

HIGHER RANK NUMERICAL RANGES OF NORMAL MATRICES

HWA-LONG GAU, CHI-KWONG LI, YIU-TUNG POON, AND NUNG-SING SZE*

ABSTRACT. The higher rank numerical range is closely connected to the construction of quantum error correction code for a noisy quantum channel. It is known that if a normal matrix $A \in M_n$ has eigenvalues a_1, \dots, a_n , then its higher rank numerical range $\Lambda_k(A)$ is the intersection of convex polygons with vertices $a_{j_1}, \dots, a_{j_{n-k+1}}$, where $1 \leq j_1 < \dots < j_{n-k+1} \leq n$. In this paper, it is shown that the higher rank numerical range of a normal matrix with m distinct eigenvalues can be written as the intersection of no more than $\max\{m, 4\}$ closed half planes. In addition, given a convex polygon \mathcal{P} a construction is given for a normal matrix $A \in M_n$ with minimum n such that $\Lambda_k(A) = \mathcal{P}$. In particular, if \mathcal{P} has p vertices, with $p \geq 3$, there is a normal matrix $A \in M_n$ with $n \leq \max\{p + k - 1, 2k + 2\}$ such that $\Lambda_k(A) = \mathcal{P}$.

1. INTRODUCTION

Let M_n be the algebra of $n \times n$ complex matrices regarded as linear operators acting on the n -dimensional Hilbert space \mathbb{C}^n . The *classical numerical range* of $A \in M_n$ is defined and denoted by

$$W(A) = \{x^*Ax \in \mathbb{C} : x \in \mathbb{C}^n \text{ with } x^*x = 1\},$$

which is a useful concept in studying matrices and operators; see [6].

In the context of quantum information theory, if the quantum states are represented as matrices in M_n , then a *quantum channel* is a trace preserving completely positive map $L : M_n \rightarrow M_n$ with the following operator sum representation

$$(1.1) \quad L(A) = \sum_{j=1}^r E_j^* A E_j,$$

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where $E_1, \dots, E_r \in M_n$ satisfy $\sum_{j=1}^r E_j E_j^* = I_n$. The matrices E_1, \dots, E_r are known as the *error operators* of the quantum channel L . A subspace V of \mathbb{C}^n is a *quantum error correction code* for the channel L if and only if the orthogonal projection $P \in M_n$ with range space V satisfies $P E_i^* E_j P = \gamma_{ij} P$ for all $i, j \in \{1, \dots, r\}$; for example, see [7, 8, 9]. In this connection, for $1 \leq k < n$ researchers define the *rank- k numerical range* of $A \in M_n$ by

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} : PAP = \lambda P \text{ for some rank-}k \text{ orthogonal projection } P\},$$

and the *joint rank- k numerical range* of $A_1, \dots, A_m \in M_n$ by $\Lambda_k(A_1, \dots, A_m)$ to be the collection of complex vectors $(a_1, \dots, a_m) \in \mathbb{C}^{1 \times m}$ such that $PA_j P = a_j P$ for a rank- k orthogonal projection $P \in M_n$. Evidently, there is a quantum error correction code V of dimension k for the quantum channel L described in (1.1) if and only if $\Lambda_k(A_1, \dots, A_m)$ is non-empty for $(A_1, \dots, A_m) = (E_1^* E_1, E_1^* E_2, \dots, E_r^* E_r)$. Also, it is easy to see that if $(a_1, \dots, a_m) \in \Lambda_k(A_1, \dots, A_m)$ then $a_j \in \Lambda_k(A_j)$ for $j = 1, \dots, m$. When $k = 1$, $\Lambda_k(A)$ reduces to the classical numerical range $W(A)$.

Recently, interesting results have been obtained for the rank- k numerical range and the joint rank- k numerical range; see [1, 2, 3, 4, 5, 11, 12, 13, 14, 16]. In particular, an explicit description of the rank- k numerical range of $A \in M_n$ is given in [14], namely,

$$(1.2) \quad \Lambda_k(A) = \bigcap_{\xi \in [0, 2\pi)} \{\mu \in \mathbb{C} : e^{-i\xi} \mu + e^{i\xi} \bar{\mu} \leq \lambda_k(e^{-i\xi} A + e^{i\xi} A^*)\},$$

where $\lambda_k(X)$ is the k th largest eigenvalue of a Hermitian matrix X .

In the study of quantum error correction, there are channels such as the randomized unitary channels and Pauli channels whose error operators are commuting normal matrices. Thus, it is of interest to study the rank- k numerical ranges of normal matrices. Although the error operators of a generic quantum channel may not commute, a good understanding of the special case would lead to deeper insights and more proof techniques for the general case.

Given $S \subseteq \mathbb{C}$, let $\text{conv } S$ denote the smallest convex subset of \mathbb{C} containing S . For a normal matrix $A \in M_n$ with eigenvalues a_1, \dots, a_n , it was conjectured in [3, 4] that

$$(1.3) \quad \Lambda_k(A) = \bigcap_{1 \leq j_1 < \dots < j_{n-k+1} \leq n} \text{conv} \{a_{j_1}, \dots, a_{j_{n-k+1}}\},$$

which is a convex polygon including its interior (if it is non-empty). This conjecture was confirmed in [14] using the description of $\Lambda_k(A)$ in (1.2).

In our discussion, a polygon would always mean a convex polygon with its interior.

In this paper, we improve the description (1.3) of the rank- k numerical range of a normal matrix. In particular, in Section 2 we show that for a normal matrix A with m distinct eigenvalues, $\Lambda_k(A)$ can be written as the intersection of no more than $\max\{m, 4\}$ closed half planes in \mathbb{C} . Moreover, if $\Lambda_k(A) \neq \emptyset$, then it is a polygon with no more than m vertices. We then consider the “inverse” problem, namely, for a given polygon \mathcal{P} , construct a normal matrix $A \in M_n$ with $\Lambda_k(A) = \mathcal{P}$. In other words, we study the necessary condition for the existence of quantum channels whose error operators have prescribed rank- k numerical ranges. It is easy to check that $\Lambda_k(\tilde{A}) = \mathcal{P}$ if $\tilde{A} = A \otimes I_k$ with $W(A) = \mathcal{P}$. Our goal is to find a normal matrix \hat{A} with smallest size so that $\Lambda_k(\hat{A}) = \mathcal{P}$. To achieve this, we give a necessary and sufficient condition for the existence of a normal matrix $A \in M_n$ so that $\Lambda_k(A) = \mathcal{P}$ in terms of k -regular sets in \mathbb{C} (see Definition 3.3). Furthermore, we show that the problem of finding a desired normal matrix A is equivalent to a combinatorial problem of extending a given p element set of unimodular complex numbers to a k -regular set. We then give the solution of the problem in Section 4. As a consequence of our results, if \mathcal{P} is a polygon with p vertices, then there is a normal matrix $A \in M_n$ with

$$n \leq \max \{p + k - 1, 2k + 2\}$$

such that $\Lambda_k(A) = \mathcal{P}$. Moreover, this upper bound is best possible in the sense that there exists \mathcal{P} so that there is no matrix of smaller dimension with rank- k numerical range equal to \mathcal{P} .

2. CONSTRUCTION OF HIGHER RANK NUMERICAL RANGES

By (1.2), $\Lambda_k(A)$ can be obtained as the intersection of infinitely many closed half planes for a given $A \in M_n$. Suppose A is normal. By (1.3), one can write $\Lambda_k(A)$ as the intersection of $\binom{n}{k-1}$ convex polygons so that $\Lambda_k(A)$ is a polygon. In particular, it is well known that $\Lambda_1(A) = \text{conv} \{a_1, \dots, a_m\}$, where a_1, \dots, a_m are the distinct eigenvalues of A .

There are nice interplay between the algebraic properties of $A \in M_n$ and the geometric properties of $\Lambda_1(A) = W(A)$. For instance, $\Lambda_1(A)$ is always non-empty; $\Lambda_1(A)$ is a singleton if and only if A is a scalar matrix; $\Lambda_1(A)$ is a non-degenerate line segment if and only if A is a non-scalar normal matrix and its eigenvalues lie on a straight line. Unfortunately, these results have no analogs for $\Lambda_k(A)$ if $k > 1$. First, the set $\Lambda_k(A)$ may be empty, see [13];

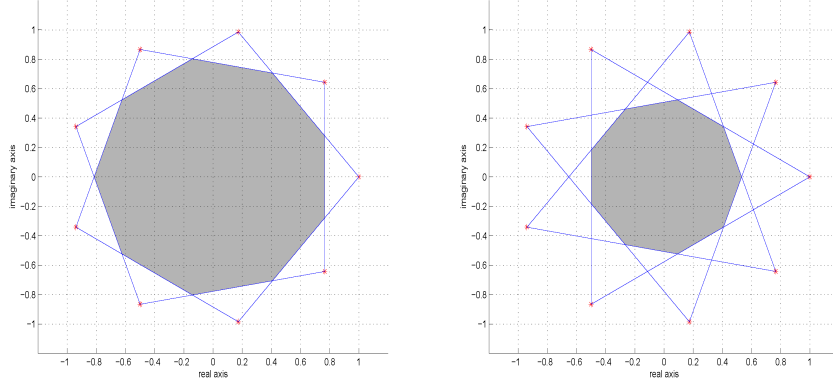
there are non-scalar matrices A such that $\Lambda_k(A)$ is a singleton; and there are non-normal matrices A such that $\Lambda_k(A)$ is a line segment. Even for a normal matrix A , it is not easy to determine whether $\Lambda_k(A)$ is empty, a point or a line segment without actually constructing the set $\Lambda_k(A)$. Moreover, there is no easy way to express the vertices of the polygon $\Lambda_k(A)$ (if it is non-empty) in terms of the eigenvalues of the normal matrix A as in the case of $\Lambda_1(A)$. Of course, one can use (1.3) to construct $\Lambda_k(A)$ for the normal matrix A , but the number of polygons needed in the construction will grow exponentially for large n and k . In the following, we will study efficient ways to generate $\Lambda_k(A)$ for a normal matrix $A \in M_n$. While it is difficult to use the eigenvalues of A to determine the set $\Lambda_k(A)$, it turns out that we can use half planes determined by the eigenvalues to generate $\Lambda_k(A)$ efficiently. In the following, we will focus on the following problem.

Problem 2.1. Determine the minimum number of half planes needed to construct $\Lambda_k(A)$ using the eigenvalues of the normal matrix $A \in M_n$.

As by-products, we will show that for a normal matrix A with m distinct eigenvalues, $\Lambda_k(A)$ is either empty or is a polygon with at most m vertices. In fact, by examining the location of the eigenvalues of A on the complex plane, one may further reduce the number of half planes needed to construct $\Lambda_k(A)$.

Suppose the eigenvalues of $A \in M_n$ are collinear. Then by a translation, followed by a rotation, we may assume that A is Hermitian with eigenvalues $a_1 \geq \dots \geq a_n$. Then we have $\Lambda_k(A) = [a_{n-k+1}, a_k]$. So we focus on those normal matrices whose eigenvalues are not collinear.

Let us motivate our result with the following examples, which can be verified by using (1.3).



$\Lambda_2(A)$ with $n = 9$ in Example 2.2 $\Lambda_3(A)$ with $n = 9$ in Example 2.2

Example 2.2. Let $A = \text{diag}(1, w, w^2, \dots, w^{n-1})$ with $w = e^{2\pi i/n}$. Then for $k \leq n/2$, we have $\Lambda_k(A) = \bigcap_{j=0}^{n-1} \mathcal{H}_j$, where

$$\mathcal{H}_j = \left\{ z \in \mathbb{C} : \text{Re} \left(e^{-\frac{(2j+k)\pi i}{n}} z \right) \leq \cos \frac{k\pi}{n} \right\},$$

and only a small part of $\text{conv}\{w^{j-1}, w^{j-1+k}\}$ lies in $\Lambda_k(A)$.

More generally, we have the following.

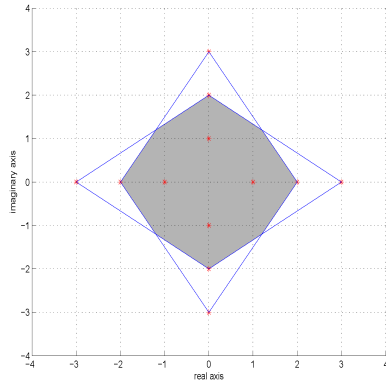
Example 2.3. Let a_1, \dots, a_n be the eigenvalues of $A \in M_n$, with $n \geq 3$. Suppose $\text{conv}\{a_1, \dots, a_n\} = \mathcal{P}$ is an n -sided convex polygon containing the origin in the interior. We may assume that a_1, \dots, a_n are arranged in the counterclockwise direction on the boundary of \mathcal{P} . For $j \in \{1, \dots, n\}$, let L_j be the line passing through a_j and a_{j+k} , where $a_{j+k} = a_{j+k-n}$ if $j+k > n$, and \mathcal{H}_j be the closed half plane determined by L_j which does not contain a_ℓ for $j < \ell < j+k$. Then

$$\Lambda_k(A) = \bigcap_{j=1}^n \mathcal{H}_j.$$

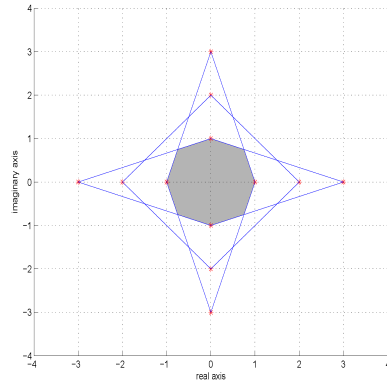
Note that each \mathcal{H}_j in Example 2.3 contains exactly $n - k + 1$ eigenvalues of A .

The situation is more complicated if $\Lambda_1(A)$ is not an n -sided convex polygon for the normal matrix $A \in M_n$.

Example 2.4. Suppose $B = \text{diag}(1, i, -1, -i, 2, 2i, -2, -2i, 3, 3i, -3, -3i)$. One can see from the figures that the eigenvalues $1, i, -1, -i$ are interior points of $\Lambda_2(B)$ while these eigenvalues are the vertices of $\Lambda_3(B)$.



$\Lambda_2(B)$



$\Lambda_3(B)$

To deal with normal matrices $A \in M_n$ as in Example 2.4 that $\Lambda_1(A)$ is not an n -side convex polygon, we need to construct some half spaces using the eigenvalues of the normal matrix A . To do this, we introduce the following. Given any two distinct complex numbers a and b , let $L(a, b)$ be the (directed) line passing through a and b . The closed half plane

$$H(a, b) = \{z \in \mathbb{C} : \operatorname{Im}((\bar{b} - \bar{a})(z - a)) \geq 0\}$$

is called the *left closed half plane* determined by $L(a, b)$. For example, $H(0, i) = \{z \in \mathbb{C} : \operatorname{Re}(z) \leq 0\}$ and $H(i, 0) = \{z \in \mathbb{C} : \operatorname{Re}(z) \geq 0\}$. Remark that in Example 2.2, the set \mathcal{H}_j is indeed the closed half plane $H(w^j, w^{j+k})$. Note that $L(a, b) \neq L(b, a)$. In our discussion, it is sometimes convenient to write

$$H(a, b) = \{z \in \mathbb{C} : \operatorname{Re}(e^{-i\xi}z) \leq \operatorname{Re}(e^{-i\xi}a)\}$$

with $\xi = \arg(b - a) - \pi/2$. Also, we use $H_0(a, b)$ to denote the *left open half plane* determined by $L(a, b)$, i.e., $H_0(a, b) = H(a, b) \setminus L(a, b)$.

We have the following result showing that for a normal matrix $A \in M_n$ with m distinct eigenvalues, $\Lambda_k(A)$ can be written as the intersection of at most $\max\{m, 4\}$ half spaces. Even without any knowledge about the final shape of the set $\Lambda_k(A)$, one can use $m(m - 1)$ half spaces to generate $\Lambda_k(A)$. Evidently, the construction is more efficient than the construction using (1.2) or (1.3). Furthermore, we can conclude that $\Lambda_k(A)$ is either an empty set, a singleton, a line segment, or a non-degenerate polygon with at most m vertices.

Theorem 2.5. *Let $A \in M_n$ be normal with distinct eigenvalues a_1, \dots, a_m that are not collinear. Let \mathcal{S} be the set of index pairs (r, s) such that $H(a_r, a_s)$ contains at least $n - k + 1$ eigenvalues (counting multiplicities) of A , and*

$$\mathcal{S}_0 = \{(r, s) \in \mathcal{S} : H_0(a_r, a_s) \text{ contains at most } n - k - 1 \text{ eigenvalues (counting multiplicities)}\}.$$

Then

$$(2.1) \quad \Lambda_k(A) = \bigcap_{(r,s) \in \mathcal{S}} H(a_r, a_s) = \bigcap_{(r,s) \in \mathcal{S}_0} H(a_r, a_s).$$

Moreover, $\Lambda_k(A)$ can be written as intersection of at most $\max\{m, 4\}$ half planes $H(a_r, a_s)$, with $(r, s) \in \mathcal{S}_0$.

Proof. In the first part of the proof, we assume that $A \in M_n$ has n eigenvalues a_1, \dots, a_n . For notational simplicity, we write $H(a_r, a_s) = H(r, s)$, $H_0(a_r, a_s) = H_0(r, s)$, and $L(a_r, a_s) = L(r, s)$ for any two distinct eigenvalues a_r and a_s of A . For each $(r, s) \in \mathcal{S}$, since $H(r, s)$ is convex and contains at least $n - k + 1$ eigenvalues of A , by (1.3), we have

$$\Lambda_k(A) = \bigcap_{1 \leq j_1 < \dots < j_{n-k+1} \leq n} \text{conv} \{a_{j_1}, \dots, a_{j_{n-k+1}}\} \subseteq H(r, s).$$

It follows that

$$(2.2) \quad \Lambda_k(A) \subseteq \bigcap_{(r,s) \in \mathcal{S}} H(r, s).$$

To prove the reverse inclusion of (2.2), note that if z is a point not in $\Lambda_k(A)$, then z will lie outside a convex polygon which equals the convex hull of $n - k + 1$ eigenvalues of A . So, it suffices to show that the convex hull \mathcal{W} of any $n - k + 1$ eigenvalues of A can be written as an intersection of half planes, $\mathcal{W} = \bigcap_{j=1}^{\ell} H(r_j, s_j)$ for some $(r_1, s_1), \dots, (r_\ell, s_\ell) \in \mathcal{S}$. We consider the following three cases.

Case 1 Suppose \mathcal{W} is a singleton. Then $\mathcal{W} = \{a_r\}$ for some eigenvalue a_r with multiplicity at least $n - k + 1$. Since the eigenvalues of A are non-collinear, there are eigenvalues a_s and a_t such that a_r, a_s , and a_t are not collinear. Then

$$\mathcal{W} = H(r, s) \cap H(s, r) \cap H(r, t) \cap H(t, r).$$

Case 2 Suppose \mathcal{W} is a non-degenerate line segment. In this case, $\mathcal{W} = \text{conv} \{a_r, a_s\}$ for some eigenvalues a_r and a_s with $a_r \neq a_s$. Since the eigenvalues of A are non-collinear, there is another eigenvalue a_t such that a_r, a_s , and a_t are not collinear. Without loss of generality, we assume that $a_t \in H(r, s)$. Otherwise, we interchange a_r and a_s . Then

$$\mathcal{W} = H(r, s) \cap H(s, r) \cap H(s, t) \cap H(t, r).$$

Case 3 Suppose \mathcal{W} is a non-degenerate polygonal disk. We may relabel the eigenvalues of A and assume that \mathcal{W} has vertices a_1, \dots, a_q arranged in the counterclockwise direction, where $q \geq 3$. For convenience of notation, we will let $a_{q+1} = a_1$ and $H(q, q+1) = H(q, 1)$. Then

$$\mathcal{W} = \bigcap_{1 \leq t \leq q} H(t, t+1).$$

Thus, the first equality in (2.1) is proved. To prove the second equality in (2.1), we claim the following.

Claim For each $(r, s) \in \mathcal{S} \setminus \mathcal{S}_0$, there exist two ordered pairs (r_1, s_1) and (r_2, s_2) in \mathcal{S}_0 such that $H(r_1, s_1) \cap H(r_2, s_2) \subseteq H_0(r, s)$.

Once the claim is proved, all the half planes $H(r, s)$ with $(r, s) \in \mathcal{S} \setminus \mathcal{S}_0$ are not needed in the intersection $\bigcap_{(r,s) \in \mathcal{S}} H(a_r, a_s)$ and hence the second equality in (2.1) holds.

To prove the claim, suppose $(r, s) \in \mathcal{S} \setminus \mathcal{S}_0$. Then $H_0(r, s)$ contains at least $n - k$ eigenvalues of A . By a translation followed by a rotation, we may assume that $H(r, s) = \{z \in \mathbb{C} : \text{Im } z \geq 0\}$ and we can relabel the index of eigenvalues so that for $1 \leq j \leq n - 1$, either $\text{Im } a_j > \text{Im } a_{j+1}$ or $\text{Im } a_j = \text{Im } a_{j+1}$ with $\text{Re } a_j \geq \text{Re } a_{j+1}$. Let

$$\mathcal{U} = \text{conv} \{a_1, \dots, a_{n-k}\} \quad \text{and} \quad \mathcal{V} = \text{conv} \{a_{n-k+1}, \dots, a_n\}.$$

Then \mathcal{U} and \mathcal{V} are disjoint if $a_{n-k} \neq a_{n-k+1}$ or $\mathcal{U} \cap \mathcal{V} = \{a_{n-k}\}$ if $a_{n-k} = a_{n-k+1}$. By the assumption, $\mathcal{U} \subseteq H_0(r, s)$ and $\{a_r, a_s\} \subseteq \mathcal{V}$. Define the set

$$\mathcal{W} = \text{conv} \{a_i - a_j : 1 \leq i \leq n - k < j \leq n\} = \{u - v : u \in \mathcal{U} \text{ and } v \in \mathcal{V}\},$$

which is a convex polygon. Note that $\mathcal{W} \subseteq \{z \in \mathbb{C} : \text{Im}(z) \geq 0\}$ since $\text{Im}(a_i - a_j) \geq 0$ for all $1 \leq i \leq n - k < j \leq n$. By the facts that \mathcal{U} and \mathcal{V} can intersect at at most one point, and the union $\mathcal{U} \cup \mathcal{V}$ cannot be contained in any line, the set \mathcal{W} does not lie in any line that passes through the origin, and the point 0 can only be either an extreme point of \mathcal{W} or is not in \mathcal{W} . Under these conditions, one can find two extreme points w_1 and w_2 in \mathcal{W} with $\text{Im}(\bar{w}_1 w_2) \neq 0$ such that

$$(2.3) \quad \text{Im}(\bar{w}_1 w) \geq 0 \geq \text{Im}(\bar{w}_2 w) \quad \text{for all } w \in \mathcal{W}.$$

Since w_1 is an extreme point in \mathcal{W} , there are eigenvalues $a_{s_1} \in \mathcal{U}$ and $a_{r_1} \in \mathcal{V}$ such that $w_1 = a_{s_1} - a_{r_1}$. Then (2.3) gives

$$\text{Im}(\bar{a}_{s_1} - \bar{a}_{r_1})(u - a_{r_1}) \geq 0 \quad \text{and} \quad \text{Im}(\bar{a}_{r_1} - \bar{a}_{s_1})(v - a_{s_1}) \geq 0$$

for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$, and thus, $\mathcal{U} \subseteq H(r_1, s_1)$ and $\mathcal{V} \subseteq H(s_1, r_1)$. With the fact that a_{r_1} and a_{s_1} lie in the line $L(r_1, s_1)$, the closed half plane $H(r_1, s_1)$ contains at least $n - k + 1$ eigenvalues of A while the open half plane $H_0(r_1, s_1)$ contains at most $n - k - 1$ eigenvalues only. Therefore, $(r_1, s_1) \in \mathcal{S}_0$. By a similar argument, one can show that there are eigenvalues $a_{r_2} \in \mathcal{U}$ and $a_{s_2} \in \mathcal{V}$ such that $w_2 = a_{r_2} - a_{s_2}$. Then (2.3) yields

$$\text{Im}(\bar{a}_{r_2} - \bar{a}_{s_2})(u - a_{s_2}) \leq 0 \quad \text{and} \quad \text{Im}(\bar{a}_{s_2} - \bar{a}_{r_2})(v - a_{r_2}) \leq 0$$

for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$, and thus, $\mathcal{U} \subseteq H(r_2, s_2)$ and $\mathcal{V} \subseteq H(s_2, r_2)$, and one can conclude that $(r_2, s_2) \in S_0$. Observe that the two lines $L(r_1, s_1)$ and $L(r_2, s_2)$ are not parallel as $\text{Im}(\bar{a}_{r_1} - \bar{a}_{s_1})(a_{s_2} - a_{r_2}) = \text{Im}(\bar{w}_1 w_2) \neq 0$. Using the fact that the two distinct eigenvalues a_r and a_s are in \mathcal{V} , which is contained in the intersection $H(s_1, r_1) \cap H(s_2, r_2)$, one can conclude that the intersection $H(r_1, s_1) \cap H(r_2, s_2)$ must lie in $H_0(r, s)$, the interior of $H(r, s)$. Therefore, the claim holds.

Next, we turn to the last part of the Theorem. It is trivial that if $\Lambda_k(A)$ is either an empty set, a singleton, or a non-degenerate line segment, then only at most 4 half planes are needed in the construction of $\Lambda_k(A)$.

Suppose A has m distinct eigenvalues a_1, \dots, a_m and $\Lambda_k(A)$ is a non-degenerate polygon. Let \mathcal{T} be a minimal subset of S_0 such that $\Lambda_k(A) = \bigcap_{(r,s) \in \mathcal{T}} H(r, s)$. Since \mathcal{T} is minimal, the half planes $H(r, s)$, $(r, s) \in \mathcal{T}$, are all distinct. We may further assume that for all $(r, s) \in \mathcal{T}$, $\{a_1, \dots, a_m\} \cap L(a_r, a_s) \subseteq \text{conv}\{a_r, a_s\}$. Since $\Lambda_k(A)$ is a non-degenerate polygon, for each $1 \leq t \leq m$, there exist at most two pairs $(r, s) \in \mathcal{T}$ such that $t \in \{r, s\}$. Therefore, \mathcal{T} contains at most m ordered pairs. \square

Example 2.6. Let $A = \text{diag}(0, 0, 1, 1, i)$. Then

$$\Lambda_2(A) = [0, 1] = H(0, 1) \cap H(1, 0) \cap H(1, i) \cap H(i, 0)$$

$$\Lambda_3(A) = \emptyset = H(1, 0) \cap H(1, i) \cap H(i, 0)$$

and the intersection of any 2 half planes $H(a_r, a_s)$ is non-empty. This example also shows that one cannot replace $\max\{m, 4\}$ by m in the conclusion in Theorem 2.5.

Example 2.7. Let $A = \text{diag}(1, -1, i, -i)$. Then

$$\Lambda_2(A) = \{0\} = H(1, -1) \cap H(-1, 1) \cap H(i, -i) \cap H(-i, i)$$

and $\Lambda_2(A)$ cannot be written as an intersection of less than 4 half planes $H(a_r, a_s)$.

Corollary 2.8. *Suppose $A \in M_n$ is normal such that $W(A)$ is an n -sided polygon containing the origin as its interior point. Let v_1, \dots, v_n be the vertices of $W(A)$ having arguments $0 \leq \xi_1 < \dots < \xi_n < 2\pi$. If $k < n/2$, then $\Lambda_k(A)$ is an n -sided convex polygon obtained by joining v_j and v_{j+k} , where $v_{j+k} = v_{j+k-n}$ if $j+k > n$.*

By Theorem 2.5, it is easy to see that the boundary of $\Lambda_k(A)$ are subsets of the union of line segments of the form $\text{conv}\{a_r, a_s\}$ such that a_r and a_s

satisfy the $H(a_r, a_s)$ condition. However, it is not easy to determine which part of the line segment actually belong to $\Lambda_k(A)$ as shown in Examples 2.2, 2.3, and 2.4. By Theorem 2.5, if the normal matrix $A \in M_n$ has m distinct eigenvalues, we need no more than $\max\{m, 4\}$ half planes $H(a_r, a_s)$ to generate $\Lambda_k(A)$. *Can one determine these half planes effectively?* We will answer this question by presenting an algorithm in Section 5 based on the discussion in this section.

3. MATRICES WITH PRESCRIBED HIGHER RANK NUMERICAL RANGES

We study the following problem in this section.

Problem 3.1. Let $k > 1$ be a positive integer, and let \mathcal{P} be a p -sided polygon in \mathbb{C} . Construct a normal matrix A with smallest size (dimension) such that $\Lambda_k(A) = \mathcal{P}$.

If \mathcal{P} degenerates to a line segment joining two points a_1 and a_2 . Then the smallest n to get a normal matrix with $\Lambda_k(A) = \mathcal{P}$ is $n = 2k$, if a_1 and a_2 are distinct and $n = k$ if $a_1 = a_2$. So we focus on the case when the polygon \mathcal{P} is non-degenerate.

A natural approach to Problem 3.1 is to reverse the construction of $\Lambda_k(A)$ in Example 2.3. Suppose we have a non-degenerate p -sided polygon \mathcal{P} , with vertices, v_1, \dots, v_p .

Without loss of generality, we may assume that 0 lies in the interior of \mathcal{P} and the arguments of v_j in $[0, 2\pi)$ are arranged in ascending order. Our goal is to use the support line L_j which passes through v_j, v_{j+1} for $j = 1, \dots, p$, where $v_{p+1} = v_1$, to construct $A = \text{diag}(a_1, \dots, a_p)$ such that $\Lambda_k(A) = \mathcal{P}$. Note that if the desired values a_1, \dots, a_p exist and are arranged in counter-clockwise direction, then (by proper numbering) the line L_j will coincide with the line passing through a_j and a_{j+k} , where $a_{j+k} = a_{j+k-p}$ if $j+k > p$. Consequently, a_j will lie at the intersection of L_j and L_{j-k} , where $L_{j-k} = L_{j-k+p}$ if $j-k < 0$. Consequently, there exists $A = \text{diag}(a_1, \dots, a_p)$ satisfying $\Lambda_k(A) = \mathcal{P}$ if the following hold.

- (1) $k < p/2$.
- (2) There exist $a_1, \dots, a_p \in \mathbb{C}$ such that
 - (2.a) $L_j \cap L_{j-k} = \{a_j\}$ for $j = 1, \dots, p$,
 - (2.b) a_1, \dots, a_p have arguments $\xi_1 < \dots < \xi_p$ in the interval $[\xi_1, \xi_1 + 2\pi)$ and 0 lie in the interior of their convex hull.

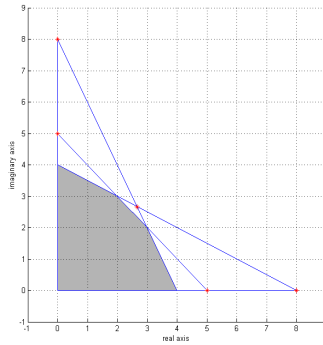
Note that by Theorem 2.5, A has the smallest dimension among all normal matrices B such that $\Lambda_k(B) = \mathcal{P}$.

Clearly, conditions (1) and (2a) are necessary in the above construction. From the following example, one can see that the above construction also fails when condition (2b) is not satisfied.

Example 3.2. Let \mathcal{P} be the 5-sided polygon with vertices $\{v_1, \dots, v_5\} = \{2 + i, 1 + 2i, -1 + 3i, -1 - i, 3 - i\}$, see the below figure. Then, with $k = 2$, we have

$$\{a_1, \dots, a_5\} = \{-1 + 4i, 7 - i, -1 + 7i, 4 - i, (5 + 5i)/3\},$$

which does not satisfy the condition (2b). Clearly, for $A = \text{diag}(a_1, \dots, a_5)$, $\Lambda_2(A)$ lies in the convex hull of $\{a_1, \dots, a_5\}$, which does not contain \mathcal{P} .



The polygon \mathcal{P}

Conditions (2a) and (2b) motivate the following definition.

Definition 3.3. Let $\Omega = \{z \in \mathbb{C} : |z| = 1\}$. A subset $\Pi = \{\alpha_1, \dots, \alpha_m\}$, with distinct $\alpha_1, \dots, \alpha_m \in \Omega$, is k -regular if every semi-circular arc of Ω without endpoints contains at least k elements in Π .

Given distinct $\alpha_1, \alpha_2 \in \Omega$, $\alpha_2/\alpha_1 = e^{i\theta}$ for a unique $0 < \theta < 2\pi$. Then $[\alpha_1, \alpha_2] = \{e^{it}\alpha_1 : 0 \leq t \leq \theta\}$ is the closed arc on Ω from α_1 to α_2 in the counterclockwise direction. Also define the open arc

$$(\alpha_1, \alpha_2) = [\alpha_1, \alpha_2] \setminus \{\alpha_1, \alpha_2\}.$$

The value θ is called the *length* of these intervals. Suppose $1 \leq k \leq n$ and $\Pi \subseteq \Omega$. Then Π is k -regular if for each $\alpha \in \Pi$, $(\alpha, -\alpha) \cap \Pi$ contains at least k elements.

Note that if $\Pi = \{e^{i\xi_j} : 1 \leq j \leq n\}$ with distinct $\xi_1, \dots, \xi_n \in [0, 2\pi)$, then Π is k -regular if and only if for each $r = 1, \dots, n$, there are $1 \leq j_1 < \dots <$

$j_k \leq n$ such that

$$(3.1) \quad e^{i\xi_{j_1}}, \dots, e^{i\xi_{j_k}} \in \left(e^{i\xi_r}, e^{i(\xi_r + \pi)} \right).$$

For this reason, a set $\{\xi_1, \dots, \xi_n\} \subseteq [0, 2\pi)$ of n distinct numbers is also called k -regular if $\{e^{i\xi_j} : 1 \leq j \leq n\}$ is k -regular as defined in Definition 3.3. For $\xi, \xi' \in [0, 2\pi)$, $[\xi, \xi']$ will denote the subset $\{t \in [0, 2\pi) : e^{it} \in [e^{i\xi}, e^{i\xi'}]\}$; the intervals $[\xi, \xi')$, $(\xi, \xi']$ and (ξ, ξ') will also be defined similarly.

In Example 2.2, a direct computation shows that for $1 \leq r, k \leq n$,

$$\xi_{r+k} - \xi_r = \begin{cases} 2k\pi/n & \text{if } r+k \leq n, \\ 2k\pi/n - 2\pi & \text{if } r+k > n. \end{cases}$$

Therefore, the set $\{\xi_1, \dots, \xi_n\}$ is k -regular and $\Lambda_k(A)$ is nonempty for $1 \leq k < n/2$. Otherwise, the set $\{\xi_1, \dots, \xi_n\}$ is not k -regular and $\Lambda_k(A)$ is either empty or a singleton.

In the following, we need an alternate formulation of (1.2). For any $d, \xi \in \mathbb{R}$, consider the closed half plane

$$(3.2) \quad \mathcal{H}(d, \xi) = \{\mu \in \mathbb{C} : \operatorname{Re}(e^{-i\xi}\mu) \leq d\},$$

and its boundary, which is the straight line

$$(3.3) \quad \mathcal{L}(d, \xi) = \partial\mathcal{H}(d, \xi) = \{\mu \in \mathbb{C} : \operatorname{Re}(e^{-i\xi}\mu) = d\}.$$

For $A \in M_n$, let $\operatorname{Re}A = (A + A^*)/2$. Then (1.2) is equivalent to

$$\Lambda_k(A) = \bigcap_{\xi \in [0, 2\pi)} \mathcal{H}(\lambda_k(\operatorname{Re}(e^{-i\xi}A)), \xi).$$

The following result is easy to verify.

Proposition 3.4. *Let $A \in M_n$ and $\Lambda_k(A) = \bigcap_{j=1}^m \mathcal{H}(d_j, \xi_j) \neq \emptyset$, where $\mathcal{H}(d_j, \xi_j)$ is defined as in (3.2) for some $d_1, \dots, d_m \in \mathbb{R}$ and distinct $\xi_1, \dots, \xi_m \in [0, 2\pi)$.*

- (a) *We have $0 \in \Lambda_k(A)$ if and only if $d_1, \dots, d_m \geq 0$, and 0 is an interior point of $\Lambda_k(A)$ if and only if $d_1, \dots, d_m > 0$.*
- (b) *If $\mu = re^{i\xi}$ with $r > 0$ and $\xi \in \mathbb{R}$, then $\Lambda_k(\mu A) = \bigcap_{j=1}^m \mathcal{H}(rd_j, \xi_j + \xi)$ and $\Lambda_k(A + \mu I) = \bigcap_{j=1}^m \mathcal{H}(d_j + r \cos(\xi - \xi_j), \xi_j)$.*

In connection to Problem 3.1, we have the following.

Theorem 3.5. *Suppose $\mathcal{P} = \bigcap_{j=1}^p \mathcal{H}(d_j, \xi_j)$ is a non-degenerate p -sided polygon, where $\mathcal{H}(d_j, \xi_j)$ is defined as in (3.2) with $d_1, \dots, d_p \in \mathbb{R}$ and distinct $\xi_1, \dots, \xi_p \in [0, 2\pi)$. Let q be a nonnegative integer. The following two statements are equivalent.*

- (I) *There is a $(p+q) \times (p+q)$ normal matrix A such that $\Lambda_k(A) = \mathcal{P}$.*
- (II) *There are distinct $\xi_{p+1}, \dots, \xi_{p+q} \in [0, 2\pi)$ such that $\{\xi_1, \dots, \xi_{p+q}\}$ is k -regular.*

Notice that a necessary condition for the set $\bigcap_{j=1}^p \mathcal{H}(d_j, \xi_j)$ to be a non-degenerate polygon is that

$$(3.4) \quad \{e^{i\xi_1}, \dots, e^{i\xi_p}\} \text{ is 1-regular.}$$

By Proposition 3.4, one may assume that 0 lies in the interior of \mathcal{P} in our proofs. However, it is equally convenient for us not to impose this assumption so that we need not verify $d_j > 0$ in $\mathcal{H}(d_j, \xi_j)$ in our proofs.

To prove Theorem 3.5, we need some lemmas.

Lemma 3.6. *Given $A = \text{diag}(a_1, \dots, a_n)$ and $1 \leq m < n$. Suppose the eigenvalues a_{m+1}, \dots, a_n are in $\Lambda_k(A)$ but not extreme points of $\Lambda_k(A)$. Then*

$$\Lambda_k(\text{diag}(a_1, \dots, a_m)) = \Lambda_k(A).$$

Proof. It suffices to show that if a_n is in $\Lambda_k(A)$ but not an extreme point of $\Lambda_k(A)$, then $\Lambda_k(\text{diag}(a_1, \dots, a_{n-1})) = \Lambda_k(A)$.

Suppose a_n satisfy the above assumption. Clearly, $\Lambda_k(\text{diag}(a_1, \dots, a_{n-1}))$ is a subset of $\Lambda_k(A)$. On the other hand, for any $1 \leq j_1 < \dots < j_{n-k} \leq n-1$, $\Lambda_k(A) \subseteq \text{conv}\{a_{j_1}, \dots, a_{j_{n-k}}, a_n\}$. Since a_n is not an extreme point of $\Lambda_k(A)$, it follows that a_n lies in $\text{conv}\{a_{j_1}, \dots, a_{j_{n-k}}, a_n\}$ but is not its extreme point. Therefore,

$$\text{conv}\{a_{j_1}, \dots, a_{j_{n-k}}\} = \text{conv}\{a_{j_1}, \dots, a_{j_{n-k}}, a_n\}.$$

Thus,

$$\begin{aligned} \Lambda_k(A) &= \bigcap \{ \text{conv}\{a_{j_1}, \dots, a_{j_{n-k+1}}\} : 1 \leq j_1 < \dots < j_{n-k} < j_{n-k+1} \leq n \} \\ &\subseteq \bigcap \{ \text{conv}\{a_{j_1}, \dots, a_{j_{n-k}}, a_n\} : 1 \leq j_1 < \dots < j_{n-k} \leq n-1 \} \\ &= \bigcap \{ \text{conv}\{a_{j_1}, \dots, a_{j_{n-k}}\} : 1 \leq j_1 < \dots < j_{n-k} \leq n-1 \} \\ &= \Lambda_k(\text{diag}(a_1, \dots, a_{n-1})). \quad \square \end{aligned}$$

The next lemma shows that if a convex polygon \mathcal{P} is the intersection of half planes $\mathcal{H}(d_j, \zeta_j)$ for $j = 1, \dots, m$, such that the set $\{\zeta_1, \dots, \zeta_m\}$ is

“almost” k -regular (in the sense that $\{\zeta_1, \dots, \zeta_m\}$ is k -regular if we count the multiplicity of each element in the set), one may replace these half planes by n other half planes $\mathcal{H}(\tilde{d}_j, \tilde{\zeta}_j)$ for $j = 1, \dots, n$, $n \geq m$, with $\tilde{\zeta}_i \neq \tilde{\zeta}_j$ for all $i \neq j$, such that $\{\tilde{\xi}_1, \dots, \tilde{\xi}_n\}$ is k -regular and the boundary $\mathcal{L}(\tilde{d}_j, \tilde{\zeta}_j)$ of $\mathcal{H}(\tilde{d}_j, \tilde{\zeta}_j)$ touches the polygon \mathcal{P} for each $j = 1, \dots, n$.

Lemma 3.7. *Suppose $\mathcal{P} = \bigcap_{j=1}^m \mathcal{H}(d_j, \zeta_j)$ such that $0 \leq \zeta_1 \leq \dots \leq \zeta_m < 2\pi$ and for each $r = 1, \dots, m$, there are $1 \leq j_1 < \dots < j_k \leq m$ such that $\zeta_{j_1}, \dots, \zeta_{j_k} \in (\zeta_r, \zeta_r + \pi)$. For every $n \geq m$, there exist $\tilde{d}_1, \dots, \tilde{d}_n \in \mathbb{R}$ and distinct $\tilde{\zeta}_1, \dots, \tilde{\zeta}_n \in [0, 2\pi)$ with $\{\tilde{\zeta}_1, \dots, \tilde{\zeta}_n\}$ being k -regular such that $\mathcal{P} = \bigcap_{j=1}^n \mathcal{H}(\tilde{d}_j, \tilde{\zeta}_j)$ and $\mathcal{P} \cap \mathcal{L}(\tilde{d}_j, \tilde{\zeta}_j) \neq \emptyset$ for each $j = 1, \dots, n$.*

Proof. Set $\tilde{\zeta}_1 = \zeta_1$, and for $s \in \{2, \dots, m\}$, let $\tilde{\zeta}_s = \zeta_s$ if $\zeta_{s-1} < \zeta_s$. For the remaining values, we have $\zeta_{s-1} = \zeta_s$ and we can set

$$\tilde{\zeta}_{s-t_1} = \zeta_{s-t_1+1} = \dots = \zeta_s = \dots = \zeta_{s+t_2} < \tilde{\zeta}_{s+t_2+1}$$

for some $t_1 \geq 1$ and $t_2 \geq 0$. Let $\ell = \min\{j : \tilde{\zeta}_j > \tilde{\zeta}_{s-t_1} + \pi\}$, then we can replace ζ_{s+j} by $\tilde{\zeta}_{s+j} = \zeta_{s+j} + \epsilon_j$ for sufficient small $\epsilon_j > 0$ for $j = -t_1 + 1, -t_1 + 2, \dots, 0, \dots, t_2$ such that

$$\tilde{\zeta}_{s-t_1} < \tilde{\zeta}_{s-t_1+1} < \dots < \tilde{\zeta}_s < \dots < \tilde{\zeta}_{s+t_2} < \min\{\tilde{\zeta}_\ell - \pi, \tilde{\zeta}_{s+t_2+1}\}.$$

After this modification, $\tilde{\zeta}_1, \dots, \tilde{\zeta}_m$ are distinct and $\{\tilde{\zeta}_1, \dots, \tilde{\zeta}_m\}$ is k -regular. If $n > m$, pick distinct $\tilde{\zeta}_{m+1}, \dots, \tilde{\zeta}_n \in [0, 2\pi) \setminus \{\tilde{\zeta}_1, \dots, \tilde{\zeta}_m\}$. Then $\{\tilde{\zeta}_1, \dots, \tilde{\zeta}_n\}$ also forms a k -regular set. Finally, let $\tilde{d}_j = \max_{\mu \in \mathcal{P}} \operatorname{Re}(e^{-i\tilde{\zeta}_j} \mu)$ for $j = 1, \dots, n$. Clearly, we have $\mathcal{P} \cap \mathcal{L}(\tilde{d}_j, \tilde{\zeta}_j) \neq \emptyset$ and $\mathcal{P} \subseteq \mathcal{H}(\tilde{d}_j, \tilde{\zeta}_j)$ for all j . By construction, $\{\zeta_1, \dots, \zeta_m\} \subset \{\tilde{\zeta}_1, \dots, \tilde{\zeta}_n\}$ and $\mathcal{P} = \bigcap_{j=1}^m \mathcal{H}(d_j, \zeta_j) = \bigcap_{j=1}^n \mathcal{H}(\tilde{d}_j, \tilde{\zeta}_j)$.

We can now present the proof of Theorem 3.5.

Proof of Theorem 3.5. Let $\mathcal{P} = \bigcap_{j=1}^p \mathcal{H}(d_j, \xi_j)$ be a non-degenerate p -sided polygon, where $d_1, \dots, d_p \in \mathbb{R}$ and $\xi_1, \dots, \xi_p \in [0, 2\pi)$.

Suppose (I) holds. We may assume that $A = \operatorname{diag}(a_1, \dots, a_{p+q})$ and $\Lambda_k(A) = \mathcal{P}$. By Lemma 3.6, one can remove the eigenvalues of A in $\Lambda_k(A)$ that are not extreme points of $\Lambda_k(A)$ to get $\tilde{A} \in M_n$ for some positive integer $n \leq p + q$ so that $\Lambda_k(A) = \Lambda_k(\tilde{A})$. We have the following.

Claim There are $f_1, \dots, f_n \in \mathbb{R}$ and $\zeta_1, \dots, \zeta_n \in [0, 2\pi)$ such that $\Lambda_k(\tilde{A}) = \bigcap_{j=1}^n \mathcal{H}(f_j, \zeta_j)$. Furthermore, for each $r = 1, \dots, n$, there exist $1 \leq j_1 < \dots < j_k \leq n$ such that $\zeta_{j_1}, \dots, \zeta_{j_k} \in (\zeta_r, \zeta_r + \pi)$.

Once the claim holds, Lemma 3.7 will ensure that $\Lambda_k(\tilde{A}) = \bigcap_{j=1}^{p+q} \mathcal{H}(\tilde{d}_j, \tilde{\xi}_j)$ for some $\tilde{d}_1, \dots, \tilde{d}_{p+q} \in \mathbb{R}$ and a k -regular set $\{\tilde{\xi}_1, \dots, \tilde{\xi}_{p+q}\}$, with

$$\bigcap_{j=1}^p \mathcal{H}(d_j, \xi_j) = \mathcal{P} = \Lambda_k(A) = \Lambda_k(\tilde{A}) = \bigcap_{j=1}^{p+q} \mathcal{H}(\tilde{d}_j, \tilde{\xi}_j).$$

Then $\{\xi_1, \dots, \xi_p\} \subseteq \{\tilde{\xi}_1, \dots, \tilde{\xi}_{p+q}\}$. Thus, we can take $\xi_{p+1}, \dots, \xi_{p+q} \in [0, 2\pi)$ so that $\{\xi_1, \dots, \xi_{p+q}\} = \{\tilde{\xi}_1, \dots, \tilde{\xi}_{p+q}\}$. Therefore, (II) holds.

For notational convenience, we assume that $A = \tilde{A}$ in the claim so that every eigenvalue of A is either an extreme point of $\Lambda_k(A)$ or does not lie in $\Lambda_k(A)$.

We first construct $\zeta_1, \dots, \zeta_n \in [0, 2\pi)$ and $f_1, \dots, f_n \in \mathbb{R}$. For $r = 1, \dots, n$, let Γ_r be the set containing all $\xi \in [0, 2\pi)$ such that the closed half plane $\mathcal{H}(\operatorname{Re}(e^{-i\xi} a_r), \xi)$ contains at least $n - k + 1$ eigenvalues of A . As a_r is either an extreme point of $\Lambda_k(A)$ or not in $\Lambda_k(A)$, there is a $\zeta \in [0, 2\pi)$ such that $\operatorname{Re}(e^{-i\zeta} a_r) \geq \lambda_k(\operatorname{Re}(e^{-i\zeta} A))$ and hence the half plane $\mathcal{H}(\operatorname{Re}(e^{-i\zeta} a_r), \zeta)$ contains at least $n - k + 1$ eigenvalues. Then Γ_r is always nonempty. Furthermore, by the definition of Γ_r , the set Γ_r is an union of closed arcs of Ω . Clearly,

$$\mathcal{P} = \Lambda_k(A) \subseteq \bigcap_{\xi \in \Gamma_r} \mathcal{H}(\operatorname{Re}(e^{-i\xi} a_r), \xi).$$

Also the above intersection, which containing \mathcal{P} , is a non-degenerate conical region. Then Γ_r is contained in some open semi-circular arc of Ω ; otherwise, the above intersection of half planes is equal to the singleton $\{a_r\}$. As Γ_r is a union of closed arcs in some open semi-circular arc of Ω , there exists a unique $\zeta_r \in \Gamma_r$ such that

$$(3.5) \quad \Gamma_r \subseteq (\zeta_r - \pi, \zeta_r].$$

Let $f_r = \operatorname{Re}(e^{-i\zeta_r} a_r)$ for $1 \leq r \leq n$. We show that $\Lambda_k(A) = \bigcap_{j=1}^n \mathcal{H}(f_j, \zeta_j)$. Suppose \mathcal{T} is a minimal subset of \mathcal{S}_0 such that $\Lambda_k(A) = \bigcap_{(r,s) \in \mathcal{T}} H(a_r, a_s)$. We may further assume that for all $(r, s) \in \mathcal{T}$, $\{a_1, \dots, a_m\} \cap L(a_r, a_s) \subseteq \operatorname{conv}\{a_r, a_s\}$. For each $(r, s) \in \mathcal{T}$, write $H(a_r, a_s) = \mathcal{H}(\operatorname{Re}(e^{-i\zeta} a_r), \zeta)$ with $\zeta = \arg(a_s - a_r) - \pi/2$. Then $\zeta \in \Gamma_r$. We claim that $\zeta = \zeta_r$. Suppose not. By the above assumption on a_r , one can see that for a sufficiently small $\epsilon > 0$,

the half plane $\mathcal{H}(\operatorname{Re}(e^{-i\hat{\zeta}}a_r), \hat{\zeta})$ with $\hat{\zeta} = \zeta - \epsilon$ will contain all eigenvalues of A that are in $H(a_r, a_s)$, i.e., $\hat{\zeta} \in \Gamma_r$. With (3.5), we have

$$\Lambda_k(A) \subseteq \mathcal{H}(\operatorname{Re}(e^{-i\zeta_r}a_r), \zeta_r) \cap \mathcal{H}(\operatorname{Re}(e^{-i\hat{\zeta}}a_r), \hat{\zeta}) \subseteq H_0(a_r, a_s) \cup \{a_r\}.$$

So $L(a_r, a_s) \cap \Lambda_k(A)$ contains at most one point. But this contradicts the fact that (r, s) is an element in the minimal subset \mathcal{T} . Therefore, $\zeta = \zeta_r$. Then for each $(r, s) \in \mathcal{T}$, $H(a_r, a_s) = \mathcal{H}(f_r, \zeta_r)$ and so

$$\bigcap_{j=1}^n \mathcal{H}(f_j, \zeta_j) \subseteq \bigcap_{(r,s) \in \mathcal{T}} H(a_r, a_s) = \Lambda_k(A) \subseteq \bigcap_{j=1}^n \mathcal{H}(f_j, \zeta_j).$$

Thus, $\Lambda_k(A) = \bigcap_{j=1}^n \mathcal{H}(f_j, \zeta_j)$ and the first part of the claim holds.

To prove the second part of the claim, without loss of generality, we may assume that $a_r = 0$ and $\zeta_r = 0$. Then $f_r = 0$ and

$$\mathcal{H}(f_r, \zeta_r) = H = \{z \in \mathbb{C} : \operatorname{Re}(z) \leq 0\}.$$

Thus, the closed left half plane contains at least $n - k + 1$ eigenvalues of A . Suppose that the closed right half plane $-H$ contains eigenvalues a_{j_1}, \dots, a_{j_h} of A with $\zeta_{j_t} \neq 0$ for $t = 1, \dots, g$, and $\zeta_{j_t} = 0$ for $t = g + 1, \dots, h$ for some $g \leq h$. Fix a sufficiently small $\epsilon > 0$. We choose $g + 1 \leq \ell \leq h$ so that

$$\operatorname{Re}(e^{-i\epsilon}a_{j_\ell}) = \max_{g+1 \leq t \leq h} \operatorname{Re}(e^{-i\epsilon}a_{j_t}).$$

Then $\{a_{j_{g+1}}, \dots, a_{j_h}\} \subseteq \mathcal{H}(\operatorname{Re}(e^{-i\epsilon}a_{j_\ell}), \epsilon)$. On the other hand, this closed half plane $\mathcal{H}(\operatorname{Re}(e^{-i\epsilon}a_{j_\ell}), \epsilon)$ also contains all eigenvalues of A that are in the left open half plane. Thus, this closed half plane $\mathcal{H}(\operatorname{Re}(e^{-i\epsilon}a_{j_\ell}), \epsilon)$ has at least $n - g$ eigenvalues of A . On the other hand by (3.5), $\epsilon \notin \Gamma_{j_\ell}$ and so $\mathcal{H}(\operatorname{Re}(e^{-i\epsilon}a_{j_\ell}), \epsilon)$ can have at most $n - k$ eigenvalues. Thus, we have $g \geq k$.

Now for each $t = 1, \dots, k$, let $\hat{d}_t = \operatorname{Re}(a_{j_t})$, then $\hat{d}_t \geq 0$ and $H \subseteq \mathcal{H}(\hat{d}_t, 0)$. Thus, the closed half plane $\mathcal{H}(\hat{d}_t, 0)$ contains at least $n - k + 1$ eigenvalues of A , i.e., $\zeta_r = 0 \in \Gamma_{j_t}$. Recall that $\zeta_{j_t} \neq 0$. By (3.5), one see that

$$\zeta_r \in (\zeta_{j_t} - \pi, \zeta_{j_t}) \quad \text{for } t = 1, \dots, k.$$

Equivalently, $\zeta_{j_1}, \dots, \zeta_{j_k} \in (\zeta_r, \zeta_r + \pi)$. Thus, our **claim** is proved, and (II) holds.

Suppose now (II) holds, namely, there are distinct $\xi_{p+1}, \dots, \xi_{p+q}$ such that $\{\xi_1, \dots, \xi_{p+q}\}$ is k -regular. For $j = p + 1, \dots, p + q$, define

$$d_j = \max_{\mu \in \mathcal{P}} \operatorname{Re}(e^{-i\xi_j} \mu).$$

Then $\mathcal{P} \subseteq \mathcal{H}(d_j, \xi_j)$ and so

$$\mathcal{P} = \bigcap_{j=1}^p \mathcal{H}(d_j, \xi_j) = \bigcap_{j=1}^n \mathcal{H}(d_j, \xi_j)$$

with $n = p + q$. By Lemma 3.7, we may assume that $\mathcal{P} \cap \mathcal{L}(d_j, \xi_j) \neq \emptyset$ for all $j = 1, \dots, n$, and $0 \leq \xi_1 < \dots < \xi_n < 2\pi$ such that condition (3.1) holds. For each $r = 1, \dots, n$, let

$$a_r = \frac{i}{\sin(\xi_{r+k} - \xi_r)} \left(e^{i\xi_r} d_{r+k} - e^{i\xi_{r+k}} d_r \right)$$

and $A = \text{diag}(a_1, \dots, a_n)$. Then

$$\text{Re}(e^{-i\xi_r} a_r) = d_r \quad \text{and} \quad \text{Re}(e^{-i\xi_{r+k}} a_r) = d_{r+k}.$$

Note that $a_r \in \mathcal{L}(d_r, \xi_r) \cap \mathcal{L}(d_{r+k}, \xi_{r+k})$ is the vertex of the conical region $\mathcal{H}(d_r, \xi_r) \cap \mathcal{H}(d_{r+k}, \xi_{r+k})$, which contains \mathcal{P} . Therefore,

$$\text{Re}(e^{-i\xi_r}(a_r - \mu)) \geq 0 \quad \text{and} \quad \text{Re}(e^{-i\xi_{r+k}}(a_r - \mu)) \geq 0,$$

for all $\mu \in \mathcal{P}$. Since $\xi_{r+k} \in (\xi_r, \xi_r + \pi)$, we have

$$(3.6) \quad \text{Re}(e^{-i\xi} a_r) \geq \max_{\mu \in \mathcal{P}} \text{Re}(e^{-i\xi} \mu) \quad \text{for all } \xi \in [\xi_r, \xi_{r+k}].$$

Let $\mu_j \in \mathcal{L}(d_j, \xi_j) \cap \mathcal{P}$ for $j = r, r+k$. As $\xi_{r+k} \in (\xi_r, \xi_r + \pi)$, we have $\mu_r = a_r - ie^{i\xi_r} b_r$ and $\mu_{r+k} = a_r + ie^{i\xi_{r+k}} c_r$ for some $b_r, c_r \geq 0$. Note that

$$\text{Re}(e^{-i\xi}(\mu_r - a_r)) = b_r \sin(\xi_r - \xi) \geq 0 \quad \text{for all } \xi \in [\xi_r - \pi, \xi_r],$$

and

$$\text{Re}(e^{-i\xi}(\mu_{r+k} - a_r)) = c_r \sin(\xi - \xi_{r+k}) \geq 0 \quad \text{for all } \xi \in [\xi_{r+k}, \xi_{r+k} + \pi].$$

Since $\{\xi_1, \dots, \xi_n\}$ is k -regular, it is easily seen that

$$[0, 2\pi) \setminus [\xi_r, \xi_{r+k}] = [\xi_r - \pi, \xi_r) \cup (\xi_{r+k}, \xi_{r+k} + \pi].$$

Therefore, for $\xi \in [0, 2\pi) \setminus [\xi_r, \xi_{r+k}]$, we have

$$\max\{\text{Re}(e^{-i\xi}(\mu_r - a_r)), \text{Re}(e^{-i\xi}(\mu_{r+k} - a_r))\} \geq 0.$$

Moreover, we have

$$(3.7) \quad \max\{\text{Re}(e^{-i\xi} \mu_r), \text{Re}(e^{-i\xi} \mu_{r+k})\} \geq \text{Re}(e^{-i\xi} a_r).$$

Let $\xi \in [0, 2\pi)$. Then $\xi \in [\xi_s, \xi_{s+1})$ for some $s \in \{1, \dots, n\}$. It follows that $\xi \in [\xi_r, \xi_{r+k}]$ for $r = s - k + 1, \dots, s$, and $\xi \in [0, 2\pi) \setminus [\xi_r, \xi_{r+k}]$ for other r . By (3.6) and (3.7),

$$\min_{r \in \{s-k+1, \dots, s\}} \text{Re}(e^{-i\xi} a_r) \geq \max_{\mu \in \mathcal{P}} \text{Re}(e^{-i\xi} \mu) \geq \max_{r \notin \{s-k+1, \dots, s\}} \text{Re}(e^{-i\xi} a_r).$$

Thus, $\lambda_k(\operatorname{Re}(e^{-i\xi}A)) = \min_{r \in \{s-k+1, \dots, s\}} \operatorname{Re}(e^{-i\xi}a_r)$ and so

$$\mathcal{P} \subseteq \mathcal{H}\left(\lambda_k(\operatorname{Re}(e^{-i\xi}A)), \xi\right).$$

Hence, $\mathcal{P} \subseteq \Lambda_k(A)$. Furthermore, if $\xi = \xi_s$, then $\operatorname{Re}(e^{-i\xi_s}a_s) = d_s$. Thus

$$\lambda_k(\operatorname{Re}(e^{-i\xi_s}A)) = \min_{r \in \{s-k+1, \dots, s\}} \operatorname{Re}(e^{-i\xi_s}a_r) \leq d_s.$$

It follows that

$$\begin{aligned} \Lambda_k(A) &= \bigcap_{\xi \in [0, 2\pi)} \mathcal{H}\left(\lambda_k(\operatorname{Re}(e^{-i\xi}A)), \xi\right) \\ &\subseteq \bigcap_{1 \leq s \leq n} \mathcal{H}\left(\lambda_k(\operatorname{Re}(e^{-i\xi_s}A)), \xi_s\right) \subseteq \bigcap_{1 \leq s \leq n} \mathcal{H}(d_s, \xi_s) = \mathcal{P}. \end{aligned}$$

Thus, $\mathcal{P} = \Lambda_k(A)$. □

By Theorem 3.5, Problem 3.1 is equivalent to the following combinatorial problem, whose solution will be given in the next section.

Problem 3.8. Suppose $\{\xi_1, \dots, \xi_p\} \subseteq [0, 2\pi)$ is 1-regular. For $k > 1$, determine the smallest nonnegative integer q so that $\{\xi_1, \dots, \xi_{p+q}\}$ is k -regular for some distinct $\xi_{p+1}, \dots, \xi_{p+q} \in [0, 2\pi)$.

4. SOLUTIONS FOR PROBLEMS 3.1 AND 3.8

In this section, we give the solutions for Problems 3.1 and 3.8. Given a non-empty set $\Pi = \{\xi_1, \dots, \xi_p\} \subseteq \Omega$, Problem 3.8 is equivalent to the study of smallest nonnegative integer q so that $\{\xi_1, \dots, \xi_{p+q}\}$ is k -regular for some distinct $\xi_{p+1}, \dots, \xi_{p+q} \in \Omega$. We have the following.

Theorem 4.1. *Let $k > 1$ be a positive integer and Π be a p element subset of Ω , including s pairs of antipodal points: $\{\beta_1, -\beta_1\}, \dots, \{\beta_s, -\beta_s\}$, where $p \geq 3$ and $s \geq 0$. Suppose Π is 1-regular but not k -regular and q is the minimum number of points in Ω one can add to Π to form a k -regular set.*

(a) *If $k \geq p - s$, then*

$$(4.1) \quad q = \begin{cases} 2k + 1 - p & \text{if } s = 0, \\ 2k + 2 - p & \text{if } s > 0. \end{cases}$$

(b) *If $k < p - s$, then q is the smallest nonnegative integer t such that one can remove t non-antipodal points from Π to get a $(k - t)$ -regular set. More*

precisely,

$$(4.2) \quad q = \min\{t \in \mathbb{N} : \Pi \setminus \{\beta_1, -\beta_1, \dots, \beta_s, -\beta_s\} \text{ has a } t\text{-element} \\ \text{subset } T \text{ such that } \Pi \setminus T \text{ is } (k-t)\text{-regular}\}.$$

Consequently,

$$(4.3) \quad q \leq \min\{2k + 2 - p, k - 1\}.$$

The inequality in (4.3) becomes equality if $\Pi = \{1, i, -1, \alpha_4, \dots, \alpha_p\}$ where $\alpha_4, \dots, \alpha_p$ lie in the open lower half plane.

Several remarks concerning Theorem 4.1 are in order. If condition (a) in the theorem holds, then the value q can be determined immediately. However, it is important to consider two cases depending on whether Π has pairs of antipodal points as illustrated by the following.

Example 4.2. Suppose $S_1 = \{1, w, w^2, w^3\}$ with $w = e^{2i\pi/5}$. Then $\alpha \neq -\beta$ for any two elements $\alpha, \beta \in S_1$ and adding w^4 to S_1 results in a 2-regular set. Suppose $S_2 = \{1, -1, i, -i\}$. Then we need to add at least two points, say, $z, -z \in \Omega \setminus S_2$, to get a 2-regular set.

Suppose condition (b) in the theorem holds. We can determine the value q by taking t non-antipodal elements away from Π at a time and check whether the resulting set is $(k-t)$ -regular. The value q can then be determined in no more than $\sum_{i=0}^{p-2s} \binom{p-2s}{i} = 2^{p-2s}$ steps. The success of reducing Problem 3.8 to a problem which is solvable in finite steps depends on Lemma 4.7 and Proposition 4.8.

It would be nice to have a simple formula for q in terms of p, k, s in case (b) of the theorem. However, the following example show that the value q depends not only on the values p, k, s , but also on the relative positions of the points in Π .

Example 4.3. Let $S_1 = \{1, w, w^2, w^3, w^4, w^5\}$ with $w = e^{2\pi i/7}$ and $S_2 = \{z^2, z^3, z^7, z^8, z^{12}, z^{13}\}$ with $z = e^{2\pi i/15}$. Notice that both S_1 and S_2 contain 6 elements and have no antipodal pairs. Furthermore, both of them are 2-regular but not 3-regular. Clearly, adding w^6 to S_1 results a 3-regular set. However, as each of the open arcs $(z^3, -z^3)$, $(z^8, -z^8)$ and $(z^{13}, -z^{13})$ contains only two elements of S_2 while the intersection of this three open arcs is empty, at least two elements has to be added to S_2 to form a 3-regular set.

Note that our proofs are constructive; see Lemma 4.7 and Propositions 4.6 and 4.8. One can actually construct a subset $\Pi' \subseteq \Omega$ with q elements so that $\Pi \cup \Pi'$ is k -regular.

By Theorem 4.1, we can answer Problems 3.1 and 3.8, and obtain some additional information on their solutions. We will continue to use the notation $\mathcal{H}(d, \xi)$ defined in (3.2) in the following.

Theorem 4.4. *For Problem 3.1, if a p -sided polygon \mathcal{P} is expressed as $\mathcal{P} = \bigcap_{j=1}^p \mathcal{H}(d_j, \xi_j)$ for some $d_1, \dots, d_p \in \mathbb{R}$ and $\xi_1, \dots, \xi_p \in [0, 2\pi)$, then the minimum dimension n for the existence of a normal matrix $A \in M_n$ such that $\Lambda_k(A) = \mathcal{P}$ is equal to $p + q$, where q is determined in Theorem 4.1. Moreover,*

$$(4.4) \quad n \leq \max\{2k + 2, p + k - 1\}.$$

The inequality in (4.4) becomes equality if $(\xi_1, \xi_2, \xi_3) = (0, \pi/2, \pi)$ and ξ_4, \dots, ξ_p lie in $(\pi, 2\pi)$.

We break down the proofs of Theorems 4.1 and 4.4 in several propositions. We first give a lower bound for the number of elements in a k -regular set.

Proposition 4.5. *Suppose $S = \{\alpha_1, \dots, \alpha_n\} \subseteq \Omega$ is k -regular. Then $n \geq 2k + 1$. Furthermore, if S contains a pair of antipodal points $\{\alpha, -\alpha\}$, then $n \geq 2k + 2$.*

Proof. For any $r \in \{1, \dots, n\}$, each of the open arcs $(\alpha_r, -\alpha_r)$ and $(-\alpha_r, \alpha_r)$ contains k elements of S . Thus, $n \geq 2k + 1$. For the last statement, if we take $\alpha_r = \alpha$, then together with α and $-\alpha$, we see that $n \geq 2k + 2$. The proof of the assertion is complete. \square

As shown in Proposition 4.5, the existence of a pair of antipodal points $\{\alpha, -\alpha\}$ has implication on the size of a k -regular set Π . The next result together with Proposition 4.5 show that the lower bound in (4.1) is best possible.

Proposition 4.6. *Let $k > 1$ and Π is a p element subset of Ω containing s pairs of antipodal points, where $p \geq 3$ and $s \geq 0$. If Π is 1-regular but not k -regular and $k \geq p - s$, then one can extend Π to a k -regular set by adding $2k + 1 - p$ or $2k + 2 - p$ elements, depending whether s is zero.*

Proof. Assume $k \geq p - s$. Suppose first that $s > 0$. Let Π'' be a set containing $(k - p + s + 1)$ pairs of antipodal points such that $\Pi'' \cap \Pi$ is

empty. Take

$$\Pi' = \Pi'' \cup -(\Pi \setminus \{\beta_1, -\beta_1, \dots, \beta_s, -\beta_s\}).$$

Then Π' contains $(2k+2-p)$ elements. Furthermore, the set $\Pi \cup \Pi'$ contains exactly $k+1$ pairs of antipodal points hence it is k -regular. Thus, the result follows if $s > 0$.

Next, suppose $s = 0$. Without loss of generality, we may assume that $1 \in \Pi$. Hence, $-1 \in \Pi'$. We now modify Π' . We first delete the point -1 in Π' . Then for all other points $\alpha \in \Pi'$, we replace α by $e^{i\xi}\alpha$ if α lies in the upper open half plane $P = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$, and by $e^{-i\xi}\alpha$ if α lies in the lower open half plane $-P$, with sufficiently small $\xi > 0$. Then we see that for every $\alpha \in \Pi \cup \Pi'$, αP still contains exactly k elements. Thus, $\Pi \cup \Pi'$ is k -regular. Furthermore, the modified set Π' has one fewer point, i.e., Π' has only $2k+1-p$ elements. The proof of is complete. \square

A referee pointed out that each $(k-1)$ -regular set can be enlarged to a k -regular set by adding in not more than 2 extra elements. The following result shows that sometimes 2 may not be the minimum number needed.

Lemma 4.7. *Let $k > 1$ and Π be a subset of Ω containing at least one non-antipodal point. The following are equivalent.*

- (a) *One can add a point $\beta \notin \Pi$ so that $\Pi \cup \{\beta\}$ is k -regular.*
- (b) *One can delete a non-antipodal point $\gamma \in \Pi$ so that $\Pi \setminus \{\gamma\}$ is $(k-1)$ -regular.*

Here, an element $\alpha \in \Pi$ is called a non-antipodal point of Π if $-\alpha \notin \Pi$.

Proof. Suppose first that (b) holds. Let $P = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. Without loss of generality, we may assume that $\gamma = 1$ is a non-antipodal point in Π . Suppose $\Pi \setminus \{\gamma\} = \{e^{i\theta_1}, \dots, e^{i\theta_{p-1}}\}$ such that

$$0 < \theta_1 < \dots < \theta_m < \pi < \theta_{m+1} < \dots < \theta_{p-1} < 2\pi.$$

As $\Pi \setminus \{\gamma\}$ is $(k-1)$ -regular, by Proposition 4.5, $\Pi \setminus \{\gamma\}$ has $p-1 \geq 2(k-1)+1$ elements. Therefore, for every $\alpha \in \Omega$, the open half plane αP contains at least $k-1$ elements in $\Pi \setminus \{\gamma\}$ and either P or $-P$ contains at least k elements in $\Pi \setminus \{\gamma\}$. Hence, we have either $m = k-1$ or $k \leq m \leq p-k$.

Choose $\beta = e^{i\theta}$ where

$$\theta = \begin{cases} \max\{\pi + \theta_m, \theta_{p-1}\}/2 & \text{if } m = k-1, \\ \min\{2\pi + \theta_1, \pi + \theta_{m+1}\}/2 & \text{if } k \leq m \leq p-k. \end{cases}$$

Now for every $\alpha \neq \pm 1$, the open half plane αP contains at least $k - 1$ elements of $\Pi \setminus \{\gamma\}$ and either γ or β . Hence, αP contains at least k elements of $\Pi \cup \{\beta\}$. On the other hand, when $\alpha = \pm 1$, the open half plane αP contains either k elements of Π or $k - 1$ elements of Π and β . Again, αP contains at least k elements of $\Pi \cup \{\beta\}$. Thus, (a) holds.

Conversely, suppose (a) holds. If $-\beta \in \Pi$, then it is easy to see that the set $\Pi \setminus \{-\beta\}$ is $(k - 1)$ -regular. From now, we assume that $-\beta \notin \Pi$. Without loss of generality, we may assume that $\beta = -1$. Furthermore, by replacing Π with the set $\{\bar{\xi} : \xi \in \Pi\}$, if necessary, we can assume that the number of elements in $\Pi \cap P$ is greater or equal to the number of elements in $\Pi \cap (-P)$. Under this assumption, the upper open half plane must contain at least one non-antipodal point of Π .

Let γ be the non-antipodal point in Π such that $0 < \arg(\gamma) \leq \arg(\alpha)$ for all non-antipodal points $\alpha \in \Pi$. Then $\gamma \in P$. We show that $\Pi \setminus \{\gamma\}$ is $(k - 1)$ -regular.

Take any $\alpha \in \Pi \setminus \{\gamma\}$. Suppose $\alpha \in \beta P \cup \gamma P$. Then the open half plane αP can contain at most one of points β and γ . As the open half plane αP contains at least k elements of $\Pi \cup \{\beta\}$, αP contains at least $k - 1$ elements of $\Pi \setminus \{\gamma\}$. Thus, $\Pi \setminus \{\gamma\}$ is $(k - 1)$ -regular if $\Pi \setminus \{\gamma\} \subseteq \beta P \cup \gamma P$. Now suppose $(\Pi \setminus \{\gamma\}) \setminus (\beta P \cup \gamma P)$ is nonempty and let $\omega_1, \dots, \omega_t$ be the points in this set. Notice that all of them lie in the upper open half plane P . Therefore, we may assume that

$$0 < \arg(\omega_1) < \dots < \arg(\omega_t) < \arg(\gamma) < \pi.$$

Also by the choice of $\gamma, \omega_1, \dots, \omega_t$ cannot be non-antipodal points and hence $-\omega_1, \dots, -\omega_t$ are in Π . Clearly, each open half plane $\omega_j P$ contains at least k elements of $\Pi \cup \{\beta\}$. Notice that $(\omega_j P) \setminus P$ contains exactly j elements of $\Pi \cup \{\beta\}$, namely, $-\omega_1, \dots, -\omega_{j-1}$ and β . Also the set $P \setminus (\omega_j P)$ contains exactly j elements of $\Pi \cup \{\beta\}$, namely, $\omega_1, \dots, \omega_j$. It follows that the half plane $\omega_j P$ contains the same number of elements of $P \cup \{\beta\}$ as the upper half plane P . By Proposition 4.5, $\Pi \cup \{\beta\}$ contains at least $2k + 2$ elements. Then by assumption, the upper open half plane P contains at least $k + 1$ elements of $\Pi \cup \{\beta\}$. Thus, every open half plane $w_j P$ contains at least $k + 1$ elements of $\Pi \cup \{\beta\}$ and it contains at least $k - 1$ elements of $\Pi \setminus \{\gamma\}$. Therefore, $\Pi \setminus \{\gamma\}$ is a $(k - 1)$ -regular set and the assertion follows. \square

Applying the above lemma inductively (repeatedly), we have the following.

Proposition 4.8. *Let $k > 1$ and Π is a p element subset of Ω containing s pairs of antipodal points, where $p \geq 3$ and $s \geq 0$. Suppose $p > 2s$. For any positive $t \leq \min\{k, p - 2s, p - 1\}$, the following are equivalent.*

- (a) *One can add t points $\beta_1, \dots, \beta_t \notin \Pi$ so that $\Pi \cup \{\beta_1, \dots, \beta_t\}$ is k -regular.*
- (b) *One can delete t non-antipodal points $\gamma_1, \dots, \gamma_t \in \Pi$ so that $\Pi \setminus \{\gamma_1, \dots, \gamma_t\}$ is $(k - t)$ -regular.*

Proof. Clearly, the result holds for $t = 1$ by Proposition 4.7. Assume the statement holds for all $\ell < t$. Suppose $\Pi \cup \{\beta_1, \dots, \beta_t\}$ is k -regular. Let $\Pi_1 = \Pi \cup \{\beta_1\}$. Then $\Pi_1 \cup \{\beta_2, \dots, \beta_t\}$ is k -regular and it follows from the assumption that one can find $t - 1$ non-antipodal points $\gamma_1, \dots, \gamma_{t-1} \in \Pi \cup \{\beta_1\}$ such that $\Pi_1 \setminus \{\gamma_1, \dots, \gamma_{t-1}\}$ is $(k - t + 1)$ -regular. If $\beta_1 \notin \{\gamma_1, \dots, \gamma_{t-1}\}$, by applying Lemma 4.7 to the set $(\Pi \setminus \{\gamma_1, \dots, \gamma_{t-1}\}) \cup \{\beta_1\}$, one can find another non-antipodal point $\gamma_t \in \Pi_1 \setminus \{\gamma_1, \dots, \gamma_{t-1}\}$ so that $\Pi \setminus \{\gamma_1, \dots, \gamma_t\}$ is $(k - t)$ -regular. On the other hand, if β_1 is one of the γ_j , say $\beta_1 = \gamma_1$, then $\Pi \setminus \{\gamma_2, \dots, \gamma_{t-1}\}$ is $(k - t + 1)$ -regular. In this case, take an arbitrary element $\gamma_t \in \Pi \setminus \{\gamma_2, \dots, \gamma_{t-1}\}$ and apply Lemma 4.7 to the set $(\Pi \setminus \{\gamma_2, \dots, \gamma_t\}) \cup \{\gamma_t\}$, one can find another non-antipodal point γ_{t+1} so that the set $\Pi \setminus \{\gamma_2, \dots, \gamma_{t+1}\}$ is $(k - t)$ -regular. Then (b) follows. The proof of (b) implying (a) can also be done by induction in a similar way. \square

Suppose $k < p - s$. Given a p element subset Π of Ω containing s pairs of antipodal points, $\beta_1, -\beta_1, \dots, \beta_s, -\beta_s$ with $s > 0$, which is not k -regular, the set obtained from Π by deleting all $p - 2s$ non-antipodal points is a $(s - 1)$ -regular set. On the other hand, if Π does not have any pair of antipodal points, then $k \leq p - 1$ and one can always delete k elements to form a 0-regular set. In both cases, one see that the following minimum always exist.

$$q = \min\{t \in \mathbb{N} : \Pi \setminus \{\beta_1, -\beta_1, \dots, \beta_s, -\beta_s\} \text{ has a } t\text{-element subset } T \text{ such that } \Pi \setminus T \text{ is } (k - t)\text{-regular}\}.$$

By Proposition 4.8, one can always add this minimum number q of points to Π to form a k -regular set. Furthermore, this number q is optimal in the sense that one cannot add fewer than q elements to do so.

By definition, q is a positive integer bounded above by $\min\{k, p - 2s\}$. The following proposition gives more information about the minimum value (4.2) in Theorem 4.1.

Proposition 4.9. *Using the notation in Theorem 4.1. If $k < p - s$, then the value q in (4.2) exists and satisfies*

$$q \leq \begin{cases} k & \text{if } (p, s) = (k + 1, 0) \text{ or } (k + 2, 1), \\ \min\{k - 1, p - 2s\} & \text{otherwise.} \end{cases}$$

Also q is bounded below by $2k + 1 - p$ or $2k + 2 - p$, depending whether s is zero. Furthermore, $q = 2k + 1 - p$ if $p \leq k + 2$ with $s = 0$ and $q = 2k + 2 - p$ if $p \leq k + 3$ with $s > 0$.

Proof. The lower bound can be seen easily from Proposition 4.5. Also the case when $(p, s) = (k + 1, 0)$ or $(k + 2, 1)$ has already discussed. Now we assume that $(p, s) \notin \{(k + 1, 0), (k + 2, 1)\}$. Consider the case when $s \geq 2$. Take $t = \min\{k - 1, p - 2s\}$ and delete t non-antipodal elements in Π . Then the resulting set is $(s - 1)$ -regular and hence $(k - t)$ -regular as $k - t = \max\{1, k - p + 2s\} \leq s - 1$. Thus, $q \leq t$.

Next we consider the case when $s = 1$ and $p \geq k + 3$. Let $\{\alpha, -\alpha\}$ be the pair of antipodal points in Π . Since Π is 1-regular, there are $\alpha_1 \in (\alpha, -\alpha) \cap \Pi$ and $\alpha_2 \in (-\alpha, \alpha) \cap \Pi$. Pick another $k - 1$ non-antipodal points $\alpha_3, \dots, \alpha_{k+1}$ in Π . The set $\Pi \setminus \{\alpha_3, \dots, \alpha_{k+1}\}$ containing $\{\alpha_1, \alpha_2, \alpha, -\alpha\}$ is 1-regular. Then $q \leq k - 1$.

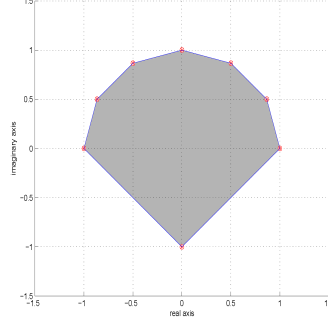
Finally consider the case when $s = 0$ and $p \geq k + 2$. We may assume that $\Pi = \{e^{i\xi_j} : 1 \leq j \leq p\}$ with $0 = \xi_1 < \dots < \xi_p < 2\pi$. Since Π is 1-regular, we can choose ℓ such that $\xi_\ell = \max\{\xi_j : 0 < \xi_j < \pi\}$. Then $S = \{\xi_1, \xi_\ell, \xi_{\ell+1}\}$ is 1-regular. Then any $p - k + 1$ subset of Π containing S is 1-regular. Thus, $q \leq k - 1$. \square

Now we are ready to present the following:

Proof of Theorems 4.1 and 4.4. The assertions on q and n follows by Propositions 4.6 and 4.8. For the last assertion in Theorem 4.1, we see that in order to get a k -regular set by adding q points to Π , we need to add at least $k - 1$ points $e^{i\xi}$, with $0 < \xi < \pi$. If $2k + 2 - p > k - 1$, then $p - 3 < k$ and we need to add an extra $k - (p - 3)$ points $e^{i\xi}$, with $\pi < \xi < 2\pi$, giving a total of $k - 1 + k - (p - 3) = 2k + 2 - p$ points. This proves the equality in (4.3). The equality in (4.4) now follows readily. \square

To close this section, let us illustrate our results by the following example.

Example 4.10. Let the polygon $\mathcal{P} = \text{conv}\{1, w, w^2, w^3, w^4, w^5, w^6, w^9\}$ with $w = e^{2\pi i/12}$, see the following.

The polygon \mathcal{P}

Then $\mathcal{P} = \bigcap_{j=1}^8 \mathcal{H}(d_j, \xi_j)$ with $d_1 = \dots = d_6 = \cos \frac{\pi}{12}$, $d_7 = d_8 = \cos \frac{\pi}{4}$, and

$$(\xi_1, \dots, \xi_8) = \left(\frac{\pi}{12}, \frac{3\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{9\pi}{12}, \frac{11\pi}{12}, \frac{15\pi}{12}, \frac{21\pi}{12} \right).$$

Thus,

$$\Pi = \{\alpha_1, \dots, \alpha_8\} = \left\{ e^{\frac{\pi i}{12}}, e^{\frac{3\pi i}{12}}, e^{\frac{5\pi i}{12}}, e^{\frac{7\pi i}{12}}, e^{\frac{9\pi i}{12}}, e^{\frac{11\pi i}{12}}, e^{\frac{15\pi i}{12}}, e^{\frac{21\pi i}{12}} \right\}.$$

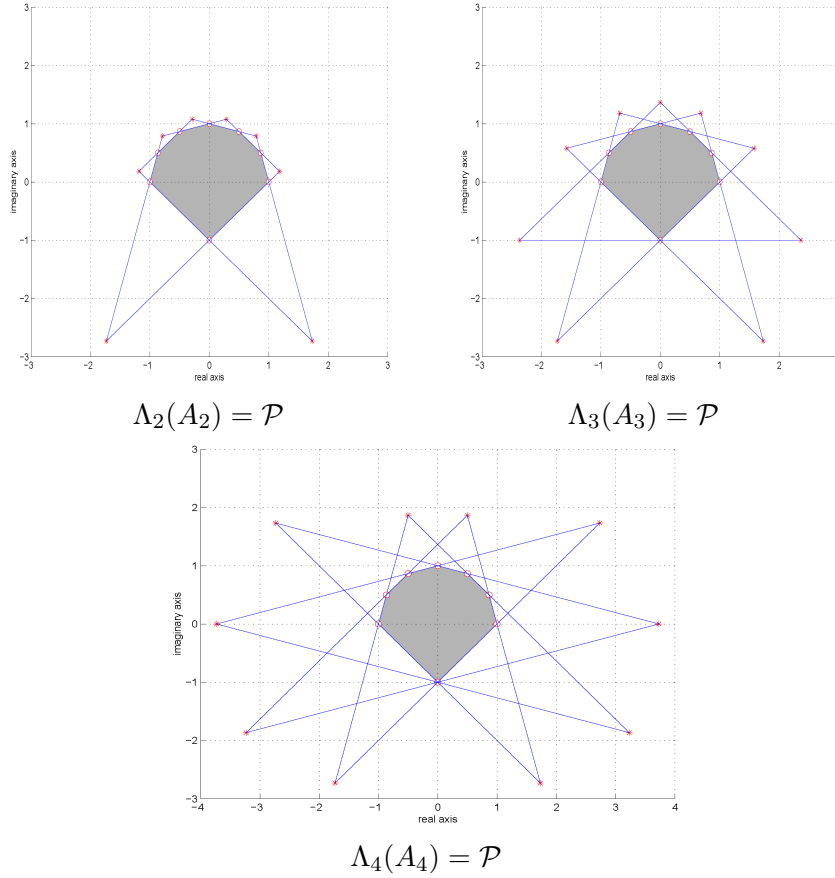
In particular, Π has two pairs of antipodal points, namely, $\left\{ e^{\frac{3\pi i}{12}}, e^{\frac{15\pi i}{12}} \right\}$ and $\left\{ e^{\frac{9\pi i}{12}}, e^{\frac{21\pi i}{12}} \right\}$, i.e., $p = 8$ and $s = 2$. By Theorem 4.4 and Proposition 4.9, for $k \geq 5$, a $(2k+2) \times (2k+2)$ normal matrix A can be constructed so that $\Lambda_k(A) = \mathcal{P}$.

It remains to consider the cases for $k \leq 4$. Clearly, Π is 2-regular. Thus, a 8×8 normal matrix A_2 can be constructed so that $\Lambda_2(A_2) = \mathcal{P}$. However, Π is not k -regular for $k \geq 3$.

Now we consider the case $k = 3$. Clearly, $\Pi \setminus \{e^{\frac{5\pi i}{12}}\}$ is 2-regular. Then Theorem 4.4 shows that there is a 9×9 normal matrix A_3 such that $\Lambda_3(A_3) = \mathcal{P}$. Indeed, following the proof of Lemma 4.7, we see that if $\Pi' = \{e^{\frac{18\pi i}{12}}\}$, $\Pi \cup \Pi'$ is 3-regular.

Finally, we turn to the case when $k = 4$. Notice that $\Pi \setminus \{e^{\frac{5\pi i}{12}}, e^{\frac{7\pi i}{12}}\}$ is 2-regular. Thus, Theorem 4.4 shows that there is a 10×10 normal matrix A_4 such that $\Lambda_4(A_4) = \mathcal{P}$.

In the following, we display the higher rank numerical ranges of A_2 , A_3 , and A_4 . In the figures, the points “o” correspond to the vertices of the polygon while the points “*” correspond to the eigenvalues of the normal matrices.



5. AN ALGORITHM

In this section, we further present a detail procedure for constructing the rank- k numerical ranges of normal matrices based on the discussion in Section 2.

Given a normal matrix A with m distinct eigenvalues a_1, \dots, a_m , one can easily construct $\Lambda_k(A)$ through the following algorithms.

Basic Algorithm First construct the set \mathcal{S}_0 . For each ordered pair (r, s) with $r < s$, count the number of eigenvalues of A (counting multiplicities) in the open planes $H_0(a_r, a_s)$ and $H_0(a_s, a_r)$.

- (1) If $H_0(a_r, a_s)$ has at most $n - k - 1$ eigenvalues while $H_0(a_s, a_r)$ has at most $k - 1$ eigenvalues, then collect the index pair (r, s) in \mathcal{S}_0 .
- (2) If $H_0(a_s, a_r)$ has at most $n - k - 1$ eigenvalues while $H_0(a_r, a_s)$ has at most $k - 1$ eigenvalues, then collect the index pair (s, r) in \mathcal{S}_0 .

Notice that one can already construct $\Lambda_k(A)$ by determine the intersection of all the half planes $H(a_r, a_s)$ with $(r, s) \in \mathcal{S}_0$. Nevertheless, one can perform the following additional steps to simplify the set \mathcal{S}_0 before constructing $\Lambda_k(A)$.

Modified Algorithm 1 Suppose in basic algorithm, there is an index pair (p, q) satisfying both (1) and (2), i.e., both pairs (p, q) and (q, p) are in \mathcal{S}_0 . Then $\Lambda_k(A)$ is a subset of a line segment. In this case, $\Lambda_k(A)$ can be constructed as follows.

Set $\hat{a}_j = (a_j - a_p)/(a_q - a_p)$ and define $\mathcal{S}_1 = \{(r, s) \in \mathcal{S}_0 : \text{Im}(\hat{a}_r) \neq \text{Im}(\hat{a}_s)\}$. If $\mathcal{S}_1 = \emptyset$, then $\Lambda_k(A) = \emptyset$. Suppose $\mathcal{S}_1 \neq \emptyset$. For each $(r, s) \in \mathcal{S}_1$, compute

$$b_{rs} = \frac{\text{Im}(\hat{a}_r) \text{Re}(\hat{a}_s) - \text{Im}(\hat{a}_s) \text{Re}(\hat{a}_r)}{\text{Im}(\hat{a}_r) - \text{Im}(\hat{a}_s)}.$$

Take

$$\begin{aligned} b_1 &= \max\{b_{rs} : (r, s) \in \mathcal{S}_1, \text{Im}(\hat{a}_r) \geq 0 \text{ and } \text{Im}(\hat{a}_s) \leq 0\}, \\ b_2 &= \min\{b_{rs} : (r, s) \in \mathcal{S}_1, \text{Im}(\hat{a}_r) \leq 0 \text{ and } \text{Im}(\hat{a}_s) \geq 0\}. \end{aligned}$$

Then $\Lambda_k(A)$ is the line segment in \mathbb{C} joining the points $(a_q - a_p)b_1 + a_p$ and $(a_q - a_p)b_2 + a_p$ if $b_1 \leq b_2$; otherwise, $\Lambda_k(A) = \emptyset$.

Modified Algorithm 2 Assume the situation mentioned in modified algorithm 1 does not hold. Check if the set \mathcal{S}_0 satisfy the following.

(5.1) There are $(r_1, s_1), \dots, (r_\ell, s_\ell) \in \mathcal{S}_0$ with $\ell \geq 3$ such that

$$\{r_1, s_1\} \cap \{r_2, s_2\} \cap \dots \cap \{r_\ell, s_\ell\} = \{t\} \quad \text{for some } 1 \leq t \leq m.$$

If yes, define

$$\theta_j = \begin{cases} \arg(a_{s_j} - t) & \text{if } r_j = t, \\ \arg(t - a_{r_j}) & \text{if } s_j = t. \end{cases}$$

Relabel the indices so that $0 \leq \theta_1 \leq \dots \leq \theta_\ell < 2\pi$. Consider the following three cases.

- (1) If $\theta_\ell - \theta_1 < \pi$, remove the all pairs (r_j, s_j) in \mathcal{S}_0 for $j \neq 1, \ell$. Then check again whether the modified set still satisfies (5.1).
- (2) If $\theta_{k+1} - \theta_k > \pi$ for some k , remove the all pairs (r_j, s_j) in \mathcal{S}_0 for $j \neq k, k+1$. Then check again whether the modified set still satisfies (5.1)
- (3) If the above two items are not satisfied, then $\Lambda_k(A)$ is either the empty set or the singleton set $\{a_t\}$. In this case, check whether a_t

lies in $H(a_r, a_s)$ for all $(r, s) \in \mathcal{S}_0$. If yes, $\Lambda_k(A)$ is the singleton set; otherwise it is the empty set.

Finally, if the modified set \mathcal{S}_0 does not satisfy (5.1), then one can construct $\Lambda_k(A)$ by determine the intersection of all the half planes $H(a_r, a_s)$ with (r, s) in the modified set \mathcal{S}_0 .

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