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HIGHER-RANK NUMERICAL RANGES AND DILATIONS

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ABSTRACT. For any n-by-n complex matrix A and any k, $1 \leq k \leq n$, let $\Lambda_k(A) = \{\lambda \in \mathbb{C} : X^*AX = \lambda I_k$ for some n-by-k X satisfying $X^*X = I_k\}$ be its rank-k numerical range. It is shown that if A is an n-by-n contraction, then

$$\Lambda_k(A) = \cap\{\Lambda_k(U) : U \text{ is an } (n + d_A)\text{-by-}(n + d_A) \text{ unitary dilation of } A\},$$

where $d_A = \operatorname{rank}(I_n - A^*A)$. This extends and refines previous results of Choi and Li on constrained unitary dilations, and a result of Mirman on S_n-matrices.

KEYWORDS: *Higher-rank numerical range, unitary dilation.*

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1. INTRODUCTION

We say that the operator A on space H *dilates* to B on K or B *compresses* to A if there is an isometry V from H to K such that $A = V^*BV$. It is easily seen that this is equivalent to B being unitarily similar to a 2-by-2 operator matrix of the form $\begin{bmatrix} A & * \\ * & * \end{bmatrix}$. The classical dilation result of Halmos asserts that *every contraction A, i.e., an A with $\|A\| \leq 1$, can be dilated to the unitary operator*

$$\begin{bmatrix} A & (I - AA^*)^{1/2} \\ (I - A^*A)^{1/2} & -A^* \end{bmatrix}$$

(cf. [11, Problem 222 (a)]). With more care, the unitary dilation can be achieved in a most economical way: *if A is a contraction on H, then A can be dilated to a unitary operator U from $H \oplus K_1$ to $H \oplus K_2$ with K_1 and K_2 of dimensions $d_{A^*} \equiv$* dim $\overline{\operatorname{ran}(I - AA^*)^{1/2}}$ *and $d_A \equiv$* dim $\overline{\operatorname{ran}(I - A^*A)^{1/2}}$, *respectively, and, moreover, in this case d_{A^*} and d_A are the smallest dimensions of such spaces K_1 and K_2.* Here d_A and d_{A^*} are called the *defect indices* of the contraction A. They provide a measure on how far A deviates from the unitary operators and play a prominent role in the unitary dilation theory. Note that $d_{A^*} = d_A$ if H is finite-dimensional.

Let M_n be the algebra of n-by-n complex matrices. In [4], the authors introduced the notion of the *rank-k numerical range* of $A \in M_n$ in connection to the study of quantum error correction; see [5]. This can be defined equivalently as

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} : X^*AX = \lambda I_k, \text{ for some } n\text{-by-}k\ X \text{ satisfying } X^*X = I_k\}.$$

Evidently, $\lambda \in \Lambda_k(A)$ if and only if λI_k dilates to A. When $k = 1$, this concept reduces to the classical numerical range. Many properties of the classical numerical range have been extended to the higher-rank numerical range; see [2, 3, 4, 5, 20]. In particular, it was shown in [13] that

$$\Lambda_k(A) = \{\mu \in \mathbb{C} : e^{it}\mu + e^{-it}\overline{\mu} \le \lambda_k(e^{it}A + e^{-it}A^*) \text{ for all } t \in [0, 2\pi)\}. \tag{1.1}$$

Here $\lambda_1(X) \ge \cdots \ge \lambda_n(X)$ denote the eigenvalues of a Hermitian $X \in M_n$. In particular, $\Lambda_k(A)$ is the intersection of closed half planes in $\mathbb{C}$, and therefore is always convex. If $N \in M_n$ is normal with eigenvalues $\lambda_1, \dots, \lambda_n$, then

$$\Lambda_k(N) = \bigcap_{1 \le j_1 < \cdots < j_{n-k+1} \le n} \operatorname{conv}\{\lambda_{j_1}, \dots, \lambda_{j_{n-k+1}}\} \tag{1.2}$$

is a polygon (including interior). In [12], it was shown that for a given positive integer n, $\Lambda_k(A)$ is nonempty for every $A \in M_n$ if and only if $n \ge 3k - 2$.

In this paper, we refine and extend a result in [6] on constrained unitary dilation by proving the following.

THEOREM 1.1. *Let $A \in M_n$ be a contraction, and $k \in \{1, \dots, n\}$. Then A has a unitary dilation $U \in M_{n+d_A}$ such that $\lambda_k(A + A^*) = \lambda_k(U + U^*)$.*

When $k = 1$, our result improves [6, Theorem 2.1] in the finite-dimensional case as [6, Theorem 2.1] requires the use of unitary dilations of $A \in M_n$ of size $2n$. The authors of [6] gave examples to demonstrate that extending [6, Theorem 2.1] in certain directions are impossible. Nevertheless, Theorem 1.1 shows that one can obtain useful generalizations of the result under a proper setting. In particular, Theorem 1.1 above can be used to deduce the following theorem, which extends a result on classical numerical range to the higher-rank numerical range.

THEOREM 1.2. *Let $A \in M_n$ be a contraction. Then, for each k, $1 \le k \le n$,*

$$\Lambda_k(A) = \cap\{\Lambda_k(U) : U \in M_{n+d_A} \text{ is a unitary dilation of } A\}.$$

When $k = 1$ and without the dimension assumption on the unitary U, Theorem 1.2 was conjectured by Halmos [10] and proved in [6]. Clearly, if $A \in M_n$ is nonzero then $A/\|A\|$ is a contraction. Thus, by Theorem 1.2, if $A \in M_n$ then $\Lambda_k(A)$ is the intersection of $\Lambda_k(\|A\|U)$, where $U \in M_{n+d_A}$ is a unitary dilation of $A/\|A\|$. Consequently, $\Lambda_k(A)$ is the intersection of polygons $\Lambda_k(N)$ of the form (1.2), where N is a (norm-preserving) normal dilation of A.

2. PROOFS

We begin with several lemmas. The first two are adaptations of Lemmas 3.2 and 3.3 in [6]. Part of the proofs are similar to those in [6]. We include the details for completeness.

LEMMA 2.1. *Let $H \in M_n$ be the leading principal submatrix of a Hermitian matrix $\widetilde{H} \in M_{n+1}$. Suppose there exists a unit vector $u \in \mathbb{C}^{n+1}$ with nonzero $(n+1)$st entry such that $\widetilde{H}u = \xi u$. For $1 \leq k \leq n$, if $\lambda_k(H) \leq \xi$, then $\lambda_k(\widetilde{H}) \leq \xi$.*

Proof. On the contrary, suppose that $\lambda_k(\widetilde{H}) > \xi$. Since ξ is an eigenvalue for $\widetilde{H}$, by the interlacing inequality [1, Corollary III.1.5], we must have $\lambda_{k+1}(\widetilde{H}) = \xi = \lambda_k(H)$. Let $v_j \in \mathbb{C}^{n+1}$ be the unit eigenvector of $\widetilde{H}$ corresponding to the eigenvalue $\lambda_j(\widetilde{H})$ for $j = 1, 2, \ldots, k$, $M = \operatorname{span}\{u, v_1, \ldots, v_k\}$ and $N = M \cap (\mathbb{C}^n \oplus \{0\})$. Then $\dim N = k$, because $u \notin \mathbb{C}^n \oplus \{0\}$. Consider the compression A of $\widetilde{H}$ on N. Since $\Lambda_1(A) \subseteq \Lambda_1(\widetilde{H}|_M) = [\xi, \lambda_1(\widetilde{H})]$, it is clear that $\lambda_k(A) \geq \xi$. On the other hand, since $N \subseteq \mathbb{C}^n \oplus \{0\}$, we also have $\xi = \lambda_k(H) \geq \lambda_k(A)$. Thus $\lambda_k(A) = \xi$. Let $y \in N$ be a unit eigenvector of A corresponding to the eigenvalue ξ. Say, $y = c_0 u + c_1 v_1 + \cdots + c_k v_k$, where $\sum_{j=0}^{k} |c_j|^2 = 1$. Since $\xi = \langle Ay, y \rangle = \langle \widetilde{H}y, y \rangle = |c_0|^2 \xi + \sum_{j=1}^{k} |c_j|^2 \lambda_j(\widetilde{H})$ and $\lambda_1(\widetilde{H}) \geq \cdots \geq \lambda_k(\widetilde{H}) > \xi$, we infer that $|c_0| = 1$ and $c_1 = \cdots = c_k = 0$. This implies that $u \in N \subseteq \mathbb{C}^n \oplus \{0\}$, a contradiction. Hence $\lambda_k(\widetilde{H}) \leq \xi$ as asserted. ∎

LEMMA 2.2. *Let $A \in M_n$ be a contraction with $d_A \geq 1$ and denote $\lambda_k(A + A^*) = 2\cos\theta$ for some $\theta \in \mathbb{R}$. Suppose neither $e^{i\theta}$ nor $e^{-i\theta}$ is an eigenvalue for A. Then A has a contractive dilation $\widetilde{A} \in M_{n+1}$ such that $\lambda_k(\widetilde{A} + \widetilde{A}^*) = \lambda_k(A + A^*)$, $d_{\widetilde{A}} = d_A - 1$ and $e^{\pm i\theta}$ are two eigenvalues for $\widetilde{A}$.*

Proof. Let v be a unit vector such that $(A + A^*)v = (2\cos\theta)v$. By [6, Lemma 3.1], we have $\|Av\| < 1$. Since

$$
\begin{aligned}
\|A^*v\|^2 - \|Av\|^2 &= v^*(AA^* - A^*A)v = v^*\{A(A + A^*) - (A + A^*)A\}v \\
&= v^*A(2\cos\theta)v - (2\cos\theta)v^*Av = 0,
\end{aligned}
$$

we have $\|A^*v\| = \|Av\|$. Let $\alpha = \sqrt{1 - \|Av\|^2} = \sqrt{1 - \|A^*v\|^2}$. Then $x = (I_n - A^*A)^{1/2}v/\alpha$ and $y = (I_n - AA^*)^{1/2}v/\alpha$ are unit vectors in $\mathbb{C}^n$. Write

$$
X = \begin{bmatrix} I_n & \overrightarrow{0_n} \\ 0_n & x \end{bmatrix}, Y = \begin{bmatrix} I_n & \overrightarrow{0_n} \\ 0_n & y \end{bmatrix}, Z = \begin{bmatrix} A & -(I_n - AA^*)^{1/2} \\ (I_n - A^*A)^{1/2} & A^* \end{bmatrix},
$$

and

$$
\widetilde{A} = X^*ZY = \begin{bmatrix} A & -(I_n - AA^*)v/\alpha \\ v^*(I_n - A^*A)/\alpha & x^*A^*y \end{bmatrix} \in M_{n+1}.
$$

Then X and Y are $2n$-by-$(n+1)$ matrices satisfying $X^*X = Y^*Y = I_{n+1}$, $Z^*Z = I_{2n}$ and $\widetilde{A}$ is a contractive dilation of A. Let $\widetilde{v} = \begin{bmatrix} v \\ 0 \end{bmatrix} \in \mathbb{C}^{n+1}$. Then

$$\widetilde{A}\widetilde{v} = \begin{bmatrix} Av \\ v^*(I_n - A^*A)v/\alpha \end{bmatrix} = \begin{bmatrix} Av \\ \alpha \end{bmatrix}$$

is a unit vector because $\alpha = \sqrt{1 - \|Av\|^2}$, and

$$(\widetilde{A} + \widetilde{A}^*)\widetilde{v} = \begin{bmatrix} (A + A^*)v \\ v^*(AA^* - A^*A)v/\alpha \end{bmatrix} = \begin{bmatrix} (2\cos\theta)v \\ 0 \end{bmatrix} = (2\cos\theta)\widetilde{v}$$

because $\|A^*v\| = \|Av\|$. It follows from [6, Lemma 3.1] that $M = \operatorname{span}\{\widetilde{v}, \widetilde{A}\widetilde{v}\}$ is a reducing subspace of $\widetilde{A}$ and the restriction of $\widetilde{A}$ on M has $e^{\pm i\theta}$ as two of its eigenvalues. So, $\widetilde{A}\widetilde{v} = \begin{bmatrix} Av \\ \alpha \end{bmatrix}$ is also an eigenvector of $\widetilde{A} + \widetilde{A}^*$ corresponding to the eigenvalue $2\cos\theta$. Note that the last entry of $\widetilde{A}\widetilde{v}$ is $\alpha \neq 0$. Applying Lemma 2.1 with $H = A + A^*$, $\widetilde{H} = \widetilde{A} + \widetilde{A}^*$ and $\xi = 2\cos\theta$, we have $\lambda_k(\widetilde{A} + \widetilde{A}^*) \leq 2\cos\theta$. By the interlacing inequality [1, Corollary III.1.5], we conclude that $\lambda_k(\widetilde{A} + \widetilde{A}^*) = 2\cos\theta$.

We now check that $d_{\widetilde{A}} = d_A - 1$. Note that the leading n-by-n principal submatrix of $\widetilde{A}^*\widetilde{A}$ equals $A^*A + ww^*$ with $w = (I_n - A^*A)v/\alpha$. Thus,

$$\begin{aligned} d_{\widetilde{A}} &= \operatorname{rank}(I_{n+1} - \widetilde{A}^*\widetilde{A}) \geq \operatorname{rank}(I_n - A^*A - ww^*) \\ &\geq \operatorname{rank}(I_n - A^*A) - 1 = d_A - 1. \end{aligned}$$

It remains to show that $d_{\widetilde{A}} \leq d_A - 1$. Let K be the eigenspace of A^*A corresponding to the eigenvalue 1. Then K has dimension $m = n - d_A$, and there is an orthonormal basis $\{u_1, \ldots, u_m\}$ for K such that $\|Au_j\| = 1$ for all $j = 1, \ldots, m$. Now, consider the vectors of the form $\tilde{u}_j = \begin{bmatrix} u_j \\ 0 \end{bmatrix} \in \mathbb{C}^{n+1}$ for $j = 1, \ldots, m$, and let $\widetilde{K}$ be the space spanned by them. Clearly, $\tilde{v} \notin \widetilde{K}$ and $\widetilde{A}\tilde{v}$ does not lie in the span of $\widetilde{K} \cup \{\tilde{v}\}$. Now, $\|\widetilde{A}w\| = 1$ for all $w \in \{\tilde{u}_1, \ldots, \tilde{u}_m, \tilde{v}, \widetilde{A}\tilde{v}\}$, which spans an $(m+2)$-dimensional subspace. Thus $\widetilde{A}^*\widetilde{A}$ has at least $m+2$ linearly independent eigenvectors for 1. So, $d_{\widetilde{A}} \leq n + 1 - (m+2) = d_A - 1$. ∎

LEMMA 2.3. *Let $A \in M_n$ be a contraction with $d_A \geq 1$ such that $\lambda_n(A + A^*) \geq \gamma$ for some $\gamma > -2$. Then A has a contractive dilation $\widetilde{A} \in M_{n+1}$ such that $d_{\widetilde{A}} = d_A - 1$, $\lambda_n(\widetilde{A} + \widetilde{A}^*) \geq \gamma$ and $-1, e^{i\theta}$ are two eigenvalues for $\widetilde{A}$, where $2\cos\theta \geq \gamma$.*

Proof. Since A is a contraction, it is unitarily similar to $U_0 \oplus A_0$, where $U_0 \in M_{n-m}$ $(1 \leq m \leq n)$ is unitary and $A_0 \in M_m$ is a contraction with no eigenvalues on the unit circle. Clearly, $d_{A_0} = d_A$. Note that $\Lambda_1(A_0)$ is a compact convex set contained in the open unit disc, and $-1 \notin \Lambda_1(A_0)$. Hence there are two chords $[-1, e^{i\theta}]$ and $[-1, e^{i\phi}]$ which are tangent to $\partial\Lambda_1(A_0)$, where $-\pi < \phi \leq \theta < \pi$. It is clear that $2\cos\theta \geq \gamma$, because $\Lambda_1(A_0)$ is contained in the closed half plane $\{z \in \mathbb{C} : z + \bar{z} \geq \gamma\}$. Let $A_0' = e^{-i(\theta+\pi)/2} A_0$. Then the line segment $[e^{i(\pi-\theta)/2}, e^{i(\theta-\pi)/2}]$ is tangent to $\partial\Lambda_1(A_0')$, and $\Lambda_1(A_0')$ is contained in the closed half plane $\{z \in \mathbb{C} : z + \bar{z} \leq 2\cos((\pi-\theta)/2)\}$. That is, $\lambda_1(A_0' + A_0'^*) = 2\cos((\pi-\theta)/2)$. By Lemma 2.2 for $k = 1$, A_0' has a contractive dilation $\widetilde{A_0'} \in M_{m+1}$ such that $d_{\widetilde{A_0'}} = d_{A_0'} - 1 = d_A - 1$, $\lambda_1(\widetilde{A_0'} + \widetilde{A_0'}^*) = 2\cos((\pi-\theta)/2)$ and $e^{\pm i(\pi-\theta)/2}$ are two eigenvalues for $\widetilde{A_0'}$. Let $\widetilde{A_0} = e^{i(\theta+\pi)/2}\widetilde{A_0'}$ and $\widetilde{A} = U_0 \oplus \widetilde{A_0}$. We deduce that $\widetilde{A}$ is a contractive dilation of A, $d_{\widetilde{A}} = d_{\widetilde{A_0'}} = d_A - 1$ and $-1, e^{i\theta}$ are two eigenvalues for $\widetilde{A}$. By the interlacing inequality, it is clear that $\lambda_n(\widetilde{A} + \widetilde{A}^*) \geq \lambda_n(A + A^*) \geq \gamma$ as desired. ∎

We are now ready for the

Proof of Theorem 1.1. We prove the result by induction on d_A. If $d_A = 0$, then $U = A$ as asserted. Assume $d_A \geq 1$ and the result holds if d_A is smaller. For convenience, say, $\lambda_k(A + A^*) = 2\cos\theta$, where $\theta \in \mathbb{R}$. It suffices to show that A has a contractive dilation $A_1 \in M_{n+1}$ such that $\lambda_k(A_1 + A_1^*) = \lambda_k(A + A^*)$ and $d_{A_1} = d_A - 1$. The result will then follow from the induction hypothesis.

Since A is a contraction, it is unitarily similar to $U_0 \oplus A_0$, where $U_0 \in M_{n-m}$ $(1 \leq m \leq n)$ is unitary and $A_0 \in M_m$ is a contraction with no eigenvalue on the unit circle. Clearly, $d_{A_0} = d_A \geq 1$. Let

$$j_0 = \max\{j : \lambda_j(A_0 + A_0^*) > 2\cos\theta\}$$

and

$$j_1 = \max\{j : \lambda_j(U_0 + U_0^*) > 2\cos\theta\}$$

with the convention that $j_0 = 0$ and $j_1 = 0$ when the corresponding set of indices is empty. Then

$$j_0 \leq m, \quad j_0 + j_1 < k \quad \text{and} \quad \lambda_{j_0+j_1+1}(A + A^*) = 2\cos\theta.$$

We consider two cases.

Case 1. Suppose $j_0 < m$. Then $2\cos\theta \geq \lambda_{j_0+1}(A_0 + A_0^*) = 2\cos\theta_0$. Note that neither $e^{i\theta_0}$ nor $e^{-i\theta_0}$ is an eigenvalue for A_0. By Lemma 2.2, A_0 has a contractive dilation $\widetilde{A}_0 \in M_{m+1}$ such that $\lambda_{j_0+1}(\widetilde{A}_0 + \widetilde{A}_0^*) = \lambda_{j_0+1}(A_0 + A_0^*) = 2\cos\theta_0 \leq 2\cos\theta$, $d_{\widetilde{A}_0} = d_{A_0} - 1$ and $e^{\pm i\theta_0}$ are two eigenvalues for $\widetilde{A}_0$. Moreover, by the interlacing inequality, $\lambda_j(\widetilde{A}_0 + \widetilde{A}_0^*) \geq \lambda_j(A_0 + A_0^*) > 2\cos\theta$ for $j \leq j_0$. Consequently, $\max\{j : \lambda_j(\widetilde{A}_0 + \widetilde{A}_0^*) > 2\cos\theta\} = j_0$. Thus, $A_1 = U_0 \oplus \widetilde{A}_0 \in M_{n+1}$ is a contractive dilation of A satisfying $d_{A_1} = d_{\widetilde{A}_0} = d_{A_0} - 1 = d_A - 1$ and $\max\{j : \lambda_j(A_1 + A_1^*) > 2\cos\theta\}$ equal to

$$\max\{j : \lambda_j(U_0 + U_0^*) > 2\cos\theta\} + \max\{j : \lambda_j(\widetilde{A}_0 + \widetilde{A}_0^*) > 2\cos\theta\} = j_1 + j_0.$$

It follows that

$$2\cos\theta \geq \lambda_{j_0+j_1+1}(A_1 + A_1^*) \geq \lambda_k(A_1 + A_1^*) \geq \lambda_k(A + A^*) = 2\cos\theta,$$

because $j_0 + j_1 < k$. Hence $\lambda_k(A_1 + A_1^*) = \lambda_k(A + A^*)$ and A_1 is a desired dilation.

Case 2. Suppose $j_0 = m$. Then $\lambda_m(A_0 + A_0^*) > 2\cos\theta$. By Lemma 2.3, A_0 has a contractive dilation $\widetilde{A}_0 \in M_{m+1}$ such that

$$\lambda_m(\widetilde{A}_0 + \widetilde{A}_0^*) > 2\cos\theta,\ d_{\widetilde{A}_0} = d_{A_0} - 1 = d_A - 1 \text{ and } \lambda_{m+1}(\widetilde{A}_0 + \widetilde{A}_0^*) = -2.$$

Then $A_1 = U_0 \oplus \widetilde{A}_0 \in M_{n+1}$ is a contractive dilation of A satisfying $d_{A_1} = d_{\widetilde{A}_0} = d_A - 1$ and

$$\begin{aligned} & \max\{j : \lambda_j(A_1 + A_1^*) > 2\cos\theta\} \\ = \ & \max\{j : \lambda_j(U_0 + U_0^*) > 2\cos\theta\} + \max\{j : \lambda_j(\widetilde{A}_0 + \widetilde{A}_0^*) > 2\cos\theta\} \\ = \ & j_1 + m = j_1 + j_0. \end{aligned}$$

It follows that

$$2\cos\theta \geq \lambda_{j_0+j_1+1}(A_1 + A_1^*) \geq \lambda_k(A_1 + A_1^*) \geq \lambda_k(A + A^*) = 2\cos\theta,$$

because $j_0 + j_1 < k$. Hence $\lambda_k(A_1 + A_1^*) = \lambda_k(A + A^*)$ and A_1 is a desired dilation. ∎

We can now use Theorem 1.1 to prove Theorem 1.2. The proof depends heavily on (1.1) and is similar to the proof of Theorem 2.4 in [6].

Proof of Theorem 1.2. Let $A \in M_n$ be a contraction. It is obvious that $\Lambda_k(A) \subseteq \Lambda_k(B)$ if B is a dilation of A. Thus, we have

$$\Lambda_k(A) \subseteq \cap\{\Lambda_k(U) : U \in M_{n+d_A} \text{ is a unitary dilation of } A\}.$$

To prove the reverse inclusion, we consider any particular $\zeta \notin \Lambda_k(A)$. Since $\Lambda_k(A)$ is a compact convex set, there exists $\theta \in [0, 2\pi)$ and $\mu \in \mathbb{R}$ such that $e^{i\theta}\zeta + e^{-i\theta}\overline{\zeta} > \mu$, while $e^{i\theta}\Lambda_k(A) = \Lambda_k(e^{i\theta}A)$ is included in the closed half plane $\{z \in \mathbb{C} : z + \overline{z} \leq \mu\}$. From (1.1), we see that $\lambda_k(e^{i\theta}A + e^{-i\theta}A^*) \leq \mu$. By Theorem 1.1, there is a unitary dilation $U \in M_{n+d_A}$ of A such that $\lambda_k(e^{i\theta}U + e^{-i\theta}U^*) \leq \mu$. By (1.1) again, $\Lambda_k(e^{i\theta}U) \subseteq \{z \in \mathbb{C} : z + \overline{z} \leq \mu\}$. Hence $e^{i\theta}\zeta \notin \Lambda_k(e^{i\theta}U)$ and $\zeta \notin \Lambda_k(U)$. This completes the proof. ∎

We end this paper by relating the rank-k numerical ranges of S_n-matrices to the Poncelet property. An n-by-n complex matrix A is said to be of *class* S_n if (i) A is a contraction, (ii) the eigenvalues of A are all in the open unit disc $\mathbb{D}$, and (iii) $d_A = 1$. In recent years, properties of the classical numerical ranges of S_n-matrices have been intensely studied (cf. [7, 8, 9, 15, 16, 17, 18, 19, 21]). Among other things, it was obtained that the boundary of the classical numerical range $\Lambda_1(A)$ of an S_n-matrix A has the $(n+1)$-Poncelet property. This means that there are infinitely many $(n+1)$-gons interscribing between the unit circle $\partial\mathbb{D}$ and the boundary $\partial\Lambda_1(A)$ or, put more precisely, for any point a on $\partial\mathbb{D}$ there is a (unique) $(n+1)$-gon with a as one of its vertices such that all its $n+1$ vertices are in $\partial\mathbb{D}$

and all its $n+1$ edges are tangent to $\partial\Lambda_1(A)$ (cf. [7, Theorem 2.1] or [15, Theorem 1]).

If A is in S_n, so is $e^{-it}A$ for any real t. Hence the eigenvalues of $(e^{-it}A + e^{it}A^*)/2$ are all distinct by [7, Corollary 2.7]. The curve Γ_j, $j = 1, \ldots, n$, is the envelope of chords

$$x\cos t + y\sin t = \lambda_j(t),$$

where $\lambda_j(t) = \lambda_j((e^{-it}A + e^{it}A^*)/2)$. Equations for the curves Γ_j are described by $\alpha_j(t) = (x_j(t), y_j(t))$ with

$$\begin{aligned} x_j(t) &= \lambda_j(t)\cos t - \lambda_j'(t)\sin t, \\ y_j(t) &= \lambda_j(t)\sin t + \lambda_j'(t)\cos t. \end{aligned}$$

These curves Γ_j are expected to have a Poncelet-type property just as $\Gamma_1 = \partial\Lambda_1(A)$ does. This is indeed the case and is proved in [15, Theorem 8]. Note that, in this case, Γ_j and Γ_{n-j+1} coincide for any j, and if $U = \mathrm{diag}\,(b_1, \ldots, b_{n+1})$ is a unitary dilation of A, where the b_j's are arranged counterclockwise around $\partial\mathbb{D}$, then, for each j, the not-necessarily-convex $(n+1)$-gon $b_1\ b_{j+1}\ b_{2j+1}\ \ldots\ b_{nj+1}$ ($b_p = b_q$ if $p \equiv q \pmod{n+1}$) has all its sides $[b_{kj+1}, b_{(k+1)j+1}]$ tangent to Γ_j. A detailed analysis of such curves, called *a package of Poncelet curves*, has been carried out by Mirman [15, 16, 18]. Note that the curve Γ_1 is convex and $\Lambda_1(A)$ is equal to the convex hull of Γ_1. Other curves Γ_j's ($2 \le j \le n-1$) are not necessarily convex (cf. [15, Example 7]), and hence $\Lambda_j(A)$ does not necessarily coincide with the convex hull of Γ_j. However, by Theorem 1.2 and [15, Theorem 8], the former is always contained in the latter and when Γ_j ($1 \le j \le n/2$) is convex, they are equal to each other.

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