THE JOINT NUMERICAL RANGE OF COMMUTING MATRICES

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Abstract. It is shown that for $n \leq 3$ the joint numerical range of a family of commuting $n \times n$ complex matrices is always convex; for $n \geq 4$ there are two commuting matrices whose joint numerical range is not convex.

1. Introduction

Let $M_{m,n}$ be the set of $m \times n$ complex matrices. For $A \in M_{m,n}$, A^* (A^t) stands for the conjugate transpose (transpose) of A; for example, see [9, 10]. Denote by $\mathbb{C}^n$ ($\mathbb{R}^n$) the set of column vectors with n complex (real) entries. Let $M_n = M_{n,n}$ and M_n^m be the set of all m-tuples of $n \times n$ matrices. We identify $\mathbb{C}^n$ with $M_{n,1}$. For notation convenience, we will also say that $\mathbf{z} \in \mathbb{C}^n$ for a complex row vector $\mathbf{z} = (z_1, \ldots, z_n)$. The joint numerical range of $\mathbf{A} = (A_1, \ldots, A_m) \in M_n^m$ is defined by

$$W(\mathbf{A}) = \{(\mathbf{x}^* A_1 \mathbf{x}, \ldots, \mathbf{x}^* A_m \mathbf{x}) : \mathbf{x} \in \mathbb{C}^n,\ \mathbf{x}^*\mathbf{x} = 1\} \subseteq \mathbb{C}^m.$$

When $m = 1$, it reduces to the classical numerical range $W(A_1)$ of $A_1 \in M_n$, which is a useful tool for studying matrices and operators; for example, see [10, Chapter 1]. The joint numerical range of m matrices is useful in studying the behavior of the family of matrices $\{A_1, \ldots, A_m\} \subseteq M_n$, and has applications in many pure and applied areas. We refer the readers to the excellent survey [14] and the paper [15] on this subject.

When $m = 1$, the Toeplitz-Hausdorff theorem asserts that $W(A_1)$ is always convex. However, for $m \geq 2$, $W(A_1, \ldots, A_m)$ may fail to be convex; see [11]. Many researchers have studied matrices with certain commutativity properties that have convex joint numerical ranges, e.g., see [3, 4, 5, 6, 11, 12, 13]. In particular, Dash [5, Proposition 2.4] proved that $W(A_1, \ldots, A_m)$ is always convex for any commuting family $\{A_1, \ldots, A_m\} \subseteq M_2$ and raised the question on the same result for $\{A_1, \ldots, A_m\} \subseteq M_n$, with $n > 2$. In [13], the author gave a simple example, which was incorporated in [15] with

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some improvements, of a commuting family $\{A_1, A_2, A_3\} \subseteq M_4$ such that $W(A_1, A_2, A_3)$ is not convex, and raised the question of whether $W(A_1, A_2)$ is convex if $A_1A_2 = A_2A_1$; see [13, Problem 2]. We will show that the answer is negative if A_1, A_2 is a commuting pair of matrices (or infinite dimensional operators) with dimension at least 4. However, for a commuting pair of matrices $A_1, A_2 \in M_3$, $W(A_1, A_2)$ is always convex. We can then deduce from the results that $W(A_1, \dots, A_m)$ is always convex for any commuting family $\{A_1, \dots, A_m\} \subseteq M_3$.

Our paper is organized as follows. In Section 2, we present some preliminary results including a short proof on the convexity of $W(A_1, \dots, A_m)$ for every commuting family $\{A_1, \dots, A_m\} \subseteq M_2$. In Section 3, we present examples of commuting matrices (or infinite dimensional operators) A_1, A_2 of dimension at least 4 such that $W(A_1, A_2)$ is not convex. We then state our main result that $W(A_1, A_2)$ is convex if $A_1, A_2 \in M_3$ commute, and deduce that $W(A_1, \dots, A_m)$ is convex for any commuting family $\{A_1, \dots, A_m\} \subseteq M_3$. The rather involved proof of the main theorem on the convexity of $W(A_1, A_2)$ for commuting pair $A_1, A_2 \in M_3$ will be given in Section 4.

2. Preliminaries and commuting families in M_2

Let $\mathcal{H}_n = \{A \in M_n : A = A^*\}$ be the real space of all $n \times n$ Hermitian matrices and I_n be the $n \times n$ identity matrix. We summarize some properties of joint numerical ranges which are useful for the sequel. We refer the interested readers to [1, 8, 11].

Proposition 2.1. *Let $\mathcal{F} = \{A_1, \dots, A_m\} \subseteq M_n$. Suppose the complex space spanned by $\{A_1, \dots, A_m\}$ has a basis $\{C_1, \dots, C_s\}$. Let $A_j = H_j + iG_j$, where $H_j, G_j \in \mathcal{H}_n$ for $j = 1, \dots, m$. Then*

(a) $W(A_1, \dots, A_m) = W(U^*A_1U, \dots, U^*A_mU)$ *for any unitary* $U \in M_n$.

(b) $W(A_1, \dots, A_m) = W(A_1^t, \dots, A_m^t)$.

(c) $W(A_1, \dots, A_m)$ *is convex if and only if* $W(C_1, \dots, C_s)$ *is convex.*

(d) *The family $\mathcal{F}$ is commuting if and only if $\{C_1, \dots, C_s\}$ is commuting.*

(e) $W(A_1, \dots, A_m) \subseteq \mathbb{C}^m$ *can be identified with* $W(H_1, G_1, \dots, H_m, G_m) \subseteq \mathbb{R}^{2m}$.

(f) *For $n = 2$ and $H_1, \dots, H_m \in \mathcal{H}_2$, $W(H_1, \dots, H_m)$ is convex if and only if* span $\{I_2, H_1, \dots, H_m\} \neq \mathcal{H}_2$.

(g) *Suppose $n \geq 3$ and $H_1, \dots, H_m \in \mathcal{H}_n$. If* span $\{I_n, H_1, \dots, H_m\}$ *has dimension at most 4, then $W(H_1, \dots, H_m)$ is convex.*

Note that (c) and (f) are given in [8, Corollary 2.4 and Example 1] and (g) is given in [1, Corollary 1]. By (e), the study of convexity of $W(A_1, \ldots, A_m)$ can be reduced to $W(H_1, G_1, \ldots, H_m, G_m)$ for Hermitian matrices $H_1, G_1, \ldots, H_m, G_m$. However, it is clear that the commutativity of $A_1, \ldots, A_m$ does not imply the commutativity of $H_1, G_1, \ldots, H_m, G_m$. In fact, if $\{H_1, G_1, \ldots, H_m, G_m\}$ is a commuting family, then $\{A_1, \ldots, A_m\}$ is a commuting family of normal matrices, and $W(A_1, \ldots, A_m)$ will be polyhedral, i.e., a convex hull of finitely many points in $\mathbb{C}^m$; see [5, Theorem 2.5]. It is clear that $(\mu_1, \ldots, \mu_m) \in W(A_1, \ldots, A_m)$ if and only if $(1, \mu_1, \ldots, \mu_m) \in W(I_n, A_1, \ldots, A_m)$ for any $(A_1, \ldots, A_m) \in M_n^m$. By Proposition 2.1, to study the convexity of $W(A_1, \ldots, A_m)$, one may focus on $W(C_1, \ldots, C_s)$ where $\{I_n, C_1, \ldots, C_s\}$ is a basis for the span of $\{I_n, A_1, \ldots, A_m\}$. It is well-known that if $\{A_1, \ldots, A_m\}$ is a commuting family of matrices then there is a unitary U such that U^*A_jU are in upper triangular form for all $j = 1, \ldots, m$; see [16]. Our proofs often use this property.

Denote by conv$S$ and ∂S the convex hull and the boundary of a set S in $\mathbb{R}^m$ or $\mathbb{C}^m$, respectively. The next result describes the intersection of support planes of conv$W(A_1, \ldots, A_m)$ with $W(A_1, \ldots, A_m)$.

Proposition 2.2. *Let $B_1, \ldots, B_r \in \mathcal{H}_n$ be Hermitian matrices. For every unit vector, $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r$, let*

$$P_{\boldsymbol{\nu}} = \{\mathbf{b} \in \mathbb{R}^r : \mathbf{b}^*\boldsymbol{\nu} \le \lambda_1(\nu_1 B_1 + \cdots + \nu_r B_r)\},$$

where $\lambda_1(H)$ denotes the largest eigenvalue of $H \in \mathcal{H}_n$ and $\mathbf{b}^*\boldsymbol{\nu} = \sum_{i=1}^r b_i\nu_i$ for $\mathbf{b} = (b_1, \ldots, b_r) \in \mathbb{R}^r$. Then

$$\text{conv}W(B_1, \ldots, B_r) = \bigcap\{P_{\boldsymbol{\nu}} : \boldsymbol{\nu} = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r,\ \boldsymbol{\nu}^*\boldsymbol{\nu} = 1\}.$$

Consequently,

$$\begin{aligned} &\partial P_{\boldsymbol{\nu}} \cap W(B_1, \ldots, B_r) \\ = \ &\{(\mathbf{x}^*B_1\mathbf{x}, \ldots, \mathbf{x}^*B_r\mathbf{x}) : \mathbf{x} \in \mathbb{C}^n,\ \mathbf{x}^*\mathbf{x} = 1,\ B_{\boldsymbol{\nu}}\mathbf{x} = \lambda_1(B_{\boldsymbol{\nu}})\mathbf{x}\}, \end{aligned}$$

where $B_{\boldsymbol{\nu}} = \sum_{j=1}^r \nu_j B_j$. Moreover, $\partial P_{\boldsymbol{\nu}} \cap W(B_1, \ldots, B_r)$ is convex if and only if

$$W(X^*B_1X, \ldots, X^*B_rX)$$

is convex, where the columns of X form an orthonormal basis for the null space of $B_{\boldsymbol{\nu}} - \lambda_1(B_{\boldsymbol{\nu}})I_n$.

Proof. If $\mathbf{x} \in \mathbb{C}^n$ is a unit vector and $\mathbf{b} = (\mathbf{x}^*B_1\mathbf{x}, \ldots, \mathbf{x}^*B_r\mathbf{x}) \in W(B_1, \ldots, B_r)$, then for any unit vector $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r$ we have

$$\mathbf{b}^*\boldsymbol{\nu} = \mathbf{x}^* \left(\sum_{j=1}^r \nu_j B_j \right) \mathbf{x} \leq \lambda_1 \left(\sum_{j=1}^r \nu_j B_j \right).$$

Thus, $W(B_1, \ldots, B_r) \subseteq P_{\boldsymbol{\nu}}$. As $P_{\boldsymbol{\nu}}$ is convex, $\mathrm{conv}W(B_1, \ldots, B_r) \subseteq P_{\boldsymbol{\nu}}$.

Conversely, suppose $\mathbf{b} = (b_1, \ldots, b_r) \notin \mathrm{conv}W(B_1, \ldots, B_r) \subseteq \mathbb{R}^r$. By the separation theorem, there exists a real unit vector $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r$ such that $\sum_{j=1}^r b_j\nu_j > \sum_{j=1}^r q_j\nu_j$ for all $(q_1, \ldots, q_r) \in W(B_1, \ldots, B_r)$, i.e., for every unit vector $\mathbf{x} \in \mathbb{C}^n$

$$\sum_{j=1}^r b_j\nu_j > \sum_{j=1}^r \nu_j(\mathbf{x}^*B_j\mathbf{x}) = \mathbf{x}^* \left(\sum_{j=1}^r \nu_j B_j \right) \mathbf{x}.$$

So, $\sum_{j=1}^r b_j\nu_j > \lambda_1(\sum_{j=1}^r \nu_j B_j)$.

The last two assertions are clear. ∎

The following result is proven in [5, Proposition 2.4]. Recently, it is also given in [2, Theorem 2.2]. We give a short proof here for completeness.

Proposition 2.3. *For any commuting family $\mathcal{F} = \{A_1, \ldots, A_m\} \subseteq M_2$, $W(A_1, \ldots, A_m)$ is convex.*

Proof. To avoid trivial considerations, suppose $\mathcal{F}$ contains a non-scalar matrix $X \in M_2$. Applying a unitary similarity, we may assume that all matrices in $\mathcal{F}$ are in upper triangular form. Let $X_0 = X - \frac{\mathrm{tr}X}{2} I_2 = \begin{pmatrix} x_1 & x_2 \\ 0 & -x_1 \end{pmatrix}$. We claim that for every $Y \in \mathcal{F}$, $Y_0 = Y - \frac{\mathrm{tr}Y}{2} I_2 = \begin{pmatrix} y_1 & y_2 \\ 0 & -y_1 \end{pmatrix}$ is a multiple of X_0 as shown in [7, Theorem II]. Thus every A_j is a linear combination of I_2, $H_1 = (X_0 + X_0^*)/2$ and $H_2 = (X_0 - X_0^*)/(2i)$. By Proposition 2.1 (f), $W(A_1, \ldots, A_m)$ is convex.

To prove our claim, note that X_0 commutes with Y_0, i.e., $x_1y_2 = x_2y_1$. Since X is non-scalar, either x_1 or $x_2 \neq 0$.

If $x_1 = 0$, then $x_2 \neq 0$ and $x_2y_1 = 0$. Thus $y_1 = 0$ and $Y_0 = (y_2/x_2)X_0$. Our claim follows.

If $x_1 \neq 0$, then $x_1y_2 = x_2y_1$ implies $Y_0 = (y_1/x_1)X_0$. Again, our claim follows. ∎

3. Convexity of commuting family of dimension at least 3

In [13], the author gave an elegant example of a commuting family $\{A_1, A_2, A_3\} \subseteq M_4$ with non-convex $W(A_1, A_2, A_3)$. The following example illustrates that $W(A_1, A_2)$ may not be convex for a commuting pair $A_1, A_2 \in M_4$.

Example 3.1. Let $A_1 = H_1 + iG_1$ and $A_2 = A_1 + A_1^2 - A_1^3 - 12I_4 = H_2 + iG_2$ with

$$H_1 = \mathrm{diag}\,(2,2,1,0), \quad G_1 = \begin{pmatrix} 1 & 0 & 2-i & -i \\ 0 & 0 & -1+i & 1-i \\ 2+i & -1-i & 0 & 0 \\ i & 1+i & 0 & 0 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} 14 & -9-7i & 8-4i & -3i \\ -9+7i & 0 & 0 & 0 \\ 8+4i & 0 & 10 & -2-4i \\ 3i & 0 & -2+4i & -9 \end{pmatrix}$$

and

$$G_2 = \begin{pmatrix} 6 & -2-2i & 12-4i & -4-6i \\ -2+2i & 0 & -3+7i & 5-i \\ 12+4i & -3-7i & 5 & 1-2i \\ -4+6i & 5+i & 1+2i & 1 \end{pmatrix}.$$

Then $A_1A_2 = A_2A_1$. Note that for the unit vector $\boldsymbol{\nu} = (1,0,0,0)$, the matrix $A_{\boldsymbol{\nu}} = \nu_1 H_1 + \nu_2 G_1 + \nu_3 H_2 + \nu_4 G_1 = H_1$ has the largest eigenvalue 2, and the null space of $A_{\boldsymbol{\nu}} - 2I_4$ is spanned by the first two columns of I_4. Let $X \in M_{4,2}$ be the matrix formed by these two columns. It is easy to check that

$$\mathrm{span}\,\{X^*H_1X, X^*G_1X, X^*H_2X, X^*G_2X\} = \mathcal{H}_2.$$

By Proposition 2.1 (e) and (f), $W(X^*A_1X, X^*A_2X)$ is not convex. Since $W(X^*A_1X, X^*A_2X) = \{(\mu_1, \mu_2) \in W(A_1, A_2) : \mathrm{Re}\,\mu_1 = 2\}$, $W(A_1, A_2)$ is not convex. Here, $\mathrm{Re}\,\mu_1$ denotes the real part of μ_1.

Remark 3.2. For $n > 4$, one can extend the above example to $\tilde{A}_1 = A_1 \oplus 0_N$ and $\tilde{A}_2 = A_2 \oplus 0_N$, where $1 \le N \le \infty$. It is clear that $\tilde{A}_1\tilde{A}_2 = \tilde{A}_2\tilde{A}_1$ and $W(\tilde{A}_1, \tilde{A}_2)$ is not convex.

For commuting $A_1, A_2 \in M_3$, we have the following.

Theorem 3.3. *Suppose $A_1, A_2 \in M_3$ commute. Then $W(A_1, A_2)$ is convex.*

The proof of the result is quite involved and technical. We will present it in the next section. From Theorem 3.3, we can deduce the following.

Theorem 3.4. *Let $\{A_1, \ldots, A_m\} \subseteq M_3$ be a commuting family of matrices. Then the complex linear span of $\{I_3, A_1, \ldots, A_m\}$ has dimension at most 3, and hence $W(A_1, \ldots, A_m)$ is convex.*

Proof. We may assume that $A_1, \ldots, A_m$ are in upper triangular form, and $\mathcal{F} = \{I_3, A_1, \ldots, A_m\}$ is linearly independent. We are going to prove by contradiction that $m \leq 2$. In the following, we will use $\operatorname{diag} A \in \mathbb{C}^n$ as the vector of diagonal entries of $A \in M_n$.

Suppose to the contrary that $m > 2$. Then $\{\operatorname{diag} I_3, \operatorname{diag} A_1, \ldots, \operatorname{diag} A_m\}$ is linearly dependent. Therefore, $\operatorname{span} \mathcal{F}$ has a nonzero nilpotent. We may assume that A_1 is a nonzero nilpotent in $\operatorname{span} \mathcal{F}$ of the largest rank. Consider the following cases:

Case 1. Rank $A_1 = 2$. Then there is an invertible S such that $S^{-1}A_1S = J$ is the upper triangular Jordan block. Then for every $2 \leq i \leq m$, $A_1A_i = A_iA_1$ implies that $S^{-1}A_iS = a_iI_3 + b_iJ + c_iJ^2$ for some $a_i, b_i, c_i \in \mathbb{C}$. Since $\{I_3, A_1, \ldots, A_m\}$ is linearly independent, we have $m \leq 2$, a contradiction.

Case 2. Rank $A_1 = 1$. So, up to a nonzero multiple and a unitary similarity transform, we may assume that $A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then for every $2 \leq i \leq m$, the condition $A_1A_i = A_iA_1$ implies that A_i is in upper triangular form with the $(1,1)$ entry equal to the $(3,3)$ entry. We may then replace A_i by $A_i - \alpha_iI_3 - \beta_iA_1$ for some $\alpha_i, \beta_i \in \mathbb{C}$ and assume that

$$A_i = \begin{pmatrix} 0 & b_i & 0 \\ 0 & a_i & c_i \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{for some } a_i, b_i, c_i \in \mathbb{C},\ i = 2, \ldots, m.$$

If $a_i = 0$ for all $2 \leq i \leq m$, then $\operatorname{span}\{A_2, A_3\}$ would contain a nonzero nilpotent of rank 2, which contradicts the assumption that A_1 has the largest rank. Therefore, we may assume that $a_2 = 1$ and $a_3 = 0$. Then $A_2A_3 = A_3A_2$ implies that $b_3 = c_3 = 0$, which contradicts $\mathcal{F}$ being linearly independent.

This shows that $m \leq 2$ and the convexity of $W(A_1, A_2)$ follows from Theorem 3.3. ∎

4. Proof of Theorem 3.3

We divide it into two subsetions. We will always assume that $A_1 = H_1 + iG_1$ and $A_2 = H_2 + iG_2$, where H_1, G_1, H_2, G_2 are Hermitian. In view of Proposition 2.1 (g), we always assume that span $\{I_n, H_1, G_1, H_2, G_2\}$ has dimension 5 to avoid trivial considerations.

4.1. span $\{I_3, A_1, A_2\} \subseteq M_3$ **does not contain a nonzero nilpotent.**

In this subsection, we assume that span $\{I_3, A_1, A_2\} \subseteq M_3$ does not contain a nonzero nilpotent. Without loss of generality, by applying unitary similarity transforms and taking linear combinations of I_3, A_1, A_2, one can assume that

$$A_1 = \begin{pmatrix} 1 & u & w_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 0 & w_2 \\ 0 & 0 & v \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{l} \text{with } u, v \geq 0, \\ w_1 + uv + w_2 = 0. \end{array} \tag{4.1}$$

The reduction can be done as follows. Since A_1 and A_2 commute, we assume without loss of generality that both A_1, A_2 are in upper triangular form. Since span $\{I_3, A_1, A_2\}$ does not contain a nonzero nilpotent matrix, $\{\operatorname{diag} I_3, \operatorname{diag} A_1, \operatorname{diag} A_2\} \subseteq \mathbb{C}^3$ is linearly independent. Replacing A_j by $\alpha_j A_1 + \beta_j A_2 + \gamma_j I_3$ with suitable $\alpha_j, \beta_j, \gamma_j \in \mathbb{C}$, $j = 1, 2$, we may assume that

$$A_1 = \begin{pmatrix} 1 & a_1 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & b_1 & b_2 \\ 0 & 0 & b_3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have

$$A_1 A_2 = \begin{pmatrix} 0 & b_1 & a_2 + b_2 + a_1 b_3 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix} = A_2 A_1 = \begin{pmatrix} 0 & 0 & a_3 b_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $a_3 = b_1 = 0 = a_2 + b_2 + a_1 b_3$. Replacing A_j by DA_jD^{-1} with a diagonal unitary matrix D, we may assume $a_1,\ b_3 \geq 0$, so that we get (4.1).

By Proposition 2.1, the convexity of $W(A_1, A_2)$ is equivalent to the convexity of the numerical range of (A_1, A_2) transformed into the form (4.1).

In the following, we will show that $W(A_1, A_2)$ is convex if $A_1, A_2 \in M_3$ are of the form in (4.1).

Proposition 4.1. *Let $A_1, A_2 \in M_3$ be of the form* (4.1)*. If* $(0, 0) \in \{(u, w_1), (v, w_2), (u, v)\}$*, then $W(A_1, A_2)$ is convex.*

Proof. If $u = w_1 = 0$, then set $(H_2, G_2) = (A_2 + A_2^*, i(A_2^* - A_2))/2$ and identify $W(A_1, A_2)$ with $W(A_1, H_2, G_2) \subseteq \mathbb{R}^3$, which is convex by Proposition 2.1 (g).

If $v = w_2 = 0$, then set $(H_1, G_1) = (A_1 + A_1^*, i(A_1^* - A_1))/2$ and identify $W(A_1, A_2)$ with $W(H_1, G_1, A_2) \subseteq \mathbb{R}^3$, which is convex.

If $u = v = 0$, then $w_1 + w_2 = 0$. By the previous argument, $A_1 + A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ so that $W(A_1 + A_2, A_2)$ is convex, and so is $W(A_1, A_2)$. ■

Next, we treat the case where $(0, 0) \notin \{(u, w_1), (v, w_2), (u, v)\}$. First, we show that $W(A_1, A_2)$ has convex boundary.

Proposition 4.2. *Let $A_1, A_2 \in M_3$ be commuting matrices of the form* (4.1) *such that $(0, 0) \notin \{(u, w_1), (v, w_2), (u, v)\}$. Then $W(A_1, A_2)$ contains all of the boundary points of* $\operatorname{conv} W(A_1, A_2)$.

Proof. Suppose A_1 and A_2 satisfy the hypothesis, and $A_1 = H_1 + iG_1$, $A_2 = H_2 + iG_2$, where $H_1, H_2, G_1, G_2 \in \mathcal{H}_3$. For every unit vector $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3, \nu_4) \in \mathbb{R}^4$, let

$$P_{\boldsymbol{\nu}} = \left\{ (b_1, \ldots, b_4) \in \mathbb{R}^4 : \sum_{i=1}^{4} b_i \nu_i \le \lambda_1(\nu_1 H_1 + \nu_2 G_1 + \nu_3 H_2 + \nu_4 G_2) \right\}.$$

By Proposition 2.2 every boundary point of $\operatorname{conv} W(A_1, A_2)$ lies in $\partial P_{\boldsymbol{\nu}}$ for some $\boldsymbol{\nu} \in \mathbb{R}^4$, and

$$\partial P_{\boldsymbol{\nu}} \cap \operatorname{conv} W(A_1, A_2) = \operatorname{conv} \left(\partial P_{\boldsymbol{\nu}} \cap W(A_1, A_2) \right).$$

We will show that $\partial P_{\boldsymbol{\nu}} \cap \operatorname{conv} W(A_1, A_2) \subseteq W(A_1, A_2)$.

Case 1. *Suppose one of the following conditions holds,*

(i) $uv = 0$,
(ii) $(w_1 - w_2)^2 = (uv)^2$, *or*
(iii) $|w_1|\sqrt{1 + v^2} \ne |w_2|\sqrt{1 + u^2}$.

In each of these cases, we will show that $\partial P_{\boldsymbol{\nu}} \cap \operatorname{conv} W(A_1, A_2)$ is a singleton lying in $W(A_1, A_2)$ for any unit vector $\boldsymbol{\nu}$.

Let $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3, \nu_4) \in \mathbb{R}^4$ be a unit vector. The matrix $B_{\boldsymbol{\nu}} = \nu_1 H_1 + \nu_2 G_1 + \nu_3 H_2 + \nu_4 G_2$ has the form

$$\begin{pmatrix} \nu_1 & \dfrac{u(\nu_1 - i\nu_2)}{2} & \dfrac{w_1(\nu_1 - i\nu_2) + w_2(\nu_3 - i\nu_4)}{2} \\ \dfrac{u(\nu_1 + i\nu_2)}{2} & 0 & \dfrac{v(\nu_3 - i\nu_4)}{2} \\ \dfrac{\overline{w}_1(\nu_1 + i\nu_2) + \overline{w}_2(\nu_3 + i\nu_4)}{2} & \dfrac{v(\nu_3 + i\nu_4)}{2} & \nu_3 \end{pmatrix}.$$

Let r be the multiplicity of $\lambda_1(B_{\boldsymbol{\nu}})$. Since $\{I_3, H_1, G_1, H_2, G_2\}$ is linearly independent, $r \le 2$.

By Proposition 2.2, if $r = 1$, then $\partial P_{\boldsymbol{\nu}} \cap W(A_1, A_2)$ is singleton and equals $\partial P_{\boldsymbol{\nu}} \cap \operatorname{conv} W(A_1, A_2)$.

We show that $r = 2$ is impossible under any one of the assumptions (i), (ii) or (iii). Assume to the contrary that $r = 2$. As $(0,0) \notin \{(u,v),(u,w_1),(v,w_2)\}$, we see that $\lambda_1(B_{\boldsymbol{\nu}}) \neq 0$. Since the (2,2) entry of $B_{\boldsymbol{\nu}}$ is 0, we have $\lambda_1(B_{\boldsymbol{\nu}}) = \lambda_2(B_{\boldsymbol{\nu}}) > 0 \geq \lambda_3(B_{\boldsymbol{\nu}})$. Thus, there is a nonzero real vector (a,b,c,d) such that

$$R = I_3 + aH_1 + bG_1 + cH_2 + dG_2 = \mathbf{z}\mathbf{z}^* \tag{4.2}$$

for some nonzero $\mathbf{z} \in \mathbb{C}^3$, so that R is a rank one positive semidefinite matrix. Let $\mathcal{S}$ be the set of all nonzero real vectors (a,b,c,d) such that R is a rank one positive semi-definite matrix. We are going to show that $S = \emptyset$, thus arriving at a contradiction. In such a case, we may assume that

$$\mathbf{z} = (z_1, z_2, z_3) = \left(\sqrt{a+1}e^{i\theta_1}, 1, \sqrt{c+1}e^{i\theta_2}\right)$$

with

$$\sqrt{a+1}e^{i\theta_1} = z_1 = z_1\bar{z}_2 = \frac{u}{2}(a-ib),\ \sqrt{c+1}e^{-i\theta_2} = \bar{z}_3 = z_2\bar{z}_3 = \frac{v}{2}(c-id),$$

and

$$\frac{uv(a-ib)(c-id)}{4} = \sqrt{(a+1)(c+1)}e^{i(\theta_1-\theta_2)} = z_1\bar{z}_3 = \frac{w_1(a-ib)+w_2(c-id)}{2}.$$

The matrix R given by (4.2) then has the form

$$R = \begin{pmatrix} a+1 & u(a-ib)/2 & uv(a-ib)(c-id)/4 \\ u(a+ib)/2 & 1 & v(c-id)/2 \\ uv(a+ib)(c+id)/4 & v(c+id)/2 & c+1 \end{pmatrix}. \tag{4.3}$$

Since R has rank 1, we have $(a,b) \neq (0,0)$ and $(c,d) \neq (0,0)$. If any one of the assumptions (i), (ii) or (iii), holds, we are going to derive a contradiction.

Suppose that (i) holds, i.e., $uv = 0$. Recall from (4.1) that u and v are nonnegative. Since $(u,v) \neq (0,0)$, we assume $u = 0 < v$ or $v = 0 < u$. Let $u = 0$ and $v > 0$. Since $(u,w_1) \neq (0,0)$, we may replace (A_1,A_2) by (D^*A_1D, D^*A_2D) for some suitable diagonal unitary matrix D and assume that $-w_2 = w_1 > 0$. Suppose there is a real vector (a,b,c,d) such that R given by (4.2) is a rank one positive semi-definite matrix of the form (4.3). Since the $(1,2)$-entry is zero, we see that $a = -1$. Now, $-w_2 = w_1 > 0$ and the $(1,3)$-entry of R is $w_1((a-ib)-(c-id)) = 0$. Thus $c = a = -1$ and $b = d$. As a result, the $(3,3)$-entry of R is zero and so must be the $(2,3)$-entry. Hence, $v = 0$, which is contradiction. Similarly, we can show that for $u > 0$ and $v = 0$, $\mathcal{S} = \emptyset$.

Suppose now that $u, v > 0$. As the matrix R in (4.3) is rank one, we have $4(a+1)/u^2 = (a^2+b^2)$ and $4(c+1)/v^2 = (c^2+d^2)$. Therefore,

$$a + ib \in \mathcal{E}_u := \{x+iy : (x-2/u^2)^2 + y^2 = 4(1/u^2 + 1/u^4)\} \tag{4.4}$$

and

$$c + id \in \mathcal{E}_v := \{x+iy : (x-2/v^2)^2 + y^2 = 4(1/v^2 + 1/v^4)\}. \tag{4.5}$$

Since $w_1+uv+w_2 = 0$, we may let $w_1 = -uv(1-\xi)/2, w_2 = -uv(1+\xi)/2$ for some $\xi \in \mathbb{C}$. As $\xi = (w_1-w_2)/uv$, assumption (ii) holds, i.e., $(w_1-w_2)^2 = (uv)^2$, if and only if $\xi = \pm 1$. Now the $(1,3)$ entry of R becomes

$$\begin{aligned} uv(a-ib)(c-id)/4 &= [w_1(a-ib) + w_2(c-id)]/2 \\ &= -[uv(a-ib)(1-\xi) + uv(c-id)(1+\xi)]/4. \end{aligned}$$

Thus, we have

$$(a-ib)(c-id) = (\xi-1)(a-ib) - (\xi+1)(c-id). \tag{4.6}$$

If $\xi = 1$, then we have $(a-ib)(c-id) = -2(c-id)$ so that $a - ib = -2$. Thus, the $(1,1)$ entry of R is -1, which is impossible. Similarly, if $\xi = -1$, then the $(3,3)$ entry of R is -1, which is impossible.

Suppose $\xi \neq \pm 1$ and (iii) holds. Substituting $w_1 = -uv(1-\xi)/2, w_2 = -uv(1+\xi)/2$, we have

$$|1-\xi|\sqrt{1+v^2} \neq |1+\xi|\sqrt{1+u^2}. \tag{4.7}$$

Since $(a-ib), (c-id) \neq 0$, (4.6) is equivalent to

$$\frac{1+\xi}{a-ib} + \frac{1-\xi}{c-id} + 1 = 0. \tag{4.8}$$

Note that $\mu \in \mathbb{C}$ lies on a circle with center $\mu_0 \geq 0$ and radius $r > \mu_0$ if and only if

$$0 = (\mu-\mu_0)(\bar{\mu}-\mu_0) - r^2 = \mu\bar{\mu} - (\mu_0\bar{\mu} + \mu_0\mu) + (\mu_0^2 - r^2).$$

Dividing by $\mu\bar{\mu}(\mu_0^2 - r^2)$, we have

$$(\mu\bar{\mu})^{-1} - \left(\frac{\mu_0}{\mu_0^2 - r^2}\mu^{-1} + \frac{\mu_0}{\mu_0^2 - r^2}\bar{\mu}^{-1}\right) = -\frac{1}{\mu_0^2 - r^2},$$

equivalently,

$$\begin{aligned} &\left(\mu^{-1} - \frac{\mu_0}{\mu_0^2 - r^2}\right)\left(\bar{\mu}^{-1} - \frac{\mu_0}{\mu_0^2 - r^2}\right) \\ &= \frac{\mu_0^2}{(\mu_0^2 - r^2)^2} - \frac{1}{\mu_0^2 - r^2} = \frac{r^2}{(\mu_0^2 - r^2)^2}. \end{aligned}$$

Applying this to the circles $\mathcal{E}_u$ and $\mathcal{E}_v$, we see that

$$\mathcal{E}_u^{-1} = \{1/\mu : \mu \in \mathcal{E}_u\} = \{-1/2 + 1/2\sqrt{1+u^2}e^{i\theta} : t \in [0, 2\pi)\},$$

and

$$\mathcal{E}_v^{-1} = \{1/\mu : \mu \in \mathcal{E}_v\} = \{-1/2 + 1/2\sqrt{1+v^2}e^{i\theta} : t \in [0, 2\pi)\}.$$

Since $c - id \in \mathcal{E}_v$ is nonzero, (4.8) yields

$$\frac{1}{c-id} = \frac{1}{\xi - 1} + \frac{\xi+1}{(\xi-1)(a-ib)} \in \tilde{\mathcal{E}}_u \cap \mathcal{E}_v^{-1}, \tag{4.9}$$

where

$$\begin{aligned} \tilde{\mathcal{E}}_u &= \left\{ \frac{1}{\xi-1} + \frac{(\xi+1)}{2(\xi-1)}(-1+\sqrt{1+u^2}e^{i\theta}) : \theta \in [0,2\pi) \right\} \\ &= \left\{ -\frac{1}{2} + \frac{(\xi+1)}{2(\xi-1)}\sqrt{1+u^2}e^{i\theta} : \theta \in [0,2\pi) \right\}. \end{aligned}$$

By (4.7), $\tilde{\mathcal{E}}_u \cap \mathcal{E}_v^{-1} = \emptyset$, a contradiction to (4.9). Thus the proof in Case 1 is complete.

Case 2. *Suppose conditions* (i), (ii) *and* (iii) *in Case 1 do not hold.*

Then $|w_1|\sqrt{1+v^2} = |w_2|\sqrt{1+u^2}$. If $m \in \mathbb{N}$, then $B_m = A_1 + E_{13}/m$ and $C_m = A_2 - E_{13}/m$ are commuting matrices in M_3 with $(1,3)$-entries $w_1 + 1/m$ and $w_2 - 1/m$, respectively. We are going to show that

$$|w_1 + 1/m|\sqrt{1+v^2} = |w_2 - 1/m|\sqrt{1+u^2} \tag{4.10}$$

for at most one m.

Note that (4.10) holds if and only if

$$\begin{aligned} &(mw_1+1)(m\overline{w}_1+1)(1+v^2) = (mw_2-1)(m\overline{w}_2-1)(1+u^2) \\ \Leftrightarrow\ & 2\left(\mathrm{Re}\,(w_1)(1+v^2) + \mathrm{Re}\,(w_2)(1+u^2)\right)m + (v^2-u^2) = 0 \end{aligned} \tag{4.11}$$

If (4.11) holds for more than one m, then $v^2 = u^2$ and $\mathrm{Re}\,(w_1) = -\mathrm{Re}\,(w_2)$. Then it follows from $u, v \geq 0$ and $w_1 + uv + w_2 = 0$ in (4.1) that $uv = 0$ and (i) holds, a contradiction.

So there exists m_0 such that $|w_1 + 1/m|\sqrt{1+v^2} \neq |w_2 - 1/m|\sqrt{1+u^2}$ for all $m \geq m_0$. By Case 1, $\partial\mathrm{conv}W(B_m, C_m) \subseteq W(B_m, C_m)$. Now, every boundary point $(\mu_1, \mu_2) \in \mathrm{conv}W(A_1, A_2)$ is the limit of a sequence of points $\{(\mu_1(m), \mu_2(m))\colon m \geq m_0\}$ with $(\mu_1(m), \mu_2(m)) \in \partial(\mathrm{conv}W(B_m, C_m)) \subseteq W(B_m, C_m)$. Note that $W(B_m, C_m) \to W(A_1, A_2)$ as $m \to \infty$ in the Hausdorff metric on compact subsets of $\mathbb{R}^2$. We have $(\mu_1, \mu_2) \in W(A_1, A_2)$. Hence, $\partial(\mathrm{conv}W(A_1, A_2)) \subseteq W(A_1, A_2)$. This finishes the proof in Case 2, and thus finishes the proof of Proposition 4.2. ■

Let $\mu_1 \in W(A_1)$ and $W(\mu_1, A_2) = \{\mu : (\mu_1, \mu) \in W(A_1, A_2)\}$. Now, we know that $W(A_1, A_2)$ has convex boundary if $A_1, A_2 \in M_3$ commute.

Therefore, to prove that $W(A_1, A_2)$ is convex, we only need to show that $W(\mu_1, A_2)$ is simply connected for every $\mu_1 \in W(A_1)$.

To prove the latter property, we will show that

$$W(\mu_1, A_2) = \{\mu : (\mu_1, \mu) \in \mathrm{conv} W(A_1, A_2)\}.$$

To this end, using linear combinations, unitary similarity and transposition of matrices, we note that the matrices A_1 and A_2 in (4.1) can be transformed as

$$A_1 = E_{11} + aE_{12},\ A_2 = \begin{pmatrix} -a \\ 1 \\ b \end{pmatrix} (0\ 1\ \xi) \text{ where } a > 0,\ b \geq 0,\ \xi \in \mathbb{C}. \tag{4.12}$$

To prove this fact, observe that if $w_1 = 0$, then $w_2 = -uv$ we can replace A_2 with

$$I_3 - (A_1 + A_2) = \begin{pmatrix} 0 & -u & uv \\ 0 & 1 & -v \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -u \\ 1 \\ 0 \end{pmatrix} (0\ 1\ -v)$$

If $w_2 = 0$, then replace (A_1, A_2) with (TA_2^tT, TA_1^tT), where $T = E_{13} + E_{22} + E_{31}$. We have

$$TA_2^tT = \begin{pmatrix} 1 & v & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad TA_1^tT = \begin{pmatrix} 0 & 0 & -uv \\ 0 & 0 & u \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we can proceed as the above case for $w_1 = 0$.

Suppose $w_1, w_2 \neq 0$. Let $a = \sqrt{u^2 + |w_1|^2}$ and $U = (1) \oplus \frac{1}{a} \begin{pmatrix} u & w_1 \\ \overline{w}_1 & -u \end{pmatrix}$ be unitary. Then

$$U^*A_1U = \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ U^*A_2U = \gamma \begin{pmatrix} 0 & -a & -ac \\ 0 & 1 & c \\ 0 & b & bc \end{pmatrix},$$

where $\gamma = -(\overline{w}_1 w_2)/a^2$, $b = (u - v\overline{w}_1)/w_2$ and $c = -u/\overline{w}_1$. Let $b = |b|e^{i\theta}$ and $D = \mathrm{diag}\,(1, 1, e^{i\theta})$. We replace (A_1, A_2) with $(D^*U^*A_1UD, \frac{1}{\gamma}D^*U^*A_2UD)$. Direct calculation gives

$$DU^*A_1UD^* = \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \frac{1}{\gamma}DU^*A_2UD^* = \begin{pmatrix} -a \\ 1 \\ |b| \end{pmatrix} (0\ 1\ \xi),$$

where $\xi = ce^{i\theta}$. If $\xi = 0 = b$, then $A_1 + A_2 = \mathrm{diag}\,(1, 1, 0)$ is Hermitian. By Proposition 2.1(g), $W(A_1, A_1 + A_2)$ is convex and hence $W(A_1, A_2)$ is also convex. So, we consider that $(b, \xi) \neq (0, 0)$.

Recall that a set $\mathcal{S}$ in $\mathbb{R}^n$ or $\mathbb{C}^n$ is *star-shaped* with a *star center* $s_0 \in \mathcal{S}$ if $ts_0 + (1-t)s \in \mathcal{S}$ for all $t \in [0,1]$ and $s \in \mathcal{S}$. We have the following.

Proposition 4.3. *Suppose that A_1, A_2 are as given by (4.1). For every $\mu_1 \in W(A_1)$, the set*

$$W(\mu_1, A_2) = \{\mu : (\mu_1, \mu) \in W(A_1, A_2)\}$$

is star-shaped. Consequently,

$$W(\mu_1, A_2) = \{\mu : (\mu_1, \mu) \in \mathrm{conv} W(A_1, A_2)\},$$

and $W(A_1, A_2)$ is convex.

Proof. Without loss of generality, we may assume that A_1 and A_2 are of the form (4.12). Suppose $\mu_1 \in W(A_1)$. We are going to show that $W(\mu_1, A_2)$ is star-shaped with a star center $1 - \mu_1$.

Let $\nu \in \mathbb{C}^3$ be a unit vector such that $\boldsymbol{\nu}^* A_1 \boldsymbol{\nu} = \mu_1$. By replacing $\boldsymbol{\nu}$ with $\tilde{\boldsymbol{\nu}} = e^{i\theta}\boldsymbol{\nu}$ for some $\theta \in \mathbb{R}$, we may assume that the first entry of $\boldsymbol{\nu}$ is non-negative. Let

$$S = \left\{ \left(p_1, p_2 e^{i\theta}, p_3 e^{i\phi}\right)^t : \theta, \phi \in [0, 2\pi),\ p_1,\ p_2,\ p_3 \geq 0,\ p_1^2 + p_2^2 + p_3^2 = 1 \right\}.$$

If $\boldsymbol{\nu} = \left(0, p_2 e^{i\theta}, p_3 e^{i\phi}\right)^t \in S$, we have $\mu_1 = \boldsymbol{\nu}^* A_1 \boldsymbol{\nu} = 0$. Moreover

$$\boldsymbol{\nu}^* A_2 \boldsymbol{\nu} \in W\left(\begin{pmatrix} 1 & \xi \\ b & b\xi \end{pmatrix}\right) \subseteq W(0, A_2).$$

As $W\left(\begin{pmatrix} 1 & \xi \\ b & b\xi \end{pmatrix}\right)$ is convex, and it contains the point $\{1\}$, we have $t + (1 - t)\boldsymbol{\nu}^* A_2 \boldsymbol{\nu} \in W(0, A_2)$ for all $t \in [0,1]$. Now assume $\nu \in S$ with $\boldsymbol{\nu}^* A_1 \boldsymbol{\nu} = \mu_1$ and $p_1 > 0$. Then

$$\mu_1 = p_1^2 + a p_1 p_2 e^{i\theta}, \qquad \text{i.e.,} \qquad p_2 e^{i\theta} = \frac{\mu_1 - p_1^2}{a p_1},$$

and

$$\begin{aligned} 1 - p_3^2 &= p_1^2 + p_2^2 = p_1^2 + \left|\frac{\mu_1 - p_1^2}{a p_1}\right|^2 = \frac{a^2 p_1^4 + |\mu_1 - p_1^2|^2}{a^2 p_1^2} \\ &= \frac{(a^2+1)p_1^4 + |\mu_1|^2 - 2(\mathrm{Re}\,\mu_1)p_1^2}{a^2 p_1^2}. \end{aligned}$$

Therefore, we have

$$-a^2 p_1^2 p_3^2 = (a^2+1)p_1^4 - (2\mathrm{Re}\,\mu_1 + a^2)p_1^2 + |\mu_1|^2 \leq 0. \tag{4.13}$$

By the above calculation, $\boldsymbol{\nu} \in S$ with positive first entry and $\boldsymbol{\nu}^* A_1 \boldsymbol{\nu} = \mu_1$ if and only if $\boldsymbol{\nu} = (p_1, (\mu_1/p_1 - p_1)/a, p_3 e^{i\phi})$ for $p_1 > 0$ satisfying inequality

(4.13), $\phi \in [0, 2\pi)$ and $p_3 = \sqrt{1 - p_1^2 - |(\mu_1/p_1 - p_1)/a|^2}$. Now

$$\begin{aligned}
\boldsymbol{\nu}^* A_2 \boldsymbol{\nu} &= (p_1 \ (\bar{\mu}_1/p_1 - p_1)/a \ p_3 e^{-i\phi}) \begin{pmatrix} 0 & -a & -a\xi \\ 0 & 1 & \xi \\ 0 & b & b\xi \end{pmatrix} \begin{pmatrix} p_1 \\ (\mu_1/p_1 - p_1)/a \\ p_3 e^{i\phi} \end{pmatrix} \\
&= \big(- ap_1 + (\bar{\mu}_1/p_1 - p_1)/a + bp_3 e^{-i\phi} \big) \big((\mu_1/p_1 - p_1)/a + \xi p_3 e^{i\phi} \big) \\
&= p_1^2 - \mu_1 + |\mu_1/p_1 - p_1|^2/a^2 + b\xi p_3^2 \\
&\quad + p_3 \{ (-ap_1 + (\bar{\mu}_1/p_1 - p_1)/a)\xi e^{i\phi} + (\mu_1/p_1 - p_1)(b/a)e^{-i\phi} \} \\
&= 1 - \mu_1 + (b\xi - 1)p_3^2 \\
&\quad + p_3 \{ (-ap_1 + (\bar{\mu}_1/p_1 - p_1)/a)\xi e^{i\phi} + (\mu_1/p_1 - p_1)(b/a)e^{-i\phi} \}.
\end{aligned}$$

For a fixed $p_1 > 0$, if we let ϕ vary in $[0, 2\pi)$, we see that $\nu^* A_2 \nu$ generates all the points of an ellipse denoted by $\mathcal{E}(p_1)$. Hence, $\mathcal{E}(p_1) \subseteq W(\mu_1, A_2)$. For a fixed $\mu_1 \in W(A_1)$, let p_u and p_ℓ be respectively the largest and smallest non-negative values of p_1 for which the inequality

$$(a^2 + 1)p_1^4 - (2\mathrm{Re}\,\mu_1 + a^2)p_1^2 + |\mu_1|^2 \le 0$$

in (4.13) is satisfied. Then

$$W(\mu_1, A_2) = \bigcup_{p \in [p_\ell, p_u]} \mathcal{E}(p).$$

Here we denote $\mathcal{E}(0) = W\left(\begin{pmatrix} 1 & \xi \\ b & b\xi \end{pmatrix}\right)$. We next show that every point inside the ellipse $\mathcal{E}(p)$ also lies in $W(\mu_1, A_2)$. As $\mu_1 \in W(A_1) = W(A_0)$ with $A_0 = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$, there is a unit vector $\tilde{\boldsymbol{\nu}} = (\tilde{p}, \nu_2) \in \mathbb{C}^2$ with $\tilde{p} \ge 0$ such that $\tilde{\boldsymbol{\nu}}^* A_0 \tilde{\boldsymbol{\nu}} = \mu_1$. Thus, with $\boldsymbol{\nu} = (\tilde{p}, \nu_2, 0) \in \mathbb{C}^3$ we have $\boldsymbol{\nu}^* A_1 \boldsymbol{\nu} = \mu_1$. The corresponding ellipse $\mathcal{E}(\tilde{p}) = \{1 - \mu_1\}$ is a singleton as $p_3 = 0$. For every $p_1 \in [p_\ell, p_u]$, we may let p_1 change continuously to $\tilde{p}$. Recall that $\boldsymbol{\nu} = (p_1, (\mu_1/p_1 - p_1)/a, p_3 e^{i\phi})$. As the entries of $\boldsymbol{\nu}$ are continuous functions in $p_1 > 0$, the ellipse $\mathcal{E}(p_1)$ will deform continuously to the singleton $\mathcal{E}(\tilde{p})$ in the set $W(\mu_1, A_2)$. Hence, by continuity all the points inside the ellipse $\mathcal{E}(p_1)$ also lie in $W(\mu_1, A_2)$, i.e.,

$$W(\mu_1, A_2) = \bigcup_{p \in [p_\ell, p_u]} \mathcal{E}(p) = \bigcup_{p \in [p_\ell, p_u]} \bar{\mathcal{E}}(p), \tag{4.14}$$

where $\overline{\mathcal{E}}(p)$ is the elliptical disk with $\mathcal{E}(p)$ as boundary.

We will show that $\bigcup_{p \in [p_\ell, p_u]} \bar{\mathcal{E}}(p)$ is star-shaped with star center $1 - \mu_1$. Solving p_3 as a function of p_1 in (4.13), we see that p_3 attains the maximum value

$$\hat{p}_3 = \sqrt{1 - p_1^2 - |(\mu_1/p_1 - p_1)/a|^2} = \frac{\sqrt{a^2 + 2(\operatorname{Re}\mu_1 - \sqrt{1+a^2}|\mu_1|)}}{a}$$

when $p_1 = \hat{p} = \sqrt{\frac{|\mu_1|}{\sqrt{1+a^2}}}$. In general, for each choice of $p_3 \in [0, \hat{p}_3]$, there are $p_1^- \in [p_\ell, \hat{p}]$ and $p_1^+ \in [\hat{p}, p_u]$ satisfying the left-hand side of (4.13). For every $0 < r < 1$ and $p_3 \in [0, \hat{p}_3]$, set $\tilde{p}_3 = rp_3$ and let $\tilde{p}_1^- \in [p_\ell, \hat{p}]$ and $\tilde{p}_1^+ \in [\hat{p}, p_u]$ satisfying equation (4.13) for p_3. With some intricate arguments presented in the Appendix, we will show that

(I) If $|\xi|^2(1+a^2) \geq b^2$, then $\overline{\mathcal{E}}(p_1^-) \subseteq \overline{\mathcal{E}}(p_1^+)$, and for every $\mu_2 \in \overline{\mathcal{E}}(p_1^+)$, $(1-r^2)(1-\mu_1) + r^2\mu_2 \in \overline{\mathcal{E}}(\tilde{p}_1^+)$.

(II) If $|\xi|^2(1+a^2) \leq b^2$, then $\overline{\mathcal{E}}(p_1^+) \subseteq \overline{\mathcal{E}}(p_1^-)$, and for every $\mu_2 \in \overline{\mathcal{E}}(p_1^-)$, $(1-r^2)(1-\mu_1) + r^2\mu_2 \in \overline{\mathcal{E}}(\tilde{p}_1^-)$.

Once (I) and (II) are proved, by (4.14) we see that $W(\mu_1, A_2)$ is star-shaped with $1-\mu_1$ as a star center, i.e., for any $\mu_2 \in W(\mu_1, A_2)$ and $t \in [0,1]$,

$$t\mu_2 + (1-t)(1-\mu_1) \in W(\mu_1, A_2).$$

Let $S = \{\mu : (\mu_1, \mu) \in \operatorname{conv} W(A_1, A_2)\}$. We have $W(\mu_1, A_2) \subseteq S$. Note that $S \subseteq \mathbb{C}$ is convex and compact. By Proposition 4.2,

$$\begin{aligned} \partial S &\subseteq \{\mu : (\mu_1, \mu) \in \partial(\operatorname{conv} W(A_1, A_2))\} \\ &\subseteq \{\mu : (\mu_1, \mu) \in W(A_1, A_2)\} = W(\mu_1, A_2). \end{aligned}$$

The star-shapedness of $W(\mu_1, A_2)$ implies that this set is simply connected. Therefore, $S \subseteq W(\mu_1, A_2)$. Hence, $S = W(\mu_1, A_2)$.

Now, we can show that $W(A_1, A_2)$ is convex as follows. Suppose (x_1, y_1), $(x_2, y_2) \in W(A_1, A_2)$, $t \in [0,1]$ and $(\mu_1, \mu_2) = t(x_1, y_1) + (1-t)(x_2, y_2)$. Then $(\mu_1, \mu_2) \in \operatorname{conv} W(A_1, A_2)$. We have $\mu_2 \in \{\mu : (\mu_1, \mu) \in \operatorname{conv} W(A_1, A_2)\} = W(\mu_1, A_2)$. Thus, $(\mu_1, \mu_2) \in W(A_1, A_2)$. So, $W(A_1, A_2)$ is convex. ∎

4.2. span $\{I_3, A_1, A_2\} \subseteq M_3$ contains a nonzero nilpotent.

Here we present the proof of Theorem 3.3 when span $\{I_3, A_1, A_2\}$ contains a nonzero nilpotent matrix. We may assume that $\{I_3, A_1, A_2\}$ is linearly independent and A_1 is nilpotent.

Similar to the case considered in Subsection 4.1, we can apply linear combinations and unitary similarity transforms to change A_1, A_2 to a simpler form. First, we show that one may assume that A_1 is rank 1. Suppose A_1 is rank 2. Then there is an invertible S such that $S^{-1}A_1S = J$ is the upper triangular Jordan block. Then $A_1A_2 = A_2A_1$ implies that

$S^{-1}A_2S = aI_3 + bJ + cJ^2$. We may replace A_2 by $A_2 - aI_3 - bA_1$. Then A_2 is a rank one nilpotent. We may then interchange the roles of A_1 and A_2. Now, A_1 is a rank one nilpotent matrix in span $\{I_3, A_1, A_2\}$. So, up to a nonzero multiple and a unitary similarity transform, we may assume that $A_1 = E_{13}$, where as before $\{E_{ij} : i, j = 1, 2, 3\}$ is the standard basis of M_3. The condition $A_1A_2 = A_2A_1$ implies that A_2 is in upper triangular form with $(1,1)$-entry equal to $(3,3)$-entry. We may then replace A_2 by $A_2 - \gamma_1 I_3 - \gamma_2 A_1$ and assume that

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and } A_2 = \begin{pmatrix} 0 & b & 0 \\ 0 & a & c \\ 0 & 0 & 0 \end{pmatrix}.$$

If necessary, we may also replace (A_1, A_2) with (DA_1^tD, DA_2^tD), where $D = E_{13} + E_{22} + E_{31}$, and assume that $|b| \geq |c|$.

If $b = 0$, then we may assume that $A_2 = E_{22}$. By Proposition 2.1, (g) and (e),

$$W(A_1, A_2) \cong W\left(\frac{(E_{13}+E_{31})}{2}, \frac{i(E_{13}-E_{31})}{2}, E_{22}\right)$$

is convex.

If $b \neq 0$, let $\zeta = |a/b|$ and $\xi = |c/b|$. Suppose $a/b = \zeta e^{i\theta}$ and $c/b = \xi e^{i\phi}$, $\theta, \phi \in [0, 2\pi)$. Let $U = \operatorname{diag}\left(1, e^{i\theta}, e^{i(2\theta-\phi)}\right)$. Replacing (A_1, A_2) with $\left(e^{i(\phi-2\theta)}U^*A_1U,\ e^{-i\theta}U^*A_2U/b\right)$, we have $(A_1, A_2) = (E_{13}, \zeta E_{22} + E_{12} + \xi E_{23})$, where $\zeta \geq 0$ and $\xi \in [0, 1]$.

Let $P_m = E_{11}/m$ and $Q_m = (E_{22} - E_{32})/m$ for $m \in \mathbb{N}$. Then

$$A_1 + P_m = \begin{pmatrix} 1/m & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad A_2 + Q_m = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \zeta + 1/m & \xi \\ 0 & -1/m & 0 \end{pmatrix}$$

commute. Moreover,

$$aI_3 + b(A_1 + P_m) + c(A_2 + Q_m) = \begin{pmatrix} a + b/m & c & b \\ 0 & a + c(\zeta + 1/m) & c\xi \\ 0 & -c/m & a \end{pmatrix}$$

is nilpotent if and only if

$$a + b/m = 0, 2a + c(\zeta + 1/m) = 0, \text{ and } a^2 + ac(\zeta + 1/m) + c^2\xi/m = 0.$$

From the last two equations, if $\zeta + 1/m \neq 0$, then

$$\frac{a}{c} = \frac{-(\zeta + 1/m)}{2} \text{ and } 0 = \left(\frac{a}{c}\right)^2 + \frac{a}{c}\left(\zeta + \frac{1}{m}\right) + \frac{\xi}{m} = \frac{\xi}{m} - \frac{1}{4}\left(\zeta + \frac{1}{m}\right)^2,$$

which can be true for at most two choices of m. Hence, except for finitely many values of m, the linear span of the set $\{I_3, A_1 + P_m, A_2 + Q_m\}$ does not contain nonzero nilpotent and $W(A_1+P_m, A_2+Q_m)$ is convex by Proposition 4.3 in Subsection 4.1.

Suppose L is the line segment joining (x^*A_1x, x^*A_2x), $(y^*A_1y, y^*A_2y) \in W(A_1, A_2)$. Let L_m be the line segment joining $(x^*(A_1+P_m)x, x^*(A_2+Q_m)x)$ and $(y^*(A_1+P_m)y, y^*(A_2+Q_m)y)$. Clearly, the endpoints of the line segments L_m converges to those of L. Thus, $L_m \to L$ in the Hausdorff metric as $m \to \infty$. Note that $L_m \subseteq W(A_1+P_m, A_2+Q_m)$ because $W(A_1+P_m, A_2+Q_m)$ is convex by Proposition 4.3 in Subsection 4.1. Since $W(A_1+B_m, A_2+Q_m) \to W(A_1, A_2)$ in the Hausdorff metric as $m \to \infty$, we infer that $L_m \to L$ as $m \to \infty$, so that $L \subseteq W(A_1, A_2)$, and therefore $W(A_1, A_2)$ is convex. ■

Appendix: Proof of (I) and (II)

We use the notation introduced in Section 4.2. For every $q \in [p_\ell, p_u]$, let

$$C_q = \begin{pmatrix} 0 & \xi((\bar{\mu}_1/q - q)/a - aq) \\ b(\mu_1/q - q)/a & 0 \end{pmatrix}.$$

If $q \in [p_\ell, p_u]$ and $q_3^2 = 1 - q^2 - |(\mu_1/q - q)/a|^2$, then

$$\bar{\mathcal{E}}(q) = 1 - \mu_1 + (b\xi - 1)q_3^2 + 2q_3 W(C_q).$$

It is clear that $W\left(C_{p_1^-}\right) \subseteq W\left(C_{p_1^+}\right)$ if and only if $\bar{\mathcal{E}}(p_1^-) \subseteq \bar{\mathcal{E}}(p_1^+)$. For every $0 < r < 1$ and $\mu_2 \in \overline{\mathcal{E}}(p_1^+)$, we have

$$(1-r^2)(1-\mu_1) + r^2\mu_2 \in 1 - \mu_1 + (b\xi - 1)(rq_3)^2 + 2(rq_3)W(rC_{p_1^+}).$$

Let $\tilde{p}_3 = rq_3$. Thus, to prove (I), it suffices to show that

$$W\left(rC_{p_1^-}\right) \subseteq W\left(rC_{p_1^+}\right) \subseteq W\left(C_{\tilde{p}_1^+}\right), \tag{4.15}$$

By Proposition 2.2, the inclusions (4.15) is equivalent to

$$r\lambda_1\left(e^{i\theta}C_{p_1^-} + e^{-i\theta}C^*_{p_1^-}\right) \le r\lambda_1\left(e^{i\theta}C_{p_1^+} + e^{-i\theta}C^*_{p_1^+}\right) \le \lambda_1\left(e^{i\theta}C_{\tilde{p}_1^+} + e^{-i\theta}C^*_{\tilde{p}_1^+}\right),$$

for every $\theta \in [0, 2\pi)$

Note that

$$\lambda_1\left(e^{i\theta}C_q + e^{-i\theta}C_q^*\right) = \sqrt{|\det(e^{i\theta}C_q + e^{-i\theta}C_q^*)|}.$$

Hence, it suffices to show that for every $\theta \in [0, 2\pi)$

$$\begin{aligned} r^2\left|\det\left(e^{i\theta}C_{p_1^-} + e^{-i\theta}C^*_{p_1^-}\right)\right| &\le r^2\left|\det\left(e^{i\theta}C_{p_1^+} + e^{-i\theta}C^*_{p_1^+}\right)\right| \\ &\le \left|\det\left(e^{i\theta}C_{\tilde{p}_1^+} + e^{-i\theta}C^*_{\tilde{p}_1^+}\right)\right|. \end{aligned} \tag{4.16}$$

For every $q \in [p_\ell, p_u]$ and $q_3^2 = 1 - q^2 - |(\mu_1/q - q)/a|^2$, we have

$$\begin{aligned} & |\det(e^{i\theta}C_q + e^{-i\theta}C_q^*)| \\ = & \left|e^{i\theta}\xi(-aq + (\bar\mu_1/q - q)/a) + e^{-i\theta}b(\bar\mu_1/q - q)/a\right|^2 \\ = & |\xi|^2|(\bar\mu_1/q - q)/a - aq|^2 + b^2|(\bar\mu_1/q - q)/a|^2 \\ & \qquad +2\mathrm{Re}\,\left(e^{2i\theta}\xi b(-aq + (\bar\mu_1/q - q)/a)(\mu_1/q - q)/a\right) \\ = & |\xi|^2(|(\bar\mu_1/q - q)/a|^2 + a^2q^2 - 2\mathrm{Re}\,(\bar\mu_1 - q^2)) + b^2|(\bar\mu_1/q - q)/a|^2 \\ & \qquad +2\mathrm{Re}\,\left(e^{2i\theta}\xi b(-aq + (\bar\mu_1/q - q)/a)(\mu_1/q - q)/a\right) \\ = & (|\xi|^2(1 + a^2) - b^2)q^2 + (|\xi|^2 + b^2)(1 - q_3^2) \\ & \qquad -2\mathrm{Re}\,(|\xi|^2\bar\mu_1 + e^{2i\theta}\xi b(1 - \mu_1 - q_3^2)). \end{aligned}$$

As

$$1 - (p_1^-)^2 - |(\mu_1/p_1^- - p_1^-)/a|^2 = 1 - (p_1^+)^2 - |(\mu_1/p_1^+ - p_1^+)/a|^2 = p_3^2,$$

the first inequality in (4.16) follows from $|\xi|^2(1 + a^2) - b^2 \ge 0$ and $p_1^+ \ge p_1^-$. Now

$$\begin{aligned} & \det\left|e^{i\theta}C_{\tilde p_1^+} + e^{-i\theta}C^*_{\tilde p_1^+}\right| - r^2\left|\det\left(e^{i\theta}C_{p_1^+} + e^{-i\theta}C^*_{p_1^+}\right)\right| \\ = & (|\xi|^2(1 + a^2) - b^2)\left((\tilde p_1^+)^2 - r^2(p_1^+)^2\right) + (1 - r^2)(|\xi|^2 + b^2) \\ & \qquad -2(1 - r^2)\mathrm{Re}\,\left(|\xi|^2\bar\mu_1 + e^{2i\theta}\xi b(1 - \mu_1)\right) \\ \ge & \left(|\xi|^2(1 + a^2) - b^2\right)(\tilde p_1^+)^2 + (|\xi|^2 + b^2) - 2\left(|\xi|^2\mathrm{Re}\,\bar\mu_1 + |\xi b(1 - \bar\mu_1)|\right) \\ & -r^2\left((|\xi|^2(1 + a^2) - b^2)(\tilde p_1^+)^2 + (|\xi|^2 + b^2) - 2\left(|\xi|^2\mathrm{Re}\,\bar\mu_1 + |\xi b(1 - \bar\mu_1)|\right)\right). \end{aligned}$$

For every $y \in [0, \hat p_3^2]$, let

$$\left(q_y^+\right)^2 = \frac{2\mathrm{Re}\,\mu_1 + a^2(1 - y) + \sqrt{(2\mathrm{Re}\,\mu_1 + a^2(1 - y))^2 - 4(a^2 + 1)|\mu_1|^2}}{2(1 + a^2)}.$$

It is not hard to see that $q_y^+ \in [\hat p, p_u]$ satisfies the left-hand side of (4.13) with $p_3 = \sqrt{y}$, i.e.,

$$-a^2(q_y^+)^2 y = (a^2 + 1)(q_y^+)^4 - (2\mathrm{Re}\,\mu_1 + a^2)(q_y^+)^2 + |\mu_1|^2.$$

Define the function $M : [0, \hat p_3^2] \to \mathbb{R}$ by

$$M(y) = \left(|\xi|^2(1 + a^2) - b^2\right)(q_y^+)^2 + \left(|\xi|^2 + b^2\right) - 2\left(|\xi|^2\mathrm{Re}\,\bar\mu_1 + |\xi b(1 - \bar\mu_1)|\right).$$

For $y = 0$, we have $(1 + a^2)(q_0^+)^4 - (2\mathrm{Re}\,\mu_1 + a^2)(q_0^+)^2 + |\mu_1|^2 = 0$ and

$$M(0) = \frac{|1 - \mu_1|^2|\xi|^2}{1 - \left(q_0^+\right)^2} - 2b|\xi||1 - \bar\mu_1| + b^2\left(1 - \left(q_0^+\right)^2\right) \ge 0.$$

We will show that M is concave so that

$$\begin{aligned} & \left|\det\left(e^{i\theta}C_{\tilde{p}_1^+}+e^{-i\theta}C^*_{\tilde{p}_1^+}\right)\right| - r^2\left|\det\left(e^{i\theta}C_{p_1^+}+e^{-i\theta}C^*_{p_1^+}\right)\right| \\ \geq\ & M(r^2p_3^2)-r^2M(p_3^2)\geq(1-r^2)M(0)\geq 0. \end{aligned}$$

Noting that $|\xi|^2(1+a^2)-b^2\geq 0$, we have

$$\begin{aligned} \frac{d^2M}{dy^2} &= \left(|\xi|^2(1+a^2)-b^2\right)\left((q_y^+)^2\right)'' \\ &= \frac{|\xi|^2(1+a^2)-b^2}{2(a^2+1)}\left(\sqrt{(2\mathrm{Re}\,\mu_1+a^2(1-y))^2-4(1+a^2)|\mu_1|^2}\right)'' \\ &= \frac{-\left(|\xi|^2(1+a^2)-b^2\right)\left(4a^4\left(a^2+1\right)\right)|\mu_1|^2}{2(a^2+1)\left(\left(2\mathrm{Re}\,\mu_1+a^2\left(1-y^2\right)\right)^2-4(a^2+1)|\mu_1|^2\right)^{3/2}}\leq 0. \end{aligned}$$

Hence M is concave.

The proof of (II) is similar, and we sketch the proof in the following. It suffices to show that for every $\theta\in[0,2\pi)$

$$\begin{aligned} r^2\left|\det\left(e^{i\theta}C_{p_1^+}+e^{-i\theta}C^*_{p_1^+}\right)\right| &\leq r^2\left|\det\left(e^{i\theta}C_{p_1^-}+e^{-i\theta}C^*_{p_1^-}\right)\right| \\ &\leq \left|\det\left(e^{i\theta}C_{\tilde{p}_1^-}+e^{-i\theta}C^*_{\tilde{p}_1^-}\right)\right|. \end{aligned} \tag{4.17}$$

Recall that

$$\begin{aligned} & |\det(e^{i\theta}C_q+e^{-i\theta}C_q^*)| \\ =&\left(|\xi|^2(1+a^2)-b^2\right)q^2+\left(|\xi|^2+b^2\right)\left(1-q_3^2\right)-2\mathrm{Re}\left(|\xi|^2\bar{\mu}_1+e^{2i\theta}\xi b(1-\mu_1-q_3^2)\right). \end{aligned}$$

Therefore, the first inequality in (4.17) follows from the inequalities $|\xi|^2(1+a^2)\leq b^2$ and $p_1^-\leq p_1^+$. The second inequality will follow from the concavity of

$$\tilde{M}(y)=\left(|\xi|^2(1+a^2)-b^2\right)(q_y^-)^2+\left(|\xi|^2+b^2\right)-2\left(|\xi|^2\mathrm{Re}\,\bar{\mu}_1+|\xi b(1-\bar{\mu}_1)|\right),$$

where

$$\left(q_y^-\right)^2=\frac{2\mathrm{Re}\,\mu_1+a^2(1-y)-\sqrt{(2\mathrm{Re}\,\mu_1+a^2(1-y))^2-4(a^2+1)|\mu_1|^2}}{2(1+a^2)}.$$

Since $|\xi|^2(1+a^2)-b^2\leq 0$,

$$\begin{aligned} \frac{d^2\tilde{M}}{dy^2} &= \left(|\xi|^2(1+a^2)-b^2\right)\left((q_y^-)^2\right)'' \\ &= \frac{|\xi|^2(1+a^2)-b^2}{2(a^2+1)}\left(-\sqrt{(2\mathrm{Re}\,\mu_1+a^2(1-y))^2-4(1+a^2)|\mu_1|^2}\right)''\leq 0. \end{aligned}$$

Thus (II) holds. ∎

Remark 4.4. It is worth pointing out that our proofs use some continuity arguments and a simple idea of homotopy (in deforming ellipses inside the numerical range of a certain matrix). In particular, intricate and involved linear algebraic arguments are used. It will be nice if a less computational proof can be found.

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