ISOMETRIES FOR KY-FAN NORMS ON BLOCK TRIANGULAR MATRIX ALGEBRAS

by

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Abstract

We characterize the linear isometries for Ky-Fan norms on the space of block triangular matrices.

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1. Introduction and Statement of the Main Theorem

Characterizing isometries of spaces of matrices or operators under various norms has been a fruitful area of research for a long time. One of the earliest results is Kadison's [Ka] characterization of isometries for the usual operator norm on $B(H)$, the space of all bounded linear operators on a Hilbert space H. Isometries of various symmetrically normed ideals were characterized (see, e.g., [So].) More recently, triangular matrix algebras and triangular operator algebras have received a great deal of attention in several contexts. In particular isometries of nest algebras or affiliated spaces have been investigated (see [AK] and the references therein).

In this article, we restrict our attention to finite dimensional spaces and to block upper triangular algebras of matrices. We characterize the isometries for Ky-Fan norms (defined below) on such algebras. The analogous question in the full matrix algebra has been dealt with in [GM].

We start by fixing notation and terminology. All matrices are over the complex field $\mathbf{C}$. Let $\mathbf{C}^n$ be the vector space of $n \times 1$ matrices equipped with the Euclidean norm $\|x\|$, and the standard basis $\{e_1, \ldots, e_n\}$.

Denote by M_n the algebra of all $n \times n$ complex matrices. For a finite sequence of positive integers $n_1, n_2, \ldots, n_t$, satisfying $n_1 + n_2 + \ldots + n_t = n$, let $\mathcal{T}(n_1, n_2, \ldots, n_t)$ be the algebra of all $n \times n$ matrices of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1t} \\ 0 & A_{22} & \ldots & A_{2t} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & A_{tt} \end{bmatrix}$$

where A_{ij} is an $n_i \times n_j$ matrix. We call such an algebra a *block upper triangular matrix algebra.* Up to algebra isomorphisms, these are all the finite dimensional nest algebras. In particular, when $n_j = 1$ for every j, we have the algebra $\mathcal{T}_n$ of upper triangular matrices.

Let $A \in M_n$. Denote the transpose and adjoint of A by A^t and A^*, respectively; furthermore, denote by $\operatorname{spec} A$ the *spectrum* of A. The singular values $s_j(A)$ of a matrix $A \in M_n$ are the eigenvalues of $(A^*A)^{1/2}$, repeated according to multiplicity and arranged in decreasing order $s_1(A) \geq s_2(A) \geq \ldots \geq s_n(A)$. For a positive integer $k \leq n$, the Ky-Fan k-norm is defined by

$$\|A\|_k = \sum_{j=1}^{k} s_j(A).$$

This is a unitarily invariant norm, that is $\|A\|_k = \|UAV\|_k$ for any unitary matrices U and V. Notice that for $k = 1$, this is the usual operator norm with matrices acting as operators on the usual unitary space $\mathbf{C}^n$, and the case $k = n$ is the so-called trace norm $\mathrm{tr}\sqrt{A^*A}$.

We turn our attention to the "usual forms" of isometries. In the case of the full matrix algebra M_n, the isometries of most unitarily invariant norms take one of the following forms $X \mapsto UXV$, and $X \mapsto UX^tV$, where U and V are unitary matrices (see [So] and [LT]). In the case of upper triangular matrices, the transpose is replaced by the map $X \mapsto X^+$, the transpose with the respect to the anti-diagonal, i.e., the "diagonal" containing the positions $(1, n)$ and $(n, 1)$. We observe that this map preserves the sequence of singular values, indeed $A^+ = JA^tJ$, where J is the involution matrix $J = [\delta_{j,1+n-j}]$, and where δ is the Kronecker delta.

Now, we state the main result of this article. To simplify the statement we make the convention that $\|A\|_k$, for $A \in M_m$ and $k > m$, will be the same as the trace norm.

Theorem. *Let $\mathcal{A} = \mathcal{T}(n_1, n_2, \ldots, n_t)$ and $\mathcal{B} = \mathcal{T}(m_1, m_2, \ldots, m_s)$ be block upper triangular algebras in M_n and M_m, respectively, and let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a surjective linear isometry for the Ky-Fan k-norm $\|A\|_k$, $(1 < k < n)$. Then $m = n$, $s = t$, and there exist unitary matrices U and V in $\mathcal{B}$ such that one of the following holds:*
(1) $n_j = m_j$ for every j, i.e., $\mathcal{B} = \mathcal{A}$, and ϕ has the form $A \mapsto UAV$, $A \in \mathcal{A}$.
(2) $n_j = m_{1+t-j}$ for every j, i.e., $\mathcal{B} = \mathcal{A}^+$, and ϕ has the form $A \mapsto UA^+V$, $A \in \mathcal{A}$.
Conversely every map of the form described above is a linear isometry with respect to any unitarily invariant norm.

Several remarks are in order.

1. A unitary matrix U in a triangular matrix algebra $\mathcal{T}(n_1, n_2, \ldots, n_k)$ must necessarily be block-diagonal, i.e., $U = U_1 \oplus U_2 \oplus \ldots \oplus U_k$ for unitary matrices $U_j \in M_{n_j}$.
2. The above results imply that $\phi : \mathcal{A} \to \mathcal{B}$ is a surjective linear isometry for the Ky-Fan k-norm, $1 < k < n$, if and only if ϕ extends to an an isometry of the full matrix algebra M_n that maps $\mathcal{A}$ onto $\mathcal{B}$.
3. One way to distinguish the two forms in the theorem is that if we multiply a map ϕ of type (1) by $\phi(I)^{-1}$ then we get a multiplicative map, while in the case (2) we get an anti-multiplicative map.
4. Another way to distinguish the two forms is that every map ϕ of form (1) is "completely isometric" in the sense that for every positive integer m, the naturally induced map $\phi \otimes id$ from $\mathcal{A} \otimes M_m$ to $\mathcal{B} \otimes M_m$ is an isometry, while maps of type (2) are not

even 2-isometric. Indeed if $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}$. where $A_{11} = e_1e_1^*$, $A_{12} = e_1e_n^*$, and $B = \begin{bmatrix} A_{11}^+ & A_{12}^+ \\ 0 & 0 \end{bmatrix}$, then it is easy to see that $\|A\|_k \neq \|B\|_k$.

5. A more thoroughgoingly triangular version of the above is that $\phi \otimes id$ is an isometry from $\mathcal{A} \otimes \mathcal{T}_m$ onto $\mathcal{B} \otimes \mathcal{T}_m$ if and only if it is of the form (1) of the Theorem.

2. Proofs and Related Questions

Two matrices A and B in M_n are called *orthogonal* if $AB^* = A^*B = 0$. We write $A \perp B$ to indicate that A and B are orthogonal.

As usual, if a matrix $A \in M_n$ is identified with the linear transformation it induces on $\mathbf{C}^n$, then the image (or range) of A is denoted by $\operatorname{ran} A$.

Lemma 1. *Let $A, B \in M_n$. The following are equivalent*

(a) $A \perp B$;

(b) $\operatorname{ran} A \perp \operatorname{ran} B$ *and* $\operatorname{ran} A^* \perp \operatorname{ran} B^*$;

(c) there exist unitary matrices $U, V \in M_n$, $C \in M_m$, and $D \in M_{n-m}$ such that

$$UAV = C \oplus 0 \quad \text{and} \quad UBV = 0 \oplus D.$$

Proof. Straightforward. ∎

The next result is well known. We include a short proof for completeness.

Lemma 2.

(a) If A and B are positive semidefinite matrices, then $\operatorname{spec}(AB) \subseteq [0, \infty)$.

(b) If A and B are positive semidefinite and if $\operatorname{spec}(AB) = \{0\}$, *then* $AB = 0$.

Proof. (a) If A is invertible, then AB is similar to $A^{1/2}BA^{1/2}$, which is positive semidefinite and the result follows. In the general case A is a limit of a sequence $\{A_j\}$ of invertible positive definite matrices and so $AB = \lim(A_jB)$, and the result follows from the continuity of the spectrum.

(b) There is nothing to prove if $A = 0$, so assume that $A \neq 0$. First if A is invertible, then again we have that the positive semidefinite matrix $A^{1/2}BA^{1/2}$ is similar to AB and thus has zero spectrum. This occurs only if $A^{1/2}BA^{1/2} = 0$ which in turn implies that $B = 0$. In the general case, we replace A and B by UAU^* and UBU^* for a unitary matrix U, and so we may assume without loss of generality that

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix}$$

where A_{11} is positive definite. Now $\operatorname{spec}(A_{11}B_{11}) = \operatorname{spec}(AB) = \{0\}$ and so $B_{11} = 0$ since A_{11} is invertible. As B is positive semidefinite, we must have that $B_{12} = 0$, and so $AB = 0$. ∎

The next lemma is Proposition 1.1 in [CHL].

Lemma 3. *Let $A, B \in M_n$ be nonzero matrices such that $\|A+B\|_k = \|A\|_k + \|B\|_k$. Then there exist unitary matrices $U, V \in M_n$ and positive semidefinite matrices $A_1, B_1 \in M_k$ such that the singular values of A_1 (respectively B_1) are the k largest singular values of A (respectively B) and*

$$UAV = A_1 \oplus A_2 \quad \text{and} \quad UBV = B_1 \oplus B_2$$

for some matrices $A_2, B_2 \in M_{n-k}$.

Lemma 4. *Let $A, B \in M_n$ be nonzero positive semidefinite matrices, and let $\omega \neq 1$ be a unimodular complex number. If $\|A+\omega B\|_k = \|A\|_k + \|B\|_k$, then A and B are orthogonal.*

Proof. The singular values of A and B are denoted by $0 \neq p_1 \geq p_2 \geq \ldots \geq p_n$ and $0 \neq q_1 \geq q_2 \geq \ldots \geq q_n$, respectively. Furthermore let $s_1 \geq s_2 \geq \ldots \geq s_n$ be singular values of $A + \omega B$. By Lemma 3, applied to A and ωB, we get that there exist unitary matrices U and V such that

$$UAV = A_0 \oplus A' \ \text{ and } \ UBV = \overline{\omega} B_0 \oplus B',$$

where $A_0, B_0 \in M_k$ are positive semidefinite matrices with singular values $p_1, \cdots, p_k$ and $q_1, \cdots, q_k$ respectively.

Now $(UAV)(UBV)^* = UABU^*$ has the same spectrum as AB, which, by Lemma 2 is included in $[0, \infty)$. Since $\omega A_0 B_0$ is a direct summand of $(UAV)(UBV)^*$, it follows that $\operatorname{spec}(\omega A_0 B_0) \subseteq [0, \infty)$. On the other hand $\operatorname{spec}(A_0 B_0) \subseteq [0, \infty)$, by Lemma 2 again. Thus, if $\mathbf{X}$ denotes the nonnegative x-axis, then $\operatorname{spec}(A_0 B_0) \subseteq \mathbf{X} \cap \overline{\omega}\mathbf{X}$ and so it must be $\{0\}$. By Lemma 2, we get that $A_0 B_0 = 0$. Since $A \neq 0$ and $B \neq 0$, we have that $p_1 \neq 0$ and $q_1 \neq 0$, and so $A_0 \neq 0$ and $B_0 \neq 0$. However $A_0 B_0 = 0$, hence both of A_0 and B_0 must be singular. Thus $p_k = q_k = 0$, and hence $p_j = q_j = 0$ for all $j > k$, i.e., $A' = B' = 0$. It now follows that $AB^* = B^* A = 0$ as required. ∎

Lemma 5. *Let $A, B \in M_n$ be nonzero matrices. Then $\|\alpha A + \beta B\|_k = |\alpha|\, \|A\|_k + |\beta|\, \|B\|_k$ for every pair of complex numbers α and β if and only if $A \perp B$ and* $\operatorname{rank} A + \operatorname{rank} B \leq k$.

Proof. The "if part" is easily verified by using Lemma 1. To prove the converse, we start by noting that due to Lemma 3, we may assume with no loss of generality that $A = A_1 \oplus A_2$, and $B = B_1 \oplus B_2$, where $A_1, B_1 \in M_k$ are positive semidefinite matrices whose eigenvalues are $p_1, \cdots, p_k$ and $q_1, \cdots, q_k$ respectively. (We use the same notation for the singular values of A, B and $A + B$ as in the proof of the preceding lemmas.)

For every unimodular complex number $\omega \neq 1$, we have $\|A_1 + \omega B_1\|_k \leq \|A + \omega B\|_k = \|A\|_k + \|B\|_k = r_1 + \cdots + r_k = (p_1 + \cdots + p_k) + (q_1 + \cdots + q_k.)$ We consider two cases according as $\|A_1 + \omega B_1\|_\ell = p_1 + q_1 + \cdots + p_\ell + q_\ell$ for some $\ell \leq k$ and some $\omega \neq 1$ or not. In the former case, we get by Lemma 4, that $A_1 \perp B_1$, which implies that both of A_1 and B_1 are singular, i.e., $p_k = q_k = 0$. Thus $A_2 = B_2 = 0$, and A and B are themselves mutually orthogonal. The orthogonality of A_1 and B_1 implies that the sum of the ranks is at most k. This completes the proof in the case that $\|A_1 + \omega B_1\|_\ell = p_1 + q_1 + \cdots + p_\ell + q_\ell$ for some $\omega \neq 1$ and some $\ell \leq k$.

Next, we consider the case when $\|A_1 + \omega B_1\|_\ell < (p_1 + q_1) + \cdots + (p_\ell + q_\ell)$ for every $\omega \neq 1$ and every $\ell \leq k$. Our aim is to show that this is not possible. Denote the singular values of $A_1 + iB_1$ and $A_2 + iB_2$ by $s_1 \geq \cdots \geq s_k$ and $\sigma_1 \geq \cdots \geq \sigma_{n-k}$ respectively. We now show that $\sigma_1, \cdots, \sigma_k$ are the k largest singular values of $A + iB$. Assume, to the contrary, that there exists a positive integer $j \leq k$ such that the k largest singular values of $A + iB$ are $\{s_1, \cdots, s_j\} \cup \{\sigma_1, \cdots, \sigma_{k-j}\}$. By our assumption we have

$$s_1 + \cdots + s_j < (p_1 + q_1) + \cdots + (p_j + q_j). \tag{1}$$

On the other hand

$$\sigma_1 + \cdots + \sigma_{k-j} \leq (p_{k+1} + q_{k+1}) + \cdots + (p_{2k-j} + q_{2k-j}) \leq (p_{j+1} + q_{j+1}) + \cdots + (p_k + q_k). \tag{2}$$

Upon adding (1) and (2), we get $\|A + iB\|_k < \|A\|_k + \|B\|_k$, which contradicts our assumption. This proves that the k largest singular values of $A + iB$ are indeed $\sigma_1, \cdots, \sigma_k$. Returning to inequalities (2), we see that the leftmost side equals the rightmost side due to the basic assumption about the norm equality. It follows that all three expressions in (2) are equal to each other. Thus $p_j = p_{k+j}$ for $1 \leq j \leq k$ and similar equalities also hold for q_j. Therefore $p_1 = \cdots = p_{2k}$ and $q_1 = \cdots = q_{2k}$. We also have that $\|A_2 + iB_2\|_k = \|A_2\|_k + \|B_2\|_k$ and so by Lemma 3, there exist unitary matrices U_2 and V_2 such that $U_2 A_2 V_2$ and $iU_2 B_2 V_2$ are themselves direct sums as in Lemma 3. The equality of the first $2k$ singular values now gives us that there exist unitary matrices U and V such that

$$UAV = \begin{bmatrix} p_1 I_k & 0 & 0 \\ 0 & p_1 I_k & 0 \\ 0 & 0 & A_3 \end{bmatrix} \quad \text{and} \quad UBV = \begin{bmatrix} q_1 I_k & 0 & 0 \\ 0 & -iq_1 I_k & 0 \\ 0 & 0 & B_3 \end{bmatrix}$$

where I_k is the identity matrix in M_k.

Now, if we combine the first two direct summands, write $UAV = A_0 \oplus A_3$ and $UBV = B_0 \oplus B_3$ and if ω is a complex number of modulus 1, $\omega \notin \{1, i\}$, and $\ell \leq k$, then $\|A_0 + \omega B_0\|_\ell < \|A_0\|_\ell + \|B_0\|_\ell$, since $\max\{|p_1 + \omega q_1|, \ |p_1 - i\omega q_1|\} < (p_1 + q_1)$. This means that we may continue and write A_3 and B_3 as direct sums. This process may be repeated indefinitely, which is absurd as n is finite. ∎

Recall that $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbf{C}^n$.

Lemma 6. *If R_1 is a rank one matrix in M_n, $n \geq 3$, then there exist $n-2$ rank one matrices $R_2, \cdots, R_{n-1} \in \mathcal{T}_n$ such that the matrices $R_1, R_2, \cdots, R_{n-1}$ are mutually orthogonal.*

Proof. Let $R_1 = xy^*$. Set $u_1 = x$. For $j = 2, \ldots, n$, let u_j be a nonzero vector in the linear span of $\{e_1, \ldots, e_j\}$ perpendicular to $u_1, \ldots, u_{j-1}$. Then

(i) $u_j \in \text{span}\ \{e_1, \ldots, e_j\}$; $(2 \leq j \leq n)$;

(ii) the vectors x, $u_2, \cdots, u_n$ are mutually orthogonal.

Similarly we may find vectors v_1, $v_2, \cdots, v_{n-1}$ such that

(a) $v_j \in \text{span}\ \{e_j, \cdots, e_n\}$; $(1 \leq j \leq n-1)$;

(b) the vectors y, $v_1, \cdots, v_{n-1}$ are mutually orthogonal.

Let $R_j = u_j v_j^*$ for $2 \leq j \leq n-1$. It is then obvious that $R_1, R_2, \cdots, R_{n-1}$ are mutually orthogonal. ∎

Proof of Theorem. Let ϕ be an isometry. We will first show that ϕ maps every rank one matrix to a rank one matrix. Assume that A is a rank one matrix in $\mathcal{A}$. By Lemma 6, there exist k mutually orthogonal rank one matrices $A_1 = A$, $A_2, \cdots, A_k \in \mathcal{A}$. By Lemma 5, $\phi(A_1), \phi(A_2), \cdots, \phi(A_k)$ are mutually orthogonal. Furthermore, if $B = A_2 + \cdots + A_k$, then again by Lemma 5, we have that $\text{rank}\phi(A) + \text{rank}\phi(B) \leq k$. Since the rank is additive on orthogonal matrices, we conclude that $\sum_1^k \text{rank}\phi(A_j) \leq k$. Thus $\text{rank}\phi(A) = 1$.

The structure of maps on triangular algebras that preserve rank one has been determined in [BS]. Indeed Theorem 4.4 in [BS] establishes that $m = n$, $\mathcal{B} = \mathcal{A}$ or $\mathcal{A}^+$ and that $\phi(A) = UAV$ for every $A \in \mathcal{A}$ or that $\phi(A) = UA^+V$ for every $A \in \mathcal{A}$ for invertible matrices U and V in $\mathcal{B}$. It remains only to prove that U and V may be chosen to be unitary. Plainly, it suffices to establish this only in the case $\phi(A) = UAV$.

First, we may multiply U by $\lambda := \|V^* e_n\|$, and V by λ^{-1}. Therefore, we may assume that $\|V^* e_n\| = 1$. For every vector $x \in \mathbf{C}^n$, the matrix xe_n^* belongs to every triangular matrix algebra. Furthermore the norm preserving property of ϕ implies that $\|Ux\| = \|x\|$ for every x. Therefore U is unitary. A similar calculation shows that V is also unitary.

As to the converse, it follows from the well known facts that every unitarily invariant norm is a function of the singular values (see [Ne]) and that the sequence of singular values of A^t is the same as that of A. ∎

We raise the question whether our result extends to the space of compact operators in a nest algebras on an infinite dimensional Hilbert space. It is known (see [LT] and [So]) that a surjective isometry for a unitarily invariant norm, which is not a multiple of the Frobenius (Hilbert-Schmidt) norm, on M_n or a symmetrically normed ideal of compact operators has the form $A \mapsto UAV$ or $A \mapsto UA^tV$ for some unitary U and V, where A^t is the transpose with respect to a fixed orthonormal basis. Our theorem and the result in [AK] show that a similar conclusion holds for surjective isometries for certain unitarily invariant norms on some triangular algebras. It would be of great interest to further extend the results to other unitarily invariant norms on finite or infinite dimensional spaces.

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