CLOSEDNESS OF THE k-NUMERICAL RANGE

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Abstract. Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, the k-numerical range of A is the set

$$W_k(A) = \left\{ \sum_{j=1}^{k} \langle Ax_j, x_j \rangle : \{x_j, \ldots, x_k\} \text{ is an orthonormal set in } \mathcal{H} \right\}.$$

In this note, we show that the closure of $W_k(A)$ can be written as the convex hull of sets involving the essential numerical range of A and $W_\ell(A)$ for $\ell \leq k$. We also show that if $W_k(A)$ is closed, then $W_\ell(A)$ is also closed for $\ell \leq k$.

Numerical range, k-numerical range, essential numerical range

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, the *numerical range* of A is the set

$$W(A) = \{\langle Ax, x \rangle : x \text{ is a unit vector in } \mathcal{H}\}.$$

It is a bounded and convex subset of the complex plane $\mathbb{C}$, but is in general not closed. Let $\mathbf{cl}\,(W(A))$ denote the closure of $W(A)$. In [5, Theorem 1], Lancaster proved that

$$\mathbf{cl}\,(W(A)) = \mathbf{conv}(W(A) \cup W_{\text{ess}}(A)),$$

where $W_{\text{ess}}(A)$ is the *essential numerical range* of A and $\mathbf{conv}(W(A) \cup W_{\text{ess}}(A))$ is the the convex hull of $W(A)$ and $W_{\text{ess}}(A)$. There are several equivalent definitions of $W_{\text{ess}}(A)$, see [2]. For our purpose, a point $\mu \in \mathbb{C}$ belongs to $W_{\text{ess}}(A)$ if and only if there is a weakly null sequence of unit vectors (or, a sequence of orthonormal vectors) $\{v_k\}$ in $\mathcal{H}$ such that $\langle Av_k, v_k \rangle \to \mu$. A consequence of Lancaster's theorem is that $W(A)$ is closed if and only if $W_{\text{ess}}(A) \subseteq W(A)$. This is an extension of an earlier result of Halmos [3, Problem 213], who showed that if A is compact, $W(A)$ is closed if and only if $0 \in W(A)$. Note that for any compact operator A, $W_{\text{ess}}(A) = \{0\}$.

There are different extensions of the notion of the numerical range. One of them is to define for each positive integer k the *k-numerical range* of $A \in \mathcal{B}(\mathcal{H})$ by

$$W_k(A) = \left\{ \sum_{j=1}^{k} \langle Ax_j, x_j \rangle : \{x_1, \ldots, x_k\} \text{ is an orthonormal set in } \mathcal{H} \right\}.$$

When $k = 1$, $W_k(A)$ reduces to the usual numerical range $W(A)$. It is well-known that $W_k(A)$ is always convex ([3, Problem 211]). But just like $W(A)$, $W_k(A)$ is not always closed. In [6] Li and Poon showed that, when $A \in \mathcal{B}(\mathcal{H})$ is compact, $W_k(A)$ is closed if and only if

$$(*) \qquad 0 \in W_k(A) \quad \text{and} \quad W_\ell(A) \subseteq W_k(A) \quad \text{for all} \quad \ell = 1, \ldots, k-1.$$

Actually their results are about the more general c-numerical range, but we shall confine our discussion to the k-numerical range.

In the next section, we give a description of $\mathbf{cl}\,(W_k(A))$ when A is not necessarily compact. More precisely, we express the closure as the convex hull of sets involving $W_{\text{ess}}(A)$ and $W_\ell(A)$

for $\ell \le k$. Another question stemming from the condition (∗) is whether there are inclusion relations between the other $W_\ell(A)$'s when $W_k(A)$ is closed. It turns out that if $W_k(A)$ is closed, $W_\ell(A)$ is also closed for $\ell \le k$. Consequently, (∗) can be written as

$$\{0\} \subseteq W_1(A) \subseteq W_2(A) \subseteq \cdots \subseteq W_k(A).$$

This is discussed in the last section.

2. Closure of $W_k(A)$

In this section we prove

Theorem 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ and k be a positive integer. Set $W_0(A) = \{0\}$. Then*

$$\mathbf{cl}\,(W_k(A)) = \mathbf{conv} \bigcup_{\ell=0}^{k} [W_\ell(A) + (k-\ell)W_{\rm ess}(A)]. \tag{1}$$

Consequently, $W_k(A)$ is closed if and only if

$$W_\ell(A) + (k-\ell)W_{\rm ess}(A) \subseteq W_k(A), \quad \ell = 0, \dots, k. \tag{2}$$

Proof. To prove (1), note that when $k = 1$, it is just [5, Theorem 1]. So, assume that $k > 1$.

"$\supseteq$" The inclusion can be deduced from [4, Theorem 3] by putting $\beta_{k-\ell,k} = \frac{k-\ell}{k}$ and $\beta_{jk} = 0$ for $j \neq k - \ell$. We include a short proof for the sake of completeness. It suffices to show that $W_\ell(A) + (k-\ell)W_{\rm ess}(A) \subseteq \mathbf{cl}\,(W_k(A))$ for $0 \le \ell \le k$. Let

$$\mu = \sum_{j=1}^{\ell} \langle Av_j, v_j\rangle + (k-\ell)\xi \in W_\ell(A) + (k-\ell)W_{\rm ess}(A),$$

for orthonormal vectors $v_1, \dots, v_\ell$ in $\mathcal{H}$ and $\xi \in W_{\rm ess}(A)$. Choosing an orthonormal basis of $\mathcal{H}$ with $\{v_1, \dots, v_\ell\}$ as the first ℓ vectors, we can represent A as the matrix $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}$, where the $\ell \times \ell$ matrix A_{11} has (j, j) entry equal to $\langle Av_j, v_j\rangle$ for $j = 1, \dots, \ell$. If F is the finite rank operator represented as $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0\end{pmatrix}$, then $\xi \in W_{\rm ess}(A - F)$ so that there is an orthonormal sequence of unit vector $\{u_1, u_2, \dots\} \subseteq \{v_1, \dots, v_\ell\}^\perp$ such that $\langle Au_j, u_j\rangle \to \xi$ [2, Theorem 5.1]. Thus,

$$\mu = \sum_{j=1}^{\ell} \langle Av_j, v_j\rangle + \lim_{m\to\infty} \sum_{j=1}^{k-\ell} \langle Au_{m+j}, u_{m+j}\rangle \in \mathbf{cl}\,(W_k(A)).$$

"$\subseteq$" Let $\mu \in \mathbf{cl}\,(W_k(A))$. There are orthonormal sets of vectors $\{v_1^{(n)}, \dots, v_k^{(n)}\}$ in $\mathcal{H}$ such that $\sum_{j=1}^{k}\langle Av_j^{(n)}, v_j^{(n)}\rangle \to \mu$. As the closed unit ball is weakly sequentially compact, by passing to subsequences, we may assume that for each j, $v_j^{(n)} \to v_j$ weakly and $\langle Av_j^{(n)}, v_j^{(n)}\rangle \to \mu_j$, for v_j in the closed unit ball of $\mathcal{H}$ and $\mu_j \in \mathbf{cl}\,(W_k(A))$. There are three possibilities (see the proof of [1, Theorem 2.1] for detail),

(i) $v_j = 0$ and $\mu_j \in W_{\rm ess}(A)$,

(ii) $\|v_j\| = 1$, $v_j^{(n)} \to v_j$ strongly and $\mu_j = \langle Av_j, v_j\rangle \in W(A)$,

(iii) $0 < \|v_j\| < 1$ and $\mu_j = \|v_j\|^2\langle A\frac{v_j}{\|v_j\|}, \frac{v_j}{\|v_j\|}\rangle + (1 - \|v_j\|^2)\xi_j$ for some $\xi_j \in W_{\rm ess}(A)$ so that μ_j is a convex combination of points in $W(A)$ and $W_{\rm ess}(A)$.

Taking any $\xi_j \in W_{\text{ess}}(A)$ in (ii), we can always write $\mu_j = \langle Av_j, v_j\rangle + (1 - \|v_j\|^2)\xi_j$. As in [6], consider the positive semidefinite operator $H = \sum_{j=1}^k \langle \cdot\,, v_j\rangle v_j$. Let $d_1 \ge d_2 \ge \cdots \ge d_k \ge 0$ be the k largest eigenvalues of H, and $\{u_1, \dots, u_k\}$ an orthonormal set of corresponding eigenvectors. For each j,

$$d_j = \langle Hu_j, u_j\rangle = \sum_{i=1}^k |\langle u_j, v_i\rangle|^2 = \lim_{n\to\infty} \sum_{i=1}^k |\langle u_j, v_i^{(n)}\rangle|^2 \le \|u_j\|^2 = 1.$$

For each $\ell = 1, \dots, k$, let $G_\ell = \sum_{j=1}^\ell \langle \cdot\,, u_j\rangle u_j$. Then

$$H = \sum_{j=1}^k d_j \langle \cdot\,, u_j\rangle u_j = d_k G_k + (d_{k-1} - d_k)G_{k-1} + \cdots + (d_1 - d_2)G_1.$$

Now, $\text{tr}AH = \sum_{j=1}^k \text{tr}A(\langle \cdot\,, v_j\rangle v_j) = \sum_{j=1}^k \langle Av_j, v_j\rangle$, and similarly, $\text{tr}AG_\ell = \sum_{j=1}^\ell \langle Au_j, u_j\rangle$. In particular, $\text{tr}AG_\ell \in W_\ell(A)$. We have

$$\sum_{j=1}^k \langle Av_j, v_j\rangle = \text{tr}AH = d_k\text{tr}AG_k + (d_{k-1} - d_k)\text{tr}AG_{k-1} + \cdots + (d_1 - d_2)\text{tr}AG_1.$$

Observe that $d_1 + \cdots + d_k = \text{tr}H = \sum_{j=1}^k \text{tr}(\langle \cdot\,, v_j\rangle v_j) = \sum_{j=1}^k \|v_j\|^2$. So,

$$\sum_{j=1}^k (1 - \|v_j\|^2)\xi_j = (k - d_1 - \cdots - d_k)\left(\frac{1 - \|v_1\|^2}{k - d_1 - \cdots - d_k}\xi_1 + \cdots + \frac{1 - \|v_k\|^2}{k - d_1 - \cdots - d_k}\xi_k\right).$$

As $W_{\text{ess}}(A)$ is convex, the number above is equal to $(k - d_1 - \cdots - d_k)\xi$ for some $\xi \in W_{\text{ess}}(A)$. We can write

$$\begin{aligned}\mu &= \textstyle\sum_{j=1}^k \mu_j = \sum_{j=1}^k \langle Av_j, v_j\rangle + \sum_{j=1}^k (1 - \|v_j\|^2)\xi_j \\ &= d_k\text{tr}G_kA + (d_{k-1} - d_k)\text{tr}G_{k-1}A + \cdots + (d_1 - d_2)\text{tr}G_1A + (k - d_1 - \cdots - d_k)\xi \\ &= d_k\text{tr}G_kA + (d_{k-1} - d_k)(\text{tr}G_{k-1}A + \xi) + \cdots + (d_1 - d_2)(\text{tr}G_1A + (k-1)\xi) \\ &\quad + (1 - d_1)(k\xi),\end{aligned}$$

where, as observed above, $\text{tr}AG_\ell \in W_\ell(A)$. The proof of (1) is complete. Statement (2) follows easily from (1). □

The following example shows that one needs to check the condition in (2) for every ℓ to conclude that $W_k(A)$ is closed.

Example 2.2. Let ℓ and k be nonnegative integers with $\ell < k$, and A the compact operator $A = -I_\ell \oplus I_k \oplus \text{diag}\,(1/2, 1/3, \dots)$ acting on $\mathcal{H} = \ell_2$. Then $W_{\text{ess}}(A) = \{0\}$. We have

$$W_j(A) + (k - j)W_{\text{ess}}(A) = \begin{cases} [-j, j] & \text{for } j = 0, 1, \dots, \ell, \\ (-\ell, j] & \text{for } j = \ell + 1, \dots, k, \end{cases}$$

and

$$W_k(A) = (-\ell, k].$$

In particular, $W_k(A)$ is not closed. In this example we have $W_j(A) + (k-j)W_{\text{ess}}(A) \subseteq W_k(A)$ for all $j \in \{0, 1, \dots, k\} \setminus \{\ell\}$. □

Actually, if $W_k(A)$ is closed, then for every $\lambda \in W_{\rm ess}(A)$, $k\lambda \in W_k(A)$ so that there are orthonormal vectors $v_1, \dots, v_k$ such that $k\lambda = \langle Av_1, v_1\rangle + \cdots + \langle Av_k, v_k\rangle$. Then $\lambda = (1/k)(\langle Av_1, v_1\rangle + \cdots + \langle Av_k, v_k\rangle) \in W(A)$. By [5, Corollary 1], $W(A)$ is closed. In the next section, we prove that indeed $W_\ell(A)$ is closed for all $1 \le \ell \le k$.

3. Closedness of $W_k(A)$ and $W_{k+1}(A)$

The main result of this section is the following.

Theorem 3.1. *Let $A \in \mathcal{B}(\mathcal{H})$ and $k \ge 1$. If $W_{k+1}(A)$ is closed, then so is $W_k(A)$.*

Note that the converse is not true. Let $\{e_j\}_{j=1}^\infty$ be the standard orthonormal basis of ℓ^2. If $A = \langle\,\cdot\,, e_1\rangle e_1 - \sum_{j=2}^\infty 2^{-j}\langle\,\cdot\,, e_j\rangle e_j$, then $W(A) = [-1/4, 1]$ is closed while $W_2 = [-3/8, 1)$ is not closed.

Proof. First consider the simpler situation when $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint. The set $W_k(A)$ is a line segment on $\mathbb{R}$, which can be computed as follows. Let

$$\lambda_\ell(A) = \inf_{\substack{W \le \mathcal{H} \\ \dim W = \ell - 1}} \sup\{\langle Ax, x\rangle : x \in W^\perp \text{ and } \|x\| = 1\}.$$

Then $\lambda_1(A) \ge \lambda_2(A) \ge \cdots$. Denote by $\sigma(A)$ the spectrum of A. Then $\lambda_1(A) = \sup \sigma(A)$ and one of the following holds.

(1) There is an orthonormal set $\{v_1, v_2, \dots\} \subseteq \mathcal{H}$ such that $Av_j = \lambda_j(A)v_j$ for $j = 1, 2, \dots$.

(2) There exists an orthonormal set $\{v_1, \dots, v_\ell\} \subseteq \mathcal{H}$, $\ell \ge 0$ satisfying $Av_j = \lambda_j(A)v_j$ for $j = 1, \dots, \ell$, and for all $j > \ell$ we have $\lambda_j(A) = \lambda_{\ell+1}(A) = \max W_{\rm ess}(A)$, which is a limit point of $\sigma(A)$. Here, if $\ell = 0$, we have $\lambda_j(A) = \max W_{\rm ess}(A)$ for all $j \ge 1$.

Let $a_k = -\sum_{j=1}^k \lambda_j(-A)$ and $b_k = \sum_{j=1}^k \lambda_j(A)$. Then $(a_k, b_k) \subseteq W_k(A) \subseteq [a_k, b_k]$. If $W_{k+1}(A)$ is closed, then $W_{k+1}(A) = [a_{k+1}, b_{k+1}]$ and there is an orthonormal set of vectors $\{v_1, \dots, v_{k+1}\}$ such that $Av_j = \lambda_j(A)v_j$. It follows that the right hand endpoint $b_k = \sum_{j=1}^k \lambda_j(A) = \sum_{j=1}^k \langle Av_j, v_j\rangle$ of $\mathbf{cl}\,(W_k(A))$ lies in $W_k(A)$. Similarly, the left hand endpoint $a_k = -\sum_{j=1}^k \lambda(-A)$ of $\mathbf{cl}\,(W_k(A))$ also lies in $W_k(A)$. Thus, $W_k(A) = [a_k, b_k]$ is closed.

Now, we turn to the case of a general operator $A \in \mathcal{B}(\mathcal{H})$. If for some $\mu \in \mathbb{C}$ and $t \in [0, 2\pi)$, $e^{it}(A - \mu I)$ is self-adjoint, then the result follows from the discussion above. So, assume that it is not the case. We prove the contra-positive, i.e., if $W_k(A)$ is not closed, then $W_{k+1}(A)$ is also not closed.

Under the assumption, there is an extreme point μ of $\mathbf{cl}\,(W_k(A))$ that does not belong to $W_k(A)$. Replacing A by $e^{it}(A - \mu I)$ for a suitable $t \in [0, 2\pi)$, we may assume that $\mu = 0$ and $W_k(A)$ lies on the left half of the complex plane and the right support line of $\mathbf{cl}\,(W_k(A))$ is the imaginary axis $L = \{iy : y \in \mathbb{R}\}$. Considering A^* instead of A if necessary, we may further assume that 0 is the upper endpoint of the line segment $\mathbf{cl}\,(W_k(A)) \cap L$.

By Theorem 2.1, $0 \in W_\ell(A) + (k - \ell)W_{\rm ess}(A)$ for some $\ell \in \{0, \dots, k\}$. That is,

$$0 = \sum_{j=1}^{\ell} \langle Av_j, v_j\rangle + (k - \ell)(h + ig),$$

for orthonormal vectors $v_1, \dots, v_\ell$ and $h + ig \in W_{\rm ess}(A)$. We shall assume that

$$\ell \text{ is the largest integer such that } 0 \in W_\ell(A) + (k - \ell)W_{\rm ess}(A). \tag{3}$$

As $0 \notin W_k(A)$, $\ell < k$. If we write $A = H + iG$ for self-adjoint H and G, then

$$0 = \left(\sum_{j=1}^{\ell} \langle Hv_j, v_j \rangle + (k-\ell)h \right) + i \left(\sum_{j=1}^{\ell} \langle Gv_j, v_j \rangle + (k-\ell)g \right).$$

If $\ell = 0$, then $0 = k(h+ig)$, or, $h+ig = 0$. So, $0 = (k+1)(h+ig) \in \mathbf{cl}\,(W_{k+1}(A))$. We will show that $0 \notin W_{k+1}(A)$ to conclude that $W_{k+1}(A)$ is not closed. Suppose on the contrary that

$$0 = \sum_{j=1}^{k+1} \langle Au_j, u_j \rangle = \sum_{j=1}^{k+1} \langle Hu_j, u_j \rangle + i \sum_{j=1}^{k+1} \langle Gu_j, u_j \rangle \in W_{k+1}(A)$$

for orthonormal vectors $u_1, \ldots, u_{k+1}$. As $W_k(A)$ lies on the left half of the complex plane, the sum of any k terms of $\langle Hu_1, u_1 \rangle, \ldots, \langle Hu_{k+1}, u_{k+1} \rangle$ is less than or equal to zero. The sum of all $k+1$ of them is zero implies that the sum of any k terms is indeed zero. We must have $\langle Hu_j, u_j \rangle = 0$ for all j. Thus, the sum of any k terms of $\langle Au_1, u_1 \rangle, \ldots, \langle Au_{k+1}, u_{k+1} \rangle$ belongs to $\mathbf{cl}\,(W_k(A)) \cap L$. As 0 is the upper endpoint of this line segment, the sum of any k terms of $\langle Gu_1, u_1 \rangle, \ldots, \langle Gu_{k+1}, u_{k+1} \rangle$ is less than or equal to zero. An argument as above yields that $\langle Gu_j, u_j \rangle = 0$ for all j. In particular, we have $0 = \sum_{j=1}^{k} \langle Au_j, u_j \rangle \in W_k(A)$, a contradiction.

So, assume in the rest of the proof that $\ell > 0$. Since $0 = \max \mathbf{cl}\,(W_k(H))$, we must have $\sum_{j=1}^{\ell} \langle Hv_j, v_j \rangle = \max W_\ell(H)$. Otherwise we can find orthonormal vectors $u_1, \ldots, u_\ell$ such that $\sum_{j=1}^{\ell} \langle Hu_j, u_j \rangle > \sum_{j=1}^{\ell} \langle Hv_j, v_j \rangle$ to get the point

$$\sum_{j=1}^{\ell} \langle Au_j, u_j \rangle + (k-\ell)(h+ig) \in \mathbf{cl}\,(W_k(A))$$

with real part $\sum_{j=1}^{\ell} \langle Hu_j, u_j \rangle + (k-\ell)h > 0$. Thus, $\sum_{j=1}^{\ell} \langle Hv_j, v_j \rangle = \sum_{j=1}^{\ell} \lambda_j(H)$. Without loss of generality, we can assume that $\langle Hv_j, v_j \rangle = \lambda_j(H)$ for each $1 \leq j \leq \ell$, so that they are the ℓ largest eigenvalues of H, counting multiplicities, with each v_j as an eigenvector corresponding to $\lambda_j(H)$. Again, it follows from $0 = \max \mathbf{cl}\,(W_k(H))$ that

$$\lambda_1(H) \geq \cdots \geq \lambda_\ell(H) \geq h \quad \text{and} \quad \lambda_j(H) = h \text{ for } j > \ell.$$

There may be $j \leq \ell$ such that $\lambda_j(H) = h$. So, let r be the smallest integer such that $\lambda_j(H) = h$ if $j > r$. We have $0 \leq r \leq \ell$.

Consider the action of G on the eigenspaces of H. If $r > 0$, let $\mathcal{H}_1 = \operatorname{span}\{v_1, \ldots, v_r\}$ be the direct sum of the eigenspaces of H corresponding to $\lambda_1(H), \ldots, \lambda_r(H)$. If $r = 0$, let $\mathcal{H}_1 = \{0\}$. On the finite dimensional subspace $\mathcal{H}_1$, we have

$$\sum_{j=1}^{r} \langle Hw_j, w_j \rangle = \sum_{j=1}^{r} \langle Hv_j, v_j \rangle \quad \text{and} \quad \sum_{j=1}^{r} \langle Gw_j, w_j \rangle = \sum_{j=1}^{r} \langle Gv_j, v_j \rangle$$

for any orthonormal basis $\{w_1, \ldots, w_r\}$ of $\mathcal{H}_1$.

Let $\mathcal{H}_2$ be the eigenspace of H corresponding to h. Then $\mathcal{H}_2$ has dimension at least $\ell - r$ and may even be infinite dimensional. Also, let $\hat{G}$ be the compression of G onto $\mathcal{H}_2$. It follows from the fact $0 = \sum_{j=1}^{\ell} \langle Gv_j, v_j \rangle + (k-\ell)g$ is the largest imaginary part of points in $\mathbf{cl}\,(W_k(A)) \cap L$ that

$$\sum_{j=r+1}^{\ell} \langle Gv_j, v_j \rangle = \sum_{j=1}^{\ell-r} \lambda_j(\hat{G}).$$

Another observation is that if $u_1, \ldots, u_{\ell-r}$ are orthonormal vectors in $\mathcal{H}_2$ satisfying

$$\sum_{j=1}^{\ell-r}\langle Gu_j,u_j\rangle=\sum_{j=r+1}^{\ell}\langle Gv_j,v_j\rangle=\sum_{j=1}^{\ell-r}\lambda_j(\hat{G})$$

and w is a unit vector in $\mathcal{H}_2$ orthogonal to $u_1,\dots,u_{\ell-r}$, then $\langle Gw,w\rangle<g$. This is because if $\langle Gw,w\rangle>g$, then

$$\sum_{j=1}^{r}\langle Av_j,v_j\rangle+\sum_{j=1}^{\ell-r}\langle Au_j,u_j\rangle+\langle Aw,w\rangle+(k-\ell-1)(h+ig) \tag{4}$$

will be a point in $\mathbf{cl}\,(W_k(A))\cap L$ with imaginary part

$$\sum_{j=1}^{\ell}\langle Gv_j,v_j\rangle+\langle Gw,w\rangle+(k-\ell-1)g>0;$$

and if $\langle Gw,w\rangle=g$, $\langle Aw,w\rangle=h+ig$ so that the sum in (4) is zero, contradicting (3).

Now consider $W_{k+1}(A)$. Note that

$$\sum_{j=1}^{k+1}\lambda_j(H)=\sum_{j=1}^{k}\lambda_j(H)+\lambda_{k+1}(H)=h.$$

So, $\hat{L}=\{h+iy:y\in\mathbb{R}\}$ is the right support line of $\mathbf{cl}\,(W_{k+1}(A))$. Let $h+i\hat{g}\in\mathbf{cl}\,(W_{k+1}(A))\cap\hat{L}$ have maximum imaginary part. Then $\hat{g}\ge g$ as by Theorem 2.1,

$$h+ig=\sum_{j=1}^{\ell}\langle Av_j,v_j\rangle+(k-\ell+1)(h+ig)\in\mathbf{cl}\,(W_{k+1}(A)).$$

We shall show that $h+i\hat{g}$ does not belong to $W_{k+1}(A)$ and therefore $W_{k+1}(A)$ is not closed. Assume the contrary that $h+i\hat{g}\in W_{k+1}(A)$. Then $h+i\hat{g}=\sum_{j=1}^{k+1}\langle Aw_j,w_j\rangle$ for orthonormal vectors $w_1,\dots,w_{k+1}$. Recall that r is the smallest integer such that $\lambda_j(H)=h$ if $j>r$. We have

$$\sum_{j=1}^{k+1}\langle Hw_j,w_j\rangle=\sum_{j=1}^{k+1}\lambda_j(H)\rangle=\sum_{j=1}^{r}\langle Hv_j,v_j\rangle+(k+1-r)h.$$

Therefore,

$$\mathcal{H}_1=\operatorname{span}\{v_1,\dots,v_r\}\subset\operatorname{span}\{w_1,\dots,w_{k+1}\}=\mathcal{H}_3\text{ and }\mathcal{H}_1^{\perp}\cap\mathcal{H}_3\subseteq\mathcal{H}_2,$$

where $\mathcal{H}_2$ is the eigenspace of H corresponding to h. Without loss of generality, we may assume that $w_j=v_j$ for $j=1,\dots,r$. Therefore, $w_{r+1},\dots,w_{k+1}\in\mathcal{H}_2$. As $h+i\hat{g}$ has the maximum imaginary part, we must have

$$\sum_{j=r+1}^{k+1}\langle Gw_j,w_j\rangle=\sum_{j=1}^{k-r+1}\lambda_j(\hat{G}),$$

where $\lambda_j(\hat{G})$ are the $k-r+1$ largest eigenvalues of the compression of G onto $\mathcal{H}_2$. Again we can assume that

$$\sum_{j=r+1}^{\ell}\langle Gw_j,w_j\rangle=\sum_{j=1}^{\ell-r}\lambda_j(\hat{G}).$$

Therefore,

$$\langle Gw_{\ell+1},w_{\ell+1}\rangle,\dots,\langle Gw_{k+1},w_{k+1}\rangle<g.$$

Hence

$$\hat{g} = \sum_{j=1}^{k+1} \langle Gw_j, w_j \rangle = \sum_{j=1}^{\ell} \langle Gw_j, w_j \rangle + \sum_{j=\ell+1}^{k+1} \langle Gw_j, w_j \rangle < \sum_{j=1}^{\ell} \langle Gw_j, w_j \rangle + (k-\ell+1)g = g,$$

which is a contradiction. □

Combining Theorems 2.1 and 3.1, we get the following criterion for the closedness of $W_k(A)$.

Corollary 3.2. *The k-numerical range $W_k(A)$ is closed if and only if*

$$kW_{\text{ess}}(A) \subseteq W_1(A) + (k-1)W_{\text{ess}}(A) \subseteq \cdots \subseteq W_{k-1}(A) + W_{\text{ess}}(A) \subseteq W_k(A).$$

In particular, if A is compact, then $W_k(A)$ is closed if and only if

$$\{0\} \subseteq W_1(A) \subseteq \cdots \subseteq W_{k-1}(A) \subseteq W_k(A).$$

Proof. By Theorem 2.1, $W_k(A)$ is closed if and only if

$$W_j(A) + (k-j)W_{\text{ess}}(A) \subseteq W_k(A) \quad \text{for} \quad j = 0, \ldots, k.$$

The implication "$\Leftarrow$" is clear.

For the converse, if $W_k(A)$ is closed, then by Theorem 3.1, $W_{k-1}(A)$ is also closed and hence $W_{k-2}(A) + W_{\text{ess}}(A) \subseteq W_{k-1}(A)$. It follows that

$$W_{k-2}(A) + 2W_{\text{ess}}(A) \subseteq W_{k-1}(A) + W_{\text{ess}}(A).$$

The other inclusions can be obtained similarly. □

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