{\|E + iF\| : (A + E) + i(B + F) \text{ is not diagonalizable by congruence}\}.$$

The infimum is not always attained.

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1 Introduction

Let M_n (respectively, H_n) be the set of $n \times n$ complex (respectively, Hermitian) matrices. Two Hermitian matrices $A, B \in H_n$ are said to be a definite pair if $|x^*(A + iB)x| \neq 0$ for every nonzero vector $x \in \mathbb{C}^n$.

Definite Hermitian pairs have useful algorithmic and theoretical properties. For example, it is known (see [4, Theorem 1.7.17]) that if (A, B) is a definite Hermitian pair, then it is diagonalizable by congruence, i.e., there is an invertible matrix $S \in M_n$ so that both S^*AS and S^*BS are diagonal matrix, equivalently, $S^*(A+iB)S$ is a diagonal matrix. This property is very useful in the analysis of the Hermitian generalized eigenvalue problem; $Ax = \lambda Bx$. If (A, B) is a definite pair, then the corresponding generalized eigenvalues are real, and can be found by solving a related Hermitian eigenvalue problem [2, §8.7.3].

Recall that the numerical range of $T \in M_n$ is

$$W(T) = \{x^*Tx : x \in \mathbb{C}^n,\ x^*x = 1\},$$

and that the numerical radius of T is

$$r(T) = \max\{|x^*Tx| : x \in \mathbb{C}^n, x^*x = 1\},$$

which is the maximum distance of a point in the numerical range to the origin.

It is known that $W(T)$ is always a compact convex set in $\mathbb{C}$, and that the numerical radius is a norm on M_n satisfying

$$r(T) \leq \|T\| \leq 2r(T) \quad \text{for all } T \in M_n, \tag{1}$$

in comparison with the spectral norm $\|T\|$; for example, see [4, 5]. Also, it is known that (A, B) is a definite Hermitian pair if and only if $W(A + iB)$ does not contain the origin, which is equivalent to the existence of $a, b \in \mathbb{R}$ such that $aA + bB$ is positive definite; see [4, p. 72]. We define the Crawford number of (A, B) by

$$c(A, B) = \min\{|x^*(A + iB)x| : x \in \mathbb{C}^n, x^*x = 1\},$$

which is the shortest distance between a point in $W(A + iB)$ and the origin. The Crawford number often appears in the study of perturbation bounds in the study of problems involving definite Hermitian pairs; see [6, Chapter VI].

It is easily shown, Proposition 1, that $c(A, B)$ is the distance to the nearest non-definite pair. The purpose of this note, Theorem 3, is to show that $c(A, B)$ is also the distance from (A, B) to the set of non-diagonalizable pairs even though diagonalizability by congruence is not equivalent to definiteness. If $c(A, B) = 0$, i.e., $0 \in W(A + iB)$, then $A + iB$ may or may not be diagonalizable by congruence, but in Proposition 2, we give an upper bound for the distance between (A, B) to the set of diagonalizable pairs.

2 Results and proofs

Proposition 1 *Let (A, B) be a definite Hermitian pair. Suppose $x \in \mathbb{C}^n$ is a unit vector such that $|x(A + iB)x| = c(A, B)$, and $(E_0, F_0) = -(x^*AxI, x^*BxI)$. Then $(A + E_0, B + F_0)$*

is not a definite pair and

$$c(A,B) = r(E_0 + iF_0) = \min\{r(E+iF) : (A+E, B+F) \text{ is not a definite pair}\}. \quad (2)$$

Furthermore, (2) is valid when $r(\cdot)$ is replaced by $\|\cdot\|$.

Proof. Let r_D denote the right hand side of (2). Let $r_{D,\|\cdot\|}$ denote the right hand side of (2) when $r(\cdot)$ is replaced by $\|\cdot\|$.

Suppose $x \in \mathbb{C}^n$ is a unit vector such that $|x^*(A+iB)x| = c(A,B)$ and $(E_0, F_0) = -(x^*AxI, x^*BxI)$. Then $0 \in W((A+E_0) + i(B+F_0))$ and hence $(A+E_0, B+F_0)$ is not definite. Since $E_0 + iF_0$ a multiple of the identity,

$$\|E_0 + iF_0\| = |(x^*Ax) + i(x^*Bx)| = c(A,B).$$

Thus $r_{D,\|\cdot\|} \leq c(A,B)$.

By (1), we have $r_D \leq r_{D,\|\cdot\|}$. Let (E,F) be a Hermitian pair such that $(A+E, B+F)$ is not definite. Consider a unit vector $y \in \mathbb{C}^n$ such that $y^*(A+E)y = y^*(B+F)y = 0$, or equivalently, $y^*Ay = -y^*Ey$ and $y^*By = -y^*Fy$. So,

$$c(A,B) \leq |y^*(A+iB)y| = |y^*(E+iF)y| \leq r(E+iF). \quad (3)$$

Thus $c(A,B) \leq r_D$. Combining this with the conclusion of the previous paragraph we have $c(A,B) = r_D = r_{D,\|\cdot\|}$. □

Proposition 2 *Let (A,B) be a Hermitian pair such that $0 \in W(A+iB)$. Then*

$$\begin{aligned} d(A,B) &= \inf\{|\mu| : \mu \notin W(A+iB)\} \\ &\geq \inf\{r(E+iF) : (A+E) + i(B+F) \text{is diagonalizable by congruence}\}. \quad (4) \end{aligned}$$

Furthermore, (4) is valid when $r(\cdot)$ is replaced by $\|\cdot\|$.

Proof. Let $T = A+iB$. Since $W(T)$ is compact, there is a boundary point μ with minimum modulus. We may replace (T,μ) by $(e^{it}T, e^{it}\mu)$ for a suitable $t \in [0, 2\pi)$ so that there is a left support line of $W(T)$ passing through μ. Then for any $\varepsilon > 0$, we can let $E + iF = (\varepsilon - \mu)I$ so that $0 \notin W(T + (E+iF))$ and hence $T + (E+iF)$ is diagonalizable by congruence. Since $\|E+iF\| = r(E+iF) \leq |\mu| + \varepsilon$ and ε is arbitrary, we get the desired inequality. □

Let (A,B) be a Hermitian pair such that $0 \in W(A+iB)$. Since $W(A+iB)$ is closed, $\inf\{|\mu| : \mu \notin W(A+iB)\}$ is not attained by any element not in $W(A+iB)$. Also,

$$\inf\{r(E+iF) : (A+E) + i(B+F) \text{ is diagonalizable by congruence}\}$$

is not always attainable. For example, if

$$A = \begin{pmatrix} 0 & 10 \\ 10 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 11 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $W(A+iB)$ is an elliptical disk with minor axis joining the numbers $11i$ and $-i$, and major axis joining the numbers $10+5i$ and $-10+5i$. Clearly, $d(A,B)=1$, and $-i$ is the boundary point of $W(A+iB)$ nearest to the origin. Suppose $E+iF$ satisfies $r(E+iF)\le 1$. We claim that $T=(A+E)+i(B+F)$ is not diagonalizable by congruence. Suppose it is not true and that $S\in M_2$ is invertible such that S^*TS is in diagonal form. Note that $0\in W(T)$. It follows that $W(S^*TS)$ is a line segment containing 0. Thus, there exists a complex unit ξ such that ξS^*TS is Hermitian. So, ξT is Hermitian and $\xi W(T)$ is a real line segment containing 0. Let $x,y,z\in\mathbb{C}^n$ be unit vectors such that $x^*(A+iB)x=11i$, $y^*(A+iB)y=10+5i$, and $z^*(A+iB)z=-10+5i$. Let $x^*Tx=\mu_1$, $y^*Ty=\mu_2$, and $z^*Tz=\mu_3$. Then $|11i-\mu_1|\le 1$, $|10+5i-\mu_2|\le 1$, and $|-10+5i-\mu_3|\le 1$. So, $W(T)$ cannot be a line segment. Hence, T is not diagonalizable.

Next, we turn to our main result.

Theorem 3 *Let (A,B) be a definite pair. Then*

$$c(A,B)=\min\{r(E+iF):(A+E)+i(B+F)\text{ is not diagonalizable by congruence}\}\quad(5)$$

and

$$c(A,B)=\inf\{\|E+iF\|:(A+E)+i(B+F)\text{ is not diagonalizable by congruence}\}.\quad(6)$$

We need two lemmas to prove Theorem 3. The first one is a standard result characterizing diagonalizability of a pair by congruence when one of the matrices is invertible. The second presents a perhaps surprising difference between the numerical radius and the spectral norm. This difference is the reason that the result in Theorem 3 contains a “min” for the numerical radius but only an “inf” for the spectral norm.

Lemma 4 [3, Table 4.5.15, part 1 (b)] *Let $A,B\in H_n$ with A invertible. Then $A+iB$ is diagonalizable by congruence if and only if $A^{-1}B$ is similar to a real diagonal matrix.*

Lemma 5 [4, Theorem 1.3.6 (b)] *Take $t\in(0,1/2]$ and set*

$$X=\begin{pmatrix}0 & it\\ it & 1\end{pmatrix}.$$

Then $r(X)=1<\|X\|$.

Proof of Theorem 3. Suppose that

$$\min\{|z|:z\in W(A+iB)\}$$

occurs at $z=re^{i\theta}$ then replacing $A+iB$ by $e^{-i\theta}(A+iB)$ if necessary we may assume that $z=i\gamma$. After a unitary similarity Now, we may assume with loss of generality that

$B = B_1 \oplus [\gamma]$ with $B_1 - \gamma I_{n-1} \in M_{n-1}$. This implies that $a_{nn} = 0$, so write $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & 0 \end{pmatrix}$ with $A_{11} \in M_{n-1}$. Let

$$E = \mathrm{diag}(d_1, \ldots, d_{n-2}) \oplus \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} \text{ and } F = 0_{n-2} \oplus \mathrm{diag}(0, -\gamma).$$

Using a Schur Complement argument for example, we can show that for any $t \neq 0$ we can choose $d_1, \ldots, d_{n-2}$ with $\gamma > d_j > 0$ such that $\tilde{A} = A + E$ is invertible. We claim that $\tilde{A} + iB$ is not diagonalizable by congruence.

Firstly, note that $\tilde{B} = B + F = B_1 \oplus 0$ has rank $n - 1$ and hence so has $\tilde{A}^{-1}\tilde{B}$.

Write

$$\tilde{A}^{-1} = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}, \quad \text{where } \ X \in M_{n-1}, \ Z \in M_1.$$

Notice that $\tilde{a}_{nn} = a_{nn} = 0$ is singular. Thus, by the Nullity Theorem [1], it follows that the complementary submatrix in $\tilde{A}^{-1}$, that is X, is also singular. Hence XB_1 has at least one zero eigenvalue. So, the rank $n - 1$ matrix

$$\tilde{A}^{-1}\tilde{B} = \begin{pmatrix} XB_1 & 0 \\ S^*\tilde{B}_1 & 0 \end{pmatrix}$$

has at most $n - 2$ nonzero eigenvalues. Thus, $\tilde{A}^{-1}\tilde{B}$ is not diagonalizable, and our claim is proved.

Now, by Lemma 5, taking $t \in (0, \gamma/2)$ ensures $r(E + iF) = \gamma$, establishing (5).

Taking $t = \epsilon > 0$ ensures $\|E + iF\| \leq \gamma + \epsilon$ and establishes (6). □

A slightly more careful argument shows that if in the proof above $A_{12} \neq 0$, then we can take $t = 0$ in constructing $(E + iF)$ such that $(A + iB) + (E + iF)$ is not diagonalizable by congruence. The resulting $(E + iF)$ will have $\|E + iF\| = \gamma$. Thus generically, the infimum in (6) is attained.

Here is an instance where the infimum in (6) is not attained. Take the 2×2 matrices $A = 0$ and $B = I$. Clearly $c(A, B) = 1$. Let E, F be Hermitian and such that

$$(A + iB) + (E + iF) \ \text{ is not diagonalizable by congruence.} \tag{7}$$

Since both A and B are invariant under unitary similarity, we may assume without loss of generality that F is diagonal. Note that $\max\{\|E\|, \|F\|\} \leq \|E + iF\|$ so if $\|E + iF\| \leq 1$ and if the pair $(A + E, B + F)$ is not definite, then F must be of the form

$$\begin{pmatrix} -1 & 0 \\ 0 & t \end{pmatrix} \text{ or } \begin{pmatrix} t & 0 \\ 0 & -1 \end{pmatrix}.$$

In either case $B + F$ is diagonal, so the condition (7) requires that $A + E = E$ has non-zero off-diagonal. However, for such E and F it is the case that $\|E + iF\| > 1$.

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