Linear Operators Preserving the Numerical Range (Radius) on Triangular Matrices

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Abstract

We characterize those linear operators on triangular or diagonal matrices preserving the numerical range or radius.

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1 Introduction

Let $\mathbf{M}_n$ be the algebra of $n \times n$ complex matrices. The *numerical range* of $A \in \mathbf{M}_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbf{C}^n, x^*x = 1\}$$

and the *numerical radius* of A is defined by

$$w(A) = \sup\{|z| : z \in W(A)\}.$$

These concepts and their generalizations have been studied extensively because of their connections and applications to many different areas (see e.g. Chapter 1 of [3]). There has been considerable interest in studying those linear operators L which preserve the numerical range or radius, i.e.,

$$W(L(A)) = W(A) \qquad \text{for all } A,$$

or

$$w(L(A)) = w(A) \qquad \text{for all } A.$$

For instance, one may see [1, 2, 4, 5, 6, 7, 9, 10, 11] and Chapter 5 of [12]. The purpose of this note is to solve these problems on $\mathbf{T}_n$, the algebra of $n \times n$ triangular matrices, and $\mathbf{D}_n$, the algebra of $n \times n$ diagonal matrices.

We will use $\{e_1, \ldots, e_n\}$ to denote the standard basis for $\mathbf{C}^n$, and use $\{E_{11}, E_{12}, \ldots, E_{nn}\}$ to denote the standard basis for $\mathbf{M}_n$.

2 Preliminaries

In the following, we collect some basic results on $W(A)$ and $w(A)$ that are useful in our study. One may see Chapter 1 of [3] for general background.

Proposition 2.1 [3, §1.2] *Let $A \in \mathbf{M}_n$.*

(a) $W(A) = W(A^t)$.

(b) $W(A) = W(U^*AU)$ *for any unitary* U.

(c) $W(\lambda A) = \lambda W(A)$ *for any* $\lambda \in \mathbf{C}$.

(d) $W(\lambda I + A) = \lambda + W(A)$ *for any* $\lambda \in \mathbf{C}$.

Proposition 2.2 [3, §1.3] *The numerical range of $A \in \mathbf{M}_n$ is always convex. In particular, if $A \in \mathbf{M}_2$ is unitarily similar to $\begin{pmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{pmatrix}$, then $W(A)$ is an elliptical disk with λ_1 and λ_2 as foci, and length of minor axis equal to $|b|$.*

Proposition 2.3 [3, 1.2.9 and 1.2.10] *If $A \in \mathbf{M}_n$ is unitarily similar to $A_1 \oplus A_2$, then $W(A) = \mathrm{conv}\,\{W(A_1) \cup W(A_2)\}$. Hence, if A is unitarily similar to $\mathrm{diag}\,(a_1, \ldots, a_n)$, then $W(A) = \mathrm{conv}\,\{a_1, \ldots, a_n\}$.*

Proposition 2.4 [3, 1.6.3 and 1.6.4] *Let $A \in \mathbf{M}_n$ and $\lambda \in \mathbf{C}$. Then λ is a non-differentiable boundary point of $W(A)$ if and only if A is unitarily similar to $\lambda I_k \oplus A_2$ such that $\lambda \notin W(A_2)$. Hence, if $W(A)$ is an n-side convex polygonal disk with vertices $\lambda_1, \ldots, \lambda_n$, then A is normal with eigenvalues $\lambda_1, \ldots, \lambda_n$.*

Proposition 2.5 [3, Problem 1 in §1.2] *Let $A \in \mathbf{M}_n$. Then $W(A) = \{\lambda\}$ if and only if $A = \lambda I$.*

Proposition 2.6 [3, 1.2.11] *Suppose B is a principal submatrix of $A \in \mathbf{M}_n$. Then $W(B) \subseteq W(A)$ and $w(B) \le w(A)$.*

Proposition 2.7 [3, 1.6.1] *Suppose $A \in \mathbf{M}_n$ is nonscalar. Then $\mathrm{tr}\, A/n$ belongs to the relative interior of $W(A)$.*

Proposition 2.8 [3, 1.2.5] *Suppose $A \in \mathbf{M}_n$ so that $A + A^*$ has λ_1 and λ_n as the largest and smallest eigenvalues. Then*

$$[\lambda_n, \lambda_1] = \{z + \bar{z} : z \in W(A)\}.$$

3 Linear Preservers of $W(A)$

This section is devoted to studying the linear preservers of $W(A)$ on $\mathbf{D}_n$ or $\mathbf{T}_n$. We begin with the following relatively easy result.

Theorem 3.1 *A linear operator $L : \mathbf{D}_n \to \mathbf{D}_n$ satisfies $W(L(A)) = W(A)$ for all $A \in \mathbf{D}_n$ if and only if there is a permutation matrix P such that L is of the form $A \mapsto P^tAP$.*

Proof. The sufficiency part is clear by Proposition 2.1. Conversely, suppose L is a linear preserver of $W(A)$ on $\mathbf{D}_n$. First, we show that $L(E_{11}) = E_{jj}$ for some $1 \le j \le n$. To this end, let $\mu = e^{i\pi/(2n)}$ and $A(t) = E_{11} + t(\sum_{j=2}^n \mu^{n+j}E_{jj})$ for $t > 0$. Then $W(A(t)) = W(L(A(t)))$ is a convex polygonal disk with vertices $1, t\mu^{n+2}, \dots, t\mu^{2n}$. By Proposition 2.4, we see that $L(A(t)) = B(t) \in \mathbf{D}_n$ has eigenvalues $1, t\mu^{n+2}, \dots, t\mu^{2n}$. Now, for any $t_1, t_2 \in (0, 1)$, we have $A(t_1), A(t_2), (A(t_1) + A(t_2))/2 = A((t_1 + t_2)/2) \in \mathbf{D}_n$ with eigenvalues $1, t\mu^{n+2}, \dots, t\mu^{2n}$ for $t = t_1, t_2$ and $(t_1 + t_2)/2$, respectively. We conclude that $B(t_1), B(t_2)$ and $(B(t_1) + B(t_2))/2$ have the same properties, and hence $B(t) = E_{j_1,j_1} + t(\sum_{s=2}^n \mu^{n+s}E_{j_s,j_s})$ for some permutation $(j_1, \dots, j_n)$ of $(1, \dots, n)$. Taking the limit $t \to 0^+$, we see that $L(E_{11}) = E_{j_1,j_1}$. By the same argument, we can show that if $r = 2, \dots, n$, then $L(E_{rr}) = E_{j_r,j_r}$ for some $1 \le j_r \le n$. We claim that $j_1, \dots, j_n$ are all different. If it is not true, i.e., $L(E_{pp}) = L(E_{qq}) = E_{j_p,j_p}$ for some $p \ne q$, then $L(E_{pp} + E_{qq}) = 2E_{j_p,j_p}$. But then $W(E_{pp} + E_{qq}) = [0, 1] \ne [0, 2] = W(2E_{j_p,j_p})$ by Proposition 2.3, which is a contradiction. Consequently, $L(\sum_{s=1}^n \mu_sE_{ss}) = \sum_{s=1}^n \mu_sE_{j_s,j_s}$. The result follows. $\square$

As pointed out by Professor Jor-Ting Chan, Theorem 3.1 actually follows easily from Theorem 4.1 in the next section, whose proof depends on the knowledge of the isometry for the sup norm on $\mathbf{C}^n$. Since the above proof of Theorem 3.1 is independent and not very long, and it contains some basic techniques illustrating how to use geometrical properties of the numerical range to study linear preservers, we include it in our discussion.

Theorem 3.2 *A linear operator $L : \mathbf{T}_n \to \mathbf{T}_n$ satisfies $W(L(A)) = W(A)$ for all $A \in \mathbf{T}_n$ if and only if there is a unitary matrix $D \in \mathbf{D}_n$ such that L is of the form $A \mapsto D^*AD$ or $A \mapsto D^*A'D$, where A' is the "transpose" of A with respect to the anti-diagonal, i.e, $A' = D_0A^tD_0$ with $D_0 = E_{1n} + E_{2,n-1} + \cdots + E_{n1}$.*

Proof. By Proposition 2.1, $W(D^*AD) = W(A)$, $W(A) = W(D_0^*AD_0) = W(D_0AD_0)$ and $W(A) = W(A^t)$. Hence, if L is of the form $A \mapsto D^*AD$ or $A \mapsto D^*D_0A^tD_0D$, then $L(\mathbf{T}_n) = \mathbf{T}_n$ and $W(L(A)) = W(A)$. The sufficiency part follows.

Conversely, let L be a numerical range preserver on $\mathbf{T}_n$. By Proposition 2.5, $W(A) = \{\lambda\}$ if and only if $A = \lambda I$. It follows that $L(I) = I$. Furthermore, $L(A) = 0$ if and only if $A = 0$, and hence L is invertible.

We prove that L is of the asserted form by induction on $n \ge 2$. Suppose $n = 2$. Then $A \in \mathbf{T}_2$ satisfies $W(A) = [0, 1]$ if and only if $A = E_{jj}$ for $j = 1$ or 2 by Proposition 2.2. Thus we see that $\{L(E_{11}), L(E_{22})\} = \{E_{11}, E_{22}\}$. Furthermore, by Proposition 2.2 again, we see that for $A \in \mathbf{T}_2$, $W(A)$ is a circular disk centered at 0 with radius one if and only if $A = \mu E_{12}$ with $|\mu| = 2$. Hence, $L(E_{12}) = \mu_0 E_{12}$ for some μ_0 with $|\mu_0| = 1$.

Let $D = E_{11} + \mu_0 E_{22}$. One easily checks that L is of the form $A \mapsto D^*AD$ or $D^*A'D$ depending on $L(E_{11}) = E_{11}$ or $L(E_{11}) = E_{22}$.

Now, suppose $n \geq 3$ and assume that numerical range preservers on $\mathbf{T}_k$ with $k \leq n-1$ are of the asserted form. Consider a numerical range preserver L on $\mathbf{T}_n$. We show that L is of the standard form by proving a number of assertions.

Assertion 1. There is a permutation matrix P such that $L(A) = P^tAP$ for any $A \in \mathbf{D}_n$.

Proof. This statement follows from Proposition 2.4 and Theorem 3.1.

By Assertion 1, we see that $L(E_{11}) = E_{jj}$ for some $1 \leq j \leq n$. For $1 \leq k \leq n$, let $\mathbf{V}_k$ be the subspace of $\mathbf{T}_n$ consisting of matrices with the kth row and kth column equal to zero row and column, respectively. Furthermore, let $X(j)$ be the matrix obtained from $X \in \mathbf{T}_n$ by removing the jth row and jth column. We have the following.

Assertion 2. Suppose $L(E_{11}) = E_{jj}$. Then $L(\mathbf{V}_1) \subseteq \mathbf{V}_j$. Moreover, the mapping $\tilde{L} : \mathbf{T}_{n-1} \to \mathbf{T}_{n-1}$ defined by $\tilde{L}(A) = L([\operatorname{tr} A/(n-1)] \oplus A)(j)$ preserves the numerical range and is of the asserted form by the induction assumption.

Proof. For any $E_{pq} \in \mathbf{V}_1$, if $A(\mu) = E_{11} + \mu E_{pq}$ with $|\mu| = 1$, then $1 = w(A(\mu)) = w(L(A(\mu)))$. If the (j,j) entry of $L(E_{pq})$ is nonzero, then we can find $\mu \in \mathbf{C}$ with $|\mu| = 1$ such that the (j,j) entry of $L(E_{11} + \mu E_{p,q})$ is a positive real number larger than 1. Consequently, $w(A(\mu)) = 1 < w(L(A(\mu)))$, which is a contradiction. So, the (j,j) entry of $L(E_{pq})$ must be 0. Now, if $L(E_{pq}) \notin \mathbf{V}_j$, then $L(E_{11} + E_{pq})$ has a nonzero entry at the (j,k) or (k,j) position for some $k \neq j$. But then the 2×2 submatrix B of $L(E_{11} + E_{pq})$ lying in the jth and kth rows and columns will be an upper triangular matrix with nonzero $(1,2)$ entry and with 1 as a diagonal entry. By Proposition 2.2, $W(B)$ is a nondegenerate elliptical disk with 1 as a focus. By Proposition 2.6, we have $w(E_{11} + E_{pq}) = 1 < w(B) \leq w(L(E_{11} + E_{pq}))$, which is a contradiction. Hence, $L(\mathbf{V}_1) \subseteq \mathbf{V}_j$ as asserted.

Now, suppose $A \in \mathbf{T}_{n-1}$. By Proposition 2.7, $\operatorname{tr} A/(n-1)$ lies in the relative interior of $W(A)$. Note that $W(L([\operatorname{tr} A/(n-1)] \oplus A)) = \operatorname{conv}\{\{\operatorname{tr} A/(n-1)\} \cup W(\tilde{L}(A))\}$ by Proposition 2.3. If $\operatorname{tr} A/(n-1) \notin W(\tilde{L}(A))$, then $\operatorname{tr} A/(n-1)$ is a non-differentiable boundary point of $W(L([\operatorname{tr} A/(n-1)] \oplus A)) = W([\operatorname{tr} A/(n-1)] \oplus A) = W(A)$, which is a contradiction. Thus, $\operatorname{tr} A/(n-1) \in W(\tilde{L}(A))$ and

$$\begin{aligned} W(A) &= W([\operatorname{tr} A/(n-1)] \oplus A) = W(L([\operatorname{tr} A/(n-1)] \oplus A)) \\ &= \operatorname{conv}\{\{\operatorname{tr} A/(n-1)\} \cup W(\tilde{L}(A))\} = W(\tilde{L}(A)). \end{aligned}$$

Assertion 3. We have $L(E_{11}) \in \{E_{11}, E_{nn}\}$ or $L(E_{nn}) \in \{E_{11}, E_{nn}\}$. Moreover, one of the following operators will preserve the numerical range and map E_{11} to E_{11}:

$$\text{(a) } A \mapsto L(A), \ \text{(b) } A \mapsto L(A)', \ \text{(c) } A \mapsto L(A'), \ \text{(d) } A \mapsto L(A')'.$$

Proof. Suppose $L(E_{11}) = E_{jj}$ such that $j \neq 1, n$. Then the operator $\tilde{L}$ defined in Assertion 2 preserves the numerical range and is of the asserted form. Suppose $\tilde{L}(B) = D_1^* B D_1$. Then $L(E_{nn}) = E_{nn}$. Suppose $\tilde{L}(B) = D_1^* B' D_1$. Then $L(E_{nn}) = E_{11}$. Hence, the first assertion is true. The second assertion follows readily from the first one.

By the above assertion, we may assume that $L(E_{11}) = E_{11}$. Otherwise, replace it by one of the three operators in (b) – (d).

Assertion 4. Suppose $L(E_{11}) = E_{11}$. Then L is of the form $A \mapsto D^* A D$ for some unitary $D \in \mathbf{D}_n$.

Proof. Define $\tilde{L}$ on $\mathbf{T}_{n-1}$ as in Assertion 2, i.e., $\tilde{L}(B) = L([\operatorname{tr} B/(n-1)] \oplus B)(1)$. Then by Assertion 2 $\tilde{L}$ preserves the numerical range, and hence is of the form

$$\text{(i) } B \mapsto D_1^* B D_1 \quad \text{or} \quad \text{(ii) } B \mapsto D_1^* B' D_1$$

for some unitary $D_1 \in \mathbf{D}_{n-1}$. In both cases, we assume that $D_1 = I$. Otherwise, replace L by the map $A \mapsto \tilde{D}_1 L(A) \tilde{D}_1^*$ with $\tilde{D}_1 = [1] \oplus D_1$. Thus, $\tilde{L}$ is the identity map, or of the form $B \mapsto B'$.

Suppose $n = 3$. We show that (ii) cannot hold. If it does hold, then $L(E_{22}) = E_{33}$ and hence $L(\mathbf{V}_2) = \mathbf{V}_3$. By arguments similar to those in the proof of Assertion 2, we see that the restriction of L from $\mathbf{V}_2$ to $\mathbf{V}_3$ induces a linear preserver of the numerical range on $\mathbf{T}_2$. Thus, there exists μ_1 with $|\mu_1| = 1$ such that $L(aE_{11} + bE_{13} + cE_{33}) = aE_{11} + \mu_1 bE_{12} + cE_{22}$ for any $a, b, c \in \mathbf{C}$. Similarly, we can consider the restriction of L on $\mathbf{V}_3$ and conclude that there exists μ_2 with $|\mu_2| = 1$ such that $L(aE_{11} + bE_{12} + cE_{22}) = aE_{11} + \mu_2 bE_{13} + cE_{33}$ for any $a, b, c \in \mathbf{C}$. Hence, for

$$A = a_1 E_{11} + a_2 E_{22} + a_3 E_{33} + b_1 E_{12} + b_2 E_{23} + c_1 E_{13},$$

we have

$$L(A) = a_1 E_{11} + a_2 E_{33} + a_3 E_{22} + \mu_2 b_1 E_{13} + b_2 E_{23} + \mu_1 c_1 E_{12}.$$

We may replace L by the mapping $A \mapsto D^* L(A) D$ with $D = \operatorname{diag}(\mu_1, 1, 1)$. Then, we have

$$L(A) = a_1 E_{11} + a_2 E_{33} + a_3 E_{22} + \mu b_1 E_{13} + b_2 E_{23} + c_1 E_{12}$$

with $\mu = \mu_2/\mu_1$. Now, consider

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad L(A) = \begin{pmatrix} 0 & 1 & \mu \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

One can check that $A + A^*$ has eigenvalues $3, 0, -1$. Since $W(A) = W(L(A))$, by Proposition 2.8 $L(A) + L(A)^*$ has 3 and -1 as the largest and smallest eigenvalues. Now $\det(L(A) + L(A)^* - 3I) = 0$ implies that $\mu = 1$. Next, consider

$$A = \begin{pmatrix} 0 & 1+i & i \\ 0 & 0 & 1-i \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad L(A) = \begin{pmatrix} 0 & i & 1+i \\ 0 & 0 & 1-i \\ 0 & 0 & 0 \end{pmatrix}.$$

One can check that $A + A^*$ has eigenvalues $\sqrt{5}, 0, -\sqrt{5}$. Since $W(A) = W(L(A))$, by Proposition 2.8 $L(A) + L(A)^*$ has $\pm\sqrt{5}$ as the largest and smallest eigenvalues. However, $\det(L(A) + L(A)^* - \sqrt{5}I) \neq 0$, which is a contradiction. Thus, (ii) cannot hold.

Now, we see that condition (i) is valid. Then $L(E_{pq}) = E_{pq}$ for $p = q$ and $(p, q) = (2, 3)$. Considering the restriction of L on $\mathbf{V}_2$ and $\mathbf{V}_3$, we conclude that $L(E_{12}) = \mu_1 E_{12}$ and $L(E_{13}) = \mu_2 E_{13}$ for some $\mu_1, \mu_2 \in \mathbf{C}$ with $|\mu_1| = |\mu_2| = 1$. Again, we may replace L by the mapping $A \mapsto D^* L(A) D$ with $D = \operatorname{diag}(\mu_1, 1, 1)$. Then, we have $L(E_{pq}) = E_{pq}$ for all $p = q$ and $(p, q) = (1, 2), (2, 3)$, and $L(E_{13}) = \mu E_{13}$ with $\mu = \mu_2/\mu_1$. Now consider

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad L(A) = \begin{pmatrix} 0 & 1 & \mu \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $A + A^*$ has eigenvalues $3, 0, -1$. By Proposition 2.8, $L(A) + L(A)^*$ has 3 and -1 as the largest and smallest eigenvalues. Since $\det(L(A) + L(A)^* - 3I) = 0$ implies that $\mu = 1$, we see that L is of the asserted form.

Next, suppose $n \geq 4$. We claim that (ii) cannot hold. If it does hold, then $L(E_{22}) = E_{nn}$ and hence $L(\mathbf{V}_2) = \mathbf{V}_n$. By arguments similar to those in the proof of Assertion 2, we see that the restriction of L from $\mathbf{V}_2$ to $\mathbf{V}_n$ induces a linear preserver of the numerical range on $\mathbf{T}_{n-1}$. By induction assumption, the induced map is of the asserted form. However, we have $L(E_{11}) = E_{11}$ and $L(E_{jj}) = E_{n-j+2,n-j+2}$ for $j = 3, \ldots, n$, which is a contradiction. Thus, condition (i) must hold.

Now, $L(\mathbf{V}_2) = \mathbf{V}_2$ and the restriction of L on $\mathbf{V}_2$ induces a linear preserver of the numerical range on $\mathbf{T}_{n-1}$. By induction assumption, the induced map is of the asserted form. Since $L(E_{ij}) = E_{ij}$ for $i, j \geq 3$, we conclude that there exist $\mu_1 \in \mathbf{C}$ with $|\mu_1| = 1$ and $X = [\mu_1] \oplus I_{n-2}$ so that the induced map is of the form $A \mapsto X^* A X$ for all $A \in \mathbf{T}_{n-1}$. Similarly, we can conclude that there exist $\mu_2 \in \mathbf{C}$ with $|\mu_2| = 1$ and $Y = [\mu_2] \oplus I_{n-2}$ so that the map induced by the restriction of L on $\mathbf{V}_3$ is of the form $A \mapsto Y^* A Y$ for all $A \in \mathbf{T}_{n-1}$. Notice that we have $L(E_{14}) = \mu_1 E_{14}$ by the induced map on $\mathbf{V}_2$ and $L(E_{14}) = \mu_2 E_{14}$ by the induced map on $\mathbf{V}_3$. Thus $\mu_1 = \mu_2$, and the result follows. $\square$

4 Numerical radius preservers

First of all, we consider the numerical radius preservers on $\mathbf{D}_n$.

Theorem 4.1 *A linear operator $L : \mathbf{D}_n \to \mathbf{D}_n$ satisfies $w(L(A)) = w(A)$ for all $A \in \mathbf{D}_n$ if and only if there is a permutation matrix P and a unitary matrix $D \in \mathbf{D}_n$ such that L is of the form $A \mapsto DPAP^t$.*

Proof. Note that L preserves the numerical radius on $\mathbf{D}_n$ if and only if the induced map $\tilde{L}$ on $\mathbf{C}^n$ defined by $\tilde{L}((\mu_1, \ldots, \mu_n)^t) = (\nu_1, \ldots, \nu_n)^t$, where $L(\sum_{j=1}^n \mu_j E_{jj}) = \sum_{j=1}^n \nu_j E_{jj}$, preserves the sup norm. It is well-known that a linear isometry $\tilde{L}$ of the sup norm on $\mathbf{C}^n$ must be of the form $v \mapsto DPv$ for some unitary $D \in \mathbf{D}_n$ and permutation matrix P. One can check that the corresponding map L on $\mathbf{D}_n$ is of the form $A \mapsto DPAP^t$. $\Box$

To characterize numerical radius preservers on $\mathbf{T}_n$, we need the following characterization of scalar matrices, which is of independent interest.

Proposition 4.2 *A matrix $A \in \mathbf{T}_n$ is a scalar matrix if and only if for any $B \in \mathbf{T}_n$ there is $\mu \in \mathbf{C}$ with $|\mu| = 1$ such that $w(A + \mu B) = w(A) + w(B)$.*

Proof. For each $A \in \mathbf{M}_n$ define

$$S_A = \{x \in \mathbf{C}^n : x^*x = 1 \text{ and } |x^*Ax| = w(A)\}.$$

Then it is clear that for any $A, B \in \mathbf{M}_n$, there exists $\mu \in \mathbf{C}$ with $|\mu| = 1$ such that $w(A + \mu B) = w(A) + w(B)$ if and only if $S_A \cap S_B \neq \emptyset$.

Suppose $A = \lambda I$. Then $S_A \cap S_B \neq \emptyset$ for any $B \in \mathbf{T}_n$. Hence there is $\mu \in \mathbf{C}$ with $|\mu| = 1$ such that $w(A + \mu B) = w(A) + w(B)$.

Conversely, suppose $A \in \mathbf{T}_n$ is such that $S_A \cap S_B \neq \emptyset$ for all $B \in \mathbf{T}_n$. For $1 \leq j \leq n$ and $B = E_{jj}$, we have $S_B = \{\mu e_j : \mu \in \mathbf{C},\ |\mu| = 1\}$. Since $S_A \cap S_B \neq \emptyset$, we conclude that the (j,j) entry a_j of A has modulus $w(A)$. Furthermore, the (j,k) entry of A must be 0 for $k > j$. Otherwise, we can let B be the 2×2 principal submatrix of A lying in the jth and kth rows and columns, and conclude that a_j is a focus of the nondegenerate elliptical disk $W(B)$. It follows that $|a_j| < w(B) \leq w(A)$, which is a contradiction.

Finally, consider $B = E_{1j}$ for $2 \leq j \leq n$. We have

$$S_B = \{\mu_1 e_1 + \mu_j e_j : |\mu_1| = |\mu_j| = 1/\sqrt{2}\}.$$

Since $S_A \cap S_B \neq \emptyset$, we have $|a_1 + a_j|/2 = w(A) = |a_1| = |a_j|$. It follows that $a_1 = a_j$ for all $2 \leq j \leq n$. $\Box$

We are now ready to state and prove the following result on numerical radius preservers.

Theorem 4.3 *A linear operator $L : \mathbf{T}_n \to \mathbf{T}_n$ satisfies $w(L(A)) = w(A)$ for all $A \in \mathbf{T}_n$ if and only if there is a unitary matrix $D \in \mathbf{D}_n$ and a complex number ξ with $|\xi| = 1$ such that L is of the form $A \mapsto \xi D^*AD$ or $A \mapsto \xi D^*A'D$, where A' is the "transpose" of A with respect to the anti-diagonal, i.e, $A' = D_0A^tD_0$ with $D_0 = E_{1n} + E_{2,n-1} + \cdots + E_{n1}$.*

Proof. The sufficiency part is clear. We consider the necessity part. Let L be a numerical radius preserver on $\mathbf{T}_n$. If $L(A) = 0$, then $0 = w(L(A)) = w(A)$, and hence $A = 0$. Thus L is invertible. Furthermore, L^{-1} also preserves the numerical radius.

Let $A = L(I)$. Then for any $B \in \mathbf{T}_n$ there exists $\mu \in \mathbf{C}$ such that

$$w(A + \mu B) = w(I + \mu L^{-1}(B)) = w(I) + w(L^{-1}(B)) = w(A) + w(B).$$

Thus $L(I) = \xi I$ for some $\xi \in \mathbf{C}$ by Proposition 4.2. Since $w(I) = w(\xi I)$, we see that $|\xi| = 1$.

We may assume that $\xi = 1$; otherwise, replace L by the mapping $A \mapsto \xi^{-1}L(A)$. We show that L actually preserves the numerical range on $\mathbf{T}_n$. The result will then follow from Theorem 3.2.

Suppose there exists $A \in \mathbf{T}_n$ such that $W(L(A)) \neq W(A)$. Then (i) there exists $\mu \in W(L(A)) \backslash W(A)$, or (ii) there exists $\mu \in W(A) \backslash W(L(A))$. Suppose (i) holds. Since $W(A)$ is convex and compact, there exists a circle with sufficiently large radius centered at a certain $\lambda \in \mathbf{C}$ so that $W(A)$ lies inside the circle, but μ lies outside the circle. Hence, for any $z \in W(A)$, we have $|z - \lambda| < |\mu - \lambda|$. Consequently,

$$w(A - \lambda I) < |\mu - \lambda| \leq w(L(A) - \lambda I) = w(L(A - \lambda I)),$$

which is a contradiction.

If (ii) holds, we can apply the same argument to L^{-1} to get a contradiction. Thus, we have $W(A) = W(L(A))$ for all $A \in \mathbf{T}_n$ as asserted. □

5 Remarks

In general, linear preservers on $\mathbf{T}_n$ may not have nice structures. For example, for spectrum preservers, invertibility preservers, etc., one can only have some information about the diagonal entries, but have no control on the strictly upper triangular part. Nonetheless, it is worth noting that linear preservers of numerical range and radius on $\mathbf{T}_n$ have nice structure.

By Theorem 2.2 in [10], L preserves the numerical range on the matrix space $\mathbf{V} = \mathbf{M}_n, \mathbf{T}_n$ or $\mathbf{D}_n$ if and only if the dual transformation L^* preserves the state space defined by

$$S = \{C \in \mathbf{V} : \operatorname{tr} C = 1 \geq |\operatorname{tr} C^*A| \text{ whenever } A \in \mathbf{V} \text{ satisfies } \|A\| \leq 1\}.$$

Similar idea has been used in [7]. In any event, one can get a corollary on the dual linear transformation for each linear preserver of $W(A)$. For $\mathbf{V} = \mathbf{M}_n$ or $\mathbf{D}_n$, there is a simple description of S, namely, it is the collection of trace one positive semidefinite matrices in $\mathbf{V}$. However, for $\mathbf{V} = \mathbf{T}_n$, there does not seem to have an easy description of S.

There are many generalizations of the numerical range and numerical radius. It would be interesting to consider the corresponding preserver problems on $\mathbf{T}_n$ and $\mathbf{D}_n$. For example, one may consider the k-numerical range and radius preservers on $\mathbf{T}_n$ (cf. [4, 9, 11]).

One may also consider extending the results to infinite dimensional context, for instance, to nested algebras.

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