# Preservers of spectral radius, numerical radius, or spectral norm of the sum on nonnegative matrices

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**Dedicated to Tom Laffey on the occasion of his 65th birthday**

**Abstract**

Let $M_n^+$ be the set of entrywise nonnegative $n \times n$ matrices. Denote by $r(A)$ the spectral radius (Perron root) of $A \in M_n^+$. Characterization is obtained for maps $f : M_n^+ \to M_n^+$ such that $r(f(A) + f(B)) = r(A + B)$ for all $A, B \in M_n^+$. In particular, it is shown that such a map has the form

$$A \mapsto S^{-1}AS \quad \text{or} \quad A \mapsto S^{-1}A^{\mathrm{tr}}\, S,$$

for some $S \in M_n^+$ with exactly one positive entry in each row and each column. Moreover, the same conclusion holds if the spectral radius is replaced by the spectrum or the peripheral spectrum. Similar results are obtained for maps on the set of $n \times n$ nonnegative symmetric matrices. Furthermore, the proofs are extended to obtain analogous results when spectral radius is replaced by the numerical range, or the spectral norm. In the case of the numerical radius, a full description of preservers of the sum is also obtained, but in this case it turns out that the standard forms do not describe all such preservers.

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# 1 Introduction

Preserver problems concern the characterization of maps on matrices or operators leaving invariant a certain function, a certain subset of a certain relation. Earlier studies focused on *linear* maps with these properties. The literature on this subject is extensive; see, for example, [12, 23] and monographs [19, 20, 21]. Recently, researchers have studied preserver problems under mild assumptions. In particular, for a given function $\nu$ on a matrix set $\mathbf{M}$ with a binary operator $A \circ B$, maps $f : \mathbf{M} \to \mathbf{M}$ have been studied, that satisfy

$$\nu(f(A) \circ f(B)) = \nu(A \circ B) \quad \forall \;\; A, B \in \mathbf{M} \tag{1.1}$$

but not a priori assumed linear or continuous; [5, 6, 13, 14, 26] is a small selection of recent works on the topic. There has been interest in studying such problems when $\nu(A)$ is the spectrum, the peripheral spectrum, the numerical radius, the spectral norm, etc. (see the definitions below). See for example the papers [7, 15, 22], where preserver problems have been studied for $\nu$ the peripheral spectrum in the context of uniform algebras; in fact, these works served as motivation for the present study of preservers on nonnegative matrices, as for nonnegative matrices the peripheral spectrum always contains the spectral radius. Moreover, the problems have also been considered for in more general contexts such as function or operator algebras [19]. It is worth noting that even without the linearity assumption, the preservers often end up to be linear and have certain "standard" or "expected" form. Although the statements of results in many cases look similar to those of linear preservers, researchers often have to develop new techniques to solve the preserver problems under mild assumptions; sometimes these assumptions involve nothing more than validity of (1.1). In some cases, one may get unexpected forms for preservers, which lead to deeper understanding and insight to the structures under consideration.

The purpose of this paper is to characterize preservers of the spectral radius, numerical radius, or spectral norm of the sum of nonnegative matrices. There are not many works in the literature on preservers in the context of real entrywise nonnegative matrices: we mention [18], where spectrum preservers are described, [1, 10, 24, 25] that deal with column rank preservers; [2] is concerned with primitivity preservers, and in [4] preserver problems that have to do with irreducibility are considered. In all these works, the linearity of the map $f$ is assumed. In contrast, in the present work we do not assume a priori any additional hypotheses on $f$ except for (1.1) for $A \circ B = A + B$ and a suitable choice of $\nu$.

Let $M_n^+$ be the set of real entrywise nonnegative matrices, and let $r(A)$ be the spectral radius of a square matrix $A$. In Section 2, we characterize maps $f : M_n^+ \to M_n^+$ such that

$$r(f(A) + f(B)) = r(A + B) \quad \forall \;\; A, B \in M_n^+.$$

In particular, it is shown that such a map has the form

$$A \mapsto S^{-1}AS \quad \text{or} \quad A \mapsto S^{-1}A^{\mathrm{tr}}\,S, \tag{1.2}$$

for some $S \in M_n^+$ with exactly one positive entry in each row and each column. Moreover, as byproducts, we show that the same conclusion holds if the spectral radius is replaced by the spectrum or the peripheral spectrum. Similar results are obtained for maps on the set of $n \times n$ nonnegative symmetric matrices in Section 3. Furthermore, the proofs are extended to obtain analogous results when spectral radius is replaced by the numerical range, radius, or the spectral norm in Section 4 and Section 5. In the case of the numerical radius, a characterization of preservers of the sum is also obtained, but in this case it turns out that the standard forms (1.2) do not describe all such preservers.

The following notation will be used throughout the paper:
$M_n$ the set of all $n \times n$ real matrices.
$K_n$ the set of all $n \times n$ real skew-symmetric matrices.
$M_n^+$ the set of $n \times n$ real matrices with nonnegative entries.
$S_n^+$ the set of symmetric matrices in $M_n^+$.
To avoid trivialities, we assume $n \geq 2$ throughout our discussion.
$\mathsf{i} = \sqrt{-1}$ complex unit
$\mathbb{C}$ and $\mathbb{R}$ stand for the complex field and the real field, respectively.
$\|x\|$ Euclidean length of a vector $x$.
$e_i$ is the $i$th coordinate vector: 1 in the $i$th position and zeros elsewhere.
$E_{ij} \in M_n^+$ the matrix unit: 1 in the $(i,j)$th position and zeros everywhere else.
$r(A)$ the spectral radius of a matrix $A$.
$\sigma(A)$ the spectrum (the set of eigenvalues) of a matrix $A$.
$\sigma_p(A) = \sigma(A) \cap \{\lambda \in \mathbb{C} \,:\, |\lambda| = r(A)\}$ the peripheral spectrum of $A$.
$A^{\mathrm{tr}}$ the transpose of $A$.
$A^*$ the conjugate transpose of $A$
$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}$ the numerical range of $A$
$w(A) = \max\{|\mu| : \mu \in W(A)\}$ the numerical radius of $A$
$\|A\| = \max\{|x^*Ay| : x, y \in \mathbb{C}^n, x^*x = y^*y = 1\}$ the spectral norm of $A$.
$X \oplus Y := \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$
$0_{p\times q}$ the $p \times q$ zero matrix
$\mathcal{P} \subset M_n^+$ the group of permutation matrices.
$\mathcal{D} \subset M_n^+$ the group of diagonal matrices with positive entries on the diagonal.
$\mathcal{PD} \subset M_n^+$ the group of matrices of the form $PD$ where $P \in \mathcal{P}$ and $D \in \mathcal{D}$.
The role of $\mathcal{PD}$ is exemplified by the following well-known fact:

**Fact** *A matrix* $A \in M_n^+$ *has the property that* $A$ *is invertible and* $A^{-1} \in M_n^+$ *if and only if* $A \in \mathcal{PD}$.

To see the fact, suppose $A$ has columns $x_1, \ldots, x_n$ and $A^{-1}$ has rows $y_1^{\text{tr}}, \ldots, y_n^{\text{tr}}$. Suppose $x_1$ has $k$ positive entries. Then for $j = 2, \ldots, n$, $y_j$ will have zero entries in the corresponding nonzero positions of $x_1$ because $y_j$ is nonnegative and $y_j^{\text{tr}} x_1 = 0$. So, all the nonzero entries of the linearly independent vectors $y_2, \ldots, y_n$ will lie in fewer than $n-k$ positions. As a result, $k \leq 1$ so that $x_1$ has only one positive entry. Similar arguments apply to the other columns. Clearly, the nonzero entries of $A$ must lie in different rows because $A$ is invertible.

# 2 Spectral radius preservers on $M_n^+$

Here is our main theorem of this section.

**Theorem 2.1** *The following statements* (1) - (4) *are equivalent for a function* $f : M_n^+ \longrightarrow M_n^+$.

(1)
$$r(A+B) = r(f(A)+f(B)), \qquad \forall \ A, B \in M_n^+. \tag{2.1}$$

(2)
$$\sigma_p(A+B) = \sigma_p(f(A)+f(B)), \qquad \forall \ A, B \in M_n^+. \tag{2.2}$$

(3)
$$\sigma(A+B) = \sigma(f(A)+f(B)), \qquad \forall \ A, B \in M_n^+. \tag{2.3}$$

(4) *There exists a matrix* $Q \in \mathcal{PD}$ *such that either*

$$f(A) = Q^{-1}AQ, \qquad \forall \ A \in M_n^+,$$

*or*

$$f(A) = Q^{-1}A^{\text{tr}}Q, \qquad \forall \ A \in M_n^+. \tag{2.4}$$

Since for $A \in M_n^+$ we always have $r(A) \in \sigma_p(A)$, the implications (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) are clear. Also, (4) $\Longrightarrow$ (3) is not difficult to see. It remains to prove (1) $\Longrightarrow$ (4).

First, we present some general results and easy observations that will be often used, sometimes without explicit reference, throughout the paper. We will use the directed graph $\Gamma(A)$ associated with $A \in M_n^+$. Recall that $\{1, 2, \ldots, n\}$ is the set of vertices of $\Gamma(A)$, and $(i,j)$ is a directed edge in $\Gamma(A)$ if and only if the $(i,j)$th entry of $A$ is positive.

A matrix $A \in M_n^+$ is said to be irreducible if there is no permutation matrix $P$ such that $PAP^{\rm tr} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ such that $A_{11}$ and $A_{22}$ are non-trivial square matrices. A useful well-known criterion for irreducibility is given in terms $\Gamma(A)$:

**Lemma 2.2** *$A \in M_n^+$ is irreducible if and only if $\Gamma(A)$ is strongly connected.*

Next, we list several well-known properties of nonnegative matrices and their spectral radii (see, for example, [8, Theorem 8.4.5] or [3]).

**Lemma 2.3** *Let $A \in M_n^+$. Then:*

(a) *$r(A) \ge r(A')$ for any principal submatrix $A'$ of $A$. In particular,*

$$r(A) \ge \max\{d : d \text{ is a diagonal entry of } A\}.$$

(b) *If $A \in M_n^+$ is nilpotent, i.e., $r(A) = 0$, then all diagonal entries of $A$ are zeros.*

(c) *If $A \in M_n^+$ is irreducible and $B \in M_n^+$ is nonzero, then $r(A + B) > r(A)$.*

(d) *If $A \in M_n^+$ is reducible, then there is a permutation matrix $P$ such that $PAP^{\rm tr}$ is upper triangular block form $[A_{ij}]_{1 \le i,j \le k}$ such that $A_{11}, \dots, A_{kk}$ are irreducible square matrices and*

$$r(A) = \max\{r(A_{jj}) : 1 \le j \le k\}.$$

(e) *If $A, B \in M_n^+$ and $A \ge B$ entrywise, then $r(A) \ge r(B)$.*

Notice that (b) is an immediate consequence of (a).

**Lemma 2.4** *Let $A_1, A_2 \in M_n^+$ have irreducible principal submatrices $B_1$ and $B_2$, respectively, such that $r(A_1) = r(B_1)$, $r(A_2) = r(B_2)$. If the row and column indices of $B_1$ and $B_2$ have non-empty intersection, then*

$$r(A_1 + A_2) > \max\{r(A_1), r(A_2)\}. \tag{2.5}$$

*Proof.* For $t_1, t_2 \in (0, 1]$ consider $t_1A_1 + t_2A_2$ and its irreducible principal submatrix $B(t_1, t_2)$ whose set of row and column indices is the union of the set of row and column indices of $B_1$ and that of $B_2$. Since row and column indices of $B_1$ and $B_2$ have non-empty intersection, the matrix $B(t_1, t_2)$ is irreducible in view of Lemma 2.2, for all $t_1, t_2 \in (0, 1]$. Now

$$\begin{aligned} r(A_1 + A_2) &\ge r(B(1, 1)) > \max\{r(B(1, 1/2)), r(B(1/2, 1))\} \\ &\ge \max\{r(B_1), r(B_2)\} = \max\{r(A_1), r(A_2)\}, \end{aligned}$$

where the strict inequality holds by Lemma 2.3 (c), and the non-strict inequalities hold in view of Lemma 2.3 (e). □

**Proof of Theorem 2.1**

We focus on the implication (1) $\Longrightarrow$ (4). Assume that the function $f$ satisfies the condition (1) of Theorem 2.1. We divide the proof into several assertions.

**Assertion 2.5** (a) *For any $A \in M_n^+$ we have $r(A) = r(f(A))$.*

(b) *$A \in M_n^+$ is nilpotent if and only if $f(A)$ is nilpotent.*

(c) *If $A$ is nonzero, then $f(A)$ is nonzero.*

*Proof.* Condition (a) follows from setting $A = B$ in (2.1).

Condition (b) follows readily from (a).

Suppose $A$ in nonzero and the $(i,j)$ entry of $A$ is nonzero. If $i = j$ then $A$ is not nilpotent and neither is $f(A)$. Thus, $f(A)$ is nonzero. If $i \neq j$, then for $B = E_{ji}$, the submatrix of $A + B$ with row and column indices $\{i, j\}$ has positive spectral radius. If $f(A) = 0$ then $r(f(A) + f(B)) = r(f(B)) = r(B) = 0$, which is a contradiction. □

**Assertion 2.6** *There is a permutation $P$ such that for any $\mu > 0$ the diagonal of the matrix $Pf(\mu E_{ii})P^{\rm tr}$ is the same as that of $\mu E_{ii}$ for $i = 1, \ldots, n$.*

*Proof.* In what follows we let $F_{ij} = f(E_{ij})$. First, consider $\mu = 1$. For each $j = 1, \ldots, n$, let $G_{jj}$ be an irreducible principal submatrix of $F_{jj}$ such that

$$r(G_{jj}) = r(F_{jj}) = 1.$$

(The existence of principal submatrices $G_{jj}$ is guaranteed by Lemma 2.3 (d).) We will show that $G_{jj} = [1]$. Note that the row (column) indices of $G_{11}, \ldots, G_{nn}$ cannot overlap. If it is not true and the row indices of $G_{ii}$ and $G_{jj}$ overlap, then by Lemma 2.4,

$$r(F_{ii} + F_{jj}) > r(F_{ii}) = 1 = r(E_{ii} + E_{jj}),$$

which is a contradiction. Thus, $G_{11}, \ldots, G_{nn}$ are one-by-one with non-overlapping row (column) indices. Since $r(G_{jj}) = 1$, we see that $G_{jj} = [1]$ for all $j = 1, \ldots, n$. Thus, there exists $P \in \mathcal{P}$ such that $PF_{jj}P^{\rm tr}$ has one in the $(j,j)$ position. Suppose $i \neq j$. the $(i,i)$ entry $PF_{jj}P^{\rm tr}$ is zero. Otherwise, the $(i,i)$ entry of $P(F_{ii} + F_{jj})P^{\rm tr}$ is larger than 1 so that by Lemma 2.3 (a),

$$r(F_{ii} + F_{jj}) = r(P(F_{ii} + F_{jj})P^{\rm tr}) > 1 = r(E_{ii} + E_{jj}).$$

For any $\mu > 0$, we can apply the preceding proof to show that there is a permutation matrix $P_\mu$ such that $P_\mu f(\mu E_{ii})P_\mu^{\rm tr}$ has $\mu$ at the $(i,i)$ position and all other diagonal entries equal to zero. If $P_\mu \neq P$, then there will be indices $i \neq j$ and $k$ so that $f(\mu E_{ii})$

has $\mu$ in the $(k,k)$ position, and $f(E_{jj})$ has one in the $(k,k)$ position. But then by Lemma 2.3 (a),

$$r(f(\mu E_{ii}) + f(E_{jj})) \geq 1 + \mu > r(\mu E_{ii} + E_{jj}),$$

which is a contradiction. □

**Assertion 2.7** *Let $P$ be the permutation satisfying the conclusion of Assertion 2.6. Then for any $i \neq j$, the $2 \times 2$ submatrix of $P(E_{ij} + E_{ji})P^{\rm tr}$ lying at rows and columns with indices $\{i, j\}$ has the form $\begin{bmatrix} 0 & g_{12} \\ g_{21} & 0 \end{bmatrix}$ with $g_{12}g_{21} = 1$.*

*Proof.* For simplicity, we assume that $P$ is the identity matrix. Otherwise, consider the map $X \mapsto Pf(X)P^{\rm tr}$.

For each $i \neq j$, let $X = E_{ij} + E_{ji}$ and let $G_{ij}$ be an irreducible principal submatrix of $f(X)$ such that

$$r(G_{ij}) = r(f(X)) = r(X) = 1. \tag{2.6}$$

We claim that $G_{ij}$ must lie in a submatrix of $f(X)$ with row and column indices in the set $\{i, j\}$. Indeed, suppose this is not true, and let $k$ be a row and column index of $G_{ij}$ different from $i$ and from $j$. Denote by $[f(E_{kk})]$ the principal submatrix of $f(E_{kk})$ having the same row and column indices as $G_{ij}$ does. Then:

$$\begin{aligned} r(f(X) + f(E_{kk})) &\geq r(G_{ij} + [f(E_{kk})]) > r(G_{ij}) \\ &= 1 = r(X + E_{kk}) = r(f(X) + f(E_{kk})), \end{aligned}$$

where the strict inequality follows from Lemma 2.3 (c). A contradiction is obtained.

Suppose $G_{ij} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$. If at least one of $g_{12}$ and $g_{21}$ is zero, then, in view of (2.6) we must have $g_{11} = 1$ or $g_{22} = 1$. But then for $Y = E_{ii}$ or $E_{jj}$, $f(X) + f(Y)$ a diagonal entry larger than or equal to 2. By Lemma 2.3 (a), we have

$$r(f(X) + f(Y)) \geq 2 > \frac{1 + \sqrt{5}}{2} = r(X + Y),$$

which is a contradiction. Thus, $g_{12}g_{21} \neq 0$.

Next, we claim that $g_{11} = 0$. If it is not true, then for sufficiently large $\mu > 0$, the matrix $f(\mu E_{ii}) + f(X)$ has $\mu + g_{11}$ at the $(i,i)$ position so that

$$r(f(\mu E_{ii}) + f(X)) \geq \mu + g_{11} > [\mu + \sqrt{\mu^2 + 4}]/2 = r(\mu E_{ii} + X),$$

which is a contradiction. Similarly, we can show that $g_{22} = 0$. Since $r(G_{ij}) = 1$, we see that $g_{12}g_{21} = 1$. □

In the rest of the proof, we assume that Assertions 2.6 and 2.7 hold with $P = I$ for simplicity.

**Assertion 2.8** *For every $\mu > 0$ and every pair of indices $i \neq j$, $f(\mu E_{ij})$ is a nonzero multiple of $E_{ij}$ or of $E_{ji}$.*

*Proof.* By Assertion 2.7, for any $i \neq j$, the matrix $f(E_{ij} + E_{ji})$ has a submatrix $G_{ij} = \begin{bmatrix} 0 & g_{ij} \\ g_{ij} & 0 \end{bmatrix}$ with row and column indices in $\{i, j\}$ such that $g_{ij} g_{ij} = 1$.

Now, suppose $\mu > 0$, $i \neq j$, and let $f(\mu E_{ij}) = [z_{rs}]_{r,s=1}^n$. Then $z_{kk} = 0$ for all $k$. Otherwise, the $(k, k)$ entry of $f(E_{kk}) + f(\mu E_{ij})$ is larger than 1 so that

$$r(f(E_{kk}) + f(\mu E_{ij})) > 1 = r(E_{kk} + \mu E_{ij}),$$

which is a contradiction. We also have $z_{pq} = 0$ if at least one of the indices $p$ and $q$ ($p \neq q$) does not belong to the two-element set $\{i, j\}$. Otherwise, the submatrix of $f(E_{pq} + E_{qp}) + f(\mu E_{ij})$ with row and column indices in $\{p, q\}$ has the form

$$C = \begin{bmatrix} 0 & g_{pq} + z_{pq} \\ g_{qp} + z_{qp} & 0 \end{bmatrix},$$

with $g_{qp} g_{pq} = 1$, so that

$$r(f(E_{pq} + E_{qp}) + f(\mu E_{ij})) \geq r(C) > 1 = r(E_{pq} + E_{qp} + \mu E_{ij}),$$

which is a contradiction.

Since $0 = r(\mu E_{ij}) = r(f(\mu E_{ij}))$, we see that $z_{ij} z_{ji} = 0$. Hence $f(\mu E_{ij})$ is a multiple of $E_{ij}$ or $E_{ji}$. Similarly, $f(\mu E_{ji})$ is a multiple of $E_{ij}$ or of $E_{ji}$. □

**Assertion 2.9** *Let $X = X_0 \oplus 0_{n-3}$, where $X_0 \in M_3^+$ is nilpotent. Then $f(X) = Z_0 \oplus 0_{n-3}$ such that $Z_0 \in M_3^+$ is nilpotent with at most 3 nonzero entries. Moreover, let*

$$S = \{E_{ij} : 1 \leq i, j \leq 3, i \neq j\}, \tag{2.7}$$

*and*

$$f(S) = \{\mu_{ij} E_{ij} : 1 \leq i, j \leq 3, i \neq j\}, \quad \text{for some } \mu_{ij} > 0.$$

*(The form of $f(S)$ follows from Assertion 2.8.) One of the following is true:*

(1) *If $f(X)$ has only one nonzero entry, then $r(f(X) + Z) > 0$ for only one matrix $Z$ in $f(S)$.*

(2) *If $f(X)$ has exactly two nonzero entries, and they lie in the same row or the same column, then $r(f(X)+Z)>0$ for exactly two matrices $Z$ in $f(S)$.*

(3) *If $f(X)$ has two or three nonzero entries such that two of them are not in the same row or column, then $r(f(X)+Z)>0$ for at least three matrices $Z$ in $f(S)$.*

*Proof* Let $f(X)=[y_{pq}]_{p,q=1}^n$. Then $f(X)$ is nilpotent so that $y_{jj}=0$ for all $j=1,\ldots,n$. Also, if $i\neq j$ and at least one of $i$ and $j$ is larger than 3, then $y_{ij}=0$. Otherwise,

$$r(f(X)+f(E_{ij}))>0=r(X+E_{ij}) \quad \text{or} \quad r(f(X)+f(E_{ji}))>0=r(X+E_{ji})$$

by the fact that $f(E_{ij})$ or $f(E_{ji})$ is a multiple of $E_{ij}$. So, if $y_{ij}\neq 0$, then $i\neq j$ and $1\leq i,j\leq 3$. Moreover, since $0=r(X)=r(f(X))$, we see that $y_{ij}y_{ji}=0$ for $i\neq j$ (otherwise, the $2\times 2$ principal submatrix of $f(X)$ with row and column indices $\{i,j\}$ would have a positive spectral radius, a contradiction with Lemma 2.3 (a)). Thus, there are at most three nonzero entries in $f(X)$, and they all lie in the leading $3\times 3$ principal submatrix of $f(X)$. Using the condition that $f(X)=[y_{ij}]_{i,j=1}^n$ with $y_{ij}y_{ji}=0$, we see that one of the condition (1) – (3) is true. □

**Assertion 2.10** *There is $D\in\mathcal{D}$ such that either*
(a) $f(\mu E_{ij})=\mu D E_{ij} D^{-1}$ *for all $\mu>0$ and all pairs $(i,j)$, or*
(b) $f(\mu E_{ij})=\mu D E_{ji} D^{-1}$ *for all $\mu>0$ and all pairs $(i,j)$.*

*Proof.* First consider the case when $\mu=1$, for all pairs of indices $(i,j)$ such that $i\neq j$.

By Assertion 2.8, $f(E_{12})=\mu_2 E_{12}$ or $f(E_{12})=\mu_2 E_{21}$ for some $\mu_2>0$. Assume $f(E_{12})=\mu_2 E_{12}$. Otherwise, replace $f$ by the map $X\mapsto f(X)^{\mathrm{tr}}$. Since

$$1=r(E_{12}+E_{21})=r(f(E_{12})+f(E_{21}))=r(\mu_2 E_{12}+f(E_{21})),$$

using the result of Assertion 2.8 again, we see that $f(E_{21})=E_{21}/\mu_2$. We get the desired conclusion for $f(E_{ij})$ with $i\neq j$ if $n=2$.

Assume $n\geq 3$. For any $j\geq 2$, we claim that $f(E_{1j})=\mu_j E_{1j}$ for some $\mu_j>0$. For simplicity, suppose that $j=3$. Let $X=E_{12}+E_{13}$ and let $f(X)=[z_{ij}]_{i,j=1}^n$. By Assertion 2.9, $z_{ij}$ can be nonzero only if $i\neq j$ and $1\leq i,j\leq 3$; also, $z_{ij}z_{ji}=0$ for all $i,j$. Since $r(X+Y)>0$ for exactly two matrices $Y\in S$ (the set $S$ is defined in (2.7)), we conclude that $r(f(X)+Z)>0$ for exactly two matrices $Z$ in $f(S)$, and therefore condition (2) of Assertion 2.9 holds. Note that

$$1=r(X+E_{21})=r(f(X)+f(E_{21}))=r(f(X)+E_{21}/\mu_2). \tag{2.8}$$

As a result, $z_{12}$ must be one of the two nonzero entries of $f(X)$ in the same row or same column. Thus, either

$$\text{(a) } z_{12}z_{13} \neq 0, \quad \text{or} \quad \text{(b) } z_{12}z_{32} \neq 0.$$

If (a) holds, then

$$1 = r(X + E_{31}) = r(f(X) + f(E_{31})).$$

Applying Assertion 2.8 for $f(E_{31})$ we see that $f(E_{31})$ is a multiple of $E_{31}$, and $f(E_{13}) = \mu_3 E_{13}$ as asserted. Suppose (b) holds. Then

$$f(E_{23} + E_{32}) - (kE_{23} + k^{-1}E_{32})$$

is nonnegative for some $k > 0$ by Assertion 2.7. It follows that

$$r(X + E_{23} + E_{32}) = 1 < r(f(X) + kE_{23} + k^{-1}E_{32}) \leq r(f(X) + f(E_{23} + E_{32}))$$

(the inequality $\leq$ holds by Lemma 2.3 (e)), which is a contradiction.

Now, we have $f(E_{1j}) = \mu_j E_{1j}$ with $\mu_j > 0$ for $j = 2, \ldots, n$. Let

$$D = \operatorname{diag}(1, \mu_2, \ldots, \mu_n).$$

We may replace $f$ by the map $X \mapsto Df(X)D^{-1}$ so that $f(E_{1j}) = E_{1j}$ for $j = 2, \ldots, n$. Since

$$1 = r(E_{1j} + E_{j1}) = r(f(E_{1j}) + f(E_{j1})) = r(E_{1j} + f(E_{j1})),$$

and since by Assertion 2.8 $f(E_{j1})$ is a multiple of either $E_{1j}$ or $E_{j1}$, we have in fact $f(E_{j1}) = E_{j1}$ for all $j = 2, \ldots, n$.

Next, we show that $f(E_{ij}) = E_{ij}$ if $i \neq j$ and $i, j \geq 2$. Assume that $(i, j) = (2, 3)$ for simplicity. Let $X = E_{12} + E_{31}$ and $f(X) = [z_{ij}]_{i,j=1}^n$. We claim that $f(X) = X$. Note that $X = X_0 \oplus 0_{n-3}$ with $X_0 \in M_3^+$ is nilpotent. Since $r(X + Y) > 0$ for at least three matrices $Y$ in $S$, it follows that $r(f(X) + \widehat{Y}) > 0$ for at least three matrices $\widehat{Y}$ in $f(S)$. Hence, $f(X)$ satisfies condition (3) of Assertion 2.9. Since

$$0 = r(X + Y) = r(f(X) + f(Y))$$

for $Y = E_{12}, E_{31}$ and $E_{32}$, we see that $z_{21} = 0$, $z_{13} = 0$ and $z_{23} = 0$, i.e.,

$$f(X) = \begin{bmatrix} 0 & z_{12} & 0 \\ 0 & 0 & 0 \\ z_{31} & z_{32} & 0 \end{bmatrix}.$$

Since

$$1 = r(X + E_{21}) = r(f(X) + E_{21}),$$

we see that $z_{12} = 1$; since

$$1 = r(X + E_{13}) = r(f(X) + E_{13}),$$

we see that $z_{31} = 1$. If $Y = E_{23} + E_{32}$, then by Assertion 2.7 there is $\nu > 0$ such that

$$f(Y) - \nu E_{23} - E_{32}/\nu \tag{2.9}$$

is nonnegative. Let

$$\widehat{Z} = E_{12} + E_{31} + z_{32}E_{32} + \nu E_{23} + E_{32}.$$

Assuming for the moment that $\nu \geq 1$, we have

$$r(f(X) + f(Y)) \geq r(\widehat{Z}) \geq r(X + Y) = r(f(X) + f(Y)), \tag{2.10}$$

where the second inequality follows by comparison between the largest roots of the characteristic polynomials $-\lambda^3 + \lambda + 1$ and $-\lambda^3 + (\nu z_{32} + 1)\lambda + \nu$ of $X + Y$ and of $\widehat{Z}$, respectively. Since the second inequality in (2.10) is an equality, we see that in fact

$$\nu = 1 \quad \text{and} \quad z_{32} = 0. \tag{2.11}$$

Hence $f(X) = X$. Now, by Assertion 2.8, $f(E_{23})$ is a multiple of $E_{23}$ or $E_{32}$. Since

$$1 = r(X + E_{23}) = r(X + f(E_{23})),$$

we see that $f(E_{23}) = E_{23}$ as asserted.

If $\nu$ of (2.9) is smaller than 1, we apply the arguments in the preceding paragraph to $\widehat{X} := E_{21} + E_{13}$ rather than to $X$, replacing $\nu$ by $\nu^{-1}$ and interchanging everywhere the subscripts. Then a contradiction with (2.11) will be obtained, thus $\nu < 1$ is not possible.

At this point, we may assume that $f(E_{ij}) = E_{ij}$ if $i \neq j$. Now consider $f(\mu E_{ij}) = [z_{pq}]_{p,q=1}^n$ for $\mu \geq 0$ and $i \neq j$. Then $z_{pp} = 0$ for all $p \in \{1, \ldots, n\}$. Otherwise, we obtain a contradiction (in the next formula $W$ stands for a matrix with zero diagonal):

$$r(f(\mu E_{ij}) + f(E_{pp})) = r(f(\mu E_{ij}) + E_{pp} + W) \geq r((1 + z_{pp})E_{pp}) > 1 = r(\mu E_{ij} + E_{pp}),$$

where the first equality follows from Assertion 2.6, and the non-strict inequality follows from Lemma 2.3 (e). Also, $z_{pq} = 0$ for $p \neq q$ if $(p, q) \neq (i, j)$. Otherwise, a contradiction again:

$$r(f(\mu E_{ij}) + f(E_{qp})) \geq r(z_{pq}E_{pq} + f(E_{qp})) = r(z_{pq}E_{pq} + E_{qp}) > 0 = r(\mu E_{ij} + E_{qp}).$$

Finally,

$$\sqrt{\mu} = r(\mu E_{ij} + E_{ji}) = r(f(\mu E_{ij}) + f(E_{ji})) = r(f(\mu E_{ij}) + E_{ji})$$

implies that

$$f(\mu E_{ij}) = \mu E_{ij}. \tag{2.12}$$

Next, consider $f(\mu E_{ii}) = [z_{rs}]_{r,s=1}^n$ for fixed $\mu > 0$ and fixed $i \in \{1, \dots, n\}$. Then $z_{ii} = \mu$ and $z_{jj} = 0$ for $j \neq i$ by Assertion 2.6. Also, $z_{pq} = 0$ for any $p \neq q$. Otherwise,

$$\begin{aligned} r(f(\mu E_{ii}) + f(\nu E_{qp})) &\geq r(z_{pq} E_{pq} + f(\nu E_{qp})) \\ &= r(z_{pq} E_{pq} + \nu E_{qp}) \qquad \text{(using (2.12))} \\ &> \mu = r(\mu E_{ii} + \nu E_{pq}) \end{aligned}$$

for a sufficiently large $\nu$. Hence $f(\mu E_{ii}) = \mu E_{ii}$. □

**Assertion 2.11** *The function $f$ has the form as in* (4) *of Theorem* 2.1.

*Proof.* Let $D \in \mathcal{D}$ satisfy the conclusion of Assertion 2.10. We may replace $f$ by the map $X \mapsto D^{-1} f(X) D$ and assume that $D = I$. We may further assume that $f(\mu E_{ij}) = \mu E_{ij}$ for all $\mu > 0$ and $(i, j)$ pairs. Otherwise, replace $f$ by the map $X \mapsto f(X)^{\mathrm{tr}}$.

Suppose $A = [a_{ij}]_{i,j=1}^n \in M_n^+$ and $f(A) = [z_{ij}]_{i,j=1}^n$. First, we show that $z_{jj} = a_{jj}$ for each $j$. For simplicity, we consider $z_{11}$. Let

$$A = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad f(A) = \begin{bmatrix} z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}.$$

Suppose $t > r(A) = r(f(A)) \geq \max\{r(A_{22}), r(Z_{22})\}$,

$$B_t := A_{12}(tI_{n-1} - A_{22})^{-1} A_{21} \quad \text{and} \quad \widehat{B}_t := Z_{12}(tI_{n-1} - Z_{22})^{-1} Z_{21}.$$

Since $\det(tI_n - (A + \mu E_{11}))$ is equal (as a function of $\mu$) to

$$-\mu(\det(tI_{n-1} - A_{22})) + \det(tI_n - A),$$

it follows that there is (unique) $\mu_t > 0$ such that

$$\det(tI_n - (A + \mu_t E_{11})) = 0.$$

Using Schur complements, we see that

$$\det(tI_n - (A + \mu_t E_{11})) = (t - a_{11} - \mu_t - B_t) \det(tI_{n-1} - A_{22}) = 0,$$

i.e.,

$$t - a_{11} - \mu_t - \sum_{k=0}^{\infty} t^{-k-1} A_{12} A_{22}^k A_{21} = 0.$$

Obviously,

$$s - a_{11} - \mu_t - \sum_{k=0}^{\infty} s^{-k-1} A_{12} A_{22}^k A_{21} > 0$$

for every $s > t$; thus

$$t = r(A + \mu_t E_{11}) = r(f(A) + \mu_t E_{11}).$$

Now

$$0 = \det(tI_n - (f(A) + \mu_t E_{11})) = (t - z_{11} - \mu_t - \widehat{B}_t)\det(tI_{n-1} - Z_{22}),$$

i.e.,

$$0 = t - z_{11} - \mu_t - \sum_{k=0}^{\infty} t^{-k-1} Z_{12} Z_{22}^k Z_{21}.$$

As a result,

$$a_{11} + \sum_{k=0}^{\infty} t^{-k-1} A_{12} A_{22}^k A_{21} = \mu_t - t = z_{11} + \sum_{k=0}^{\infty} t^{-k-1} Z_{12} Z_{22}^k Z_{21}$$

for all sufficiently large $t$, and hence $a_{11} = z_{11}$ as asserted.

Next, we show that $a_{ij} = z_{ij}$ for $i \neq j$. For simplicity, we consider $z_{12}$. First, suppose $n = 2$. Since

$$r(A + tE_{21}) = [(a_{11} + a_{22}) + \sqrt{(a_{11} - a_{22})^2 + 4a_{12}(a_{21} + t)}]/2$$

and

$$r(f(A) + tE_{21}) = [(z_{11} + z_{22}) + \sqrt{(z_{11} - z_{22})^2 + 4z_{12}(z_{21} + t)}]/2$$

are equal for all $t > 0$, and using $a_{11} = z_{11}$, $a_{22} = z_{22}$, we see that $a_{12} = b_{12}$.

Next, suppose $n > 2$. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad f(A) = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},$$

with $A_{11}, Z_{11} \in M_2^+$. Arguing by contradiction, assume that

$$\varepsilon := a_{12} - z_{12} > 0. \tag{2.13}$$

[If the opposite inequality holds, interchange the roles of $A$ and $f(A)$ in the following argument.] Suppose $t > r(A) = r(f(A))$ and

$$B_t := A_{12}(tI - A_{22})^{-1}A_{21} = \sum_{k=0}^{\infty} t^{-1}A_{12}(t^{-1}A_{22})^k A_{21}.$$

There is $T > 0$ such that each entry of $B_t$ lies in $[0, \varepsilon/3)$ whenever $t \geq T$. If

$$B_t = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad \text{and} \quad C_{t,\mu} = A_{11} + \mu E_{21} + B_t, \quad (\mu > 0),$$

then $C_{t,\mu}$ has eigenvalues

$$\left[(a_{11} + a_{22} + b_{11} + b_{22}) \pm \sqrt{(a_{11} + b_{11} - a_{22} - b_{22})^2 + 4(a_{12} + b_{12})(a_{21} + b_{21} + \mu)}\right] /2. \tag{2.14}$$

Note that

$$\det(tI_n - (A + \mu E_{21})) = \det(tI_2 - C_{t,\mu}) \det(tI_{n-2} - A_{22}) \tag{2.15}$$

so that $\det(tI_2 - C_{t,\mu}) > 0$ if $\mu = 0$. Inequality (2.13) implies that $a_{12} > 0$, which, together with formula (2.14), shows that there is (unique) $\nu_t > 0$ such that the larger eigenvalue of $C_{t,\nu_t}$ equals $t$. Moreover, for any $\lambda > t$, we have

$$\det(\lambda I - A - \nu_t E_{21}) \neq 0 = \det(tI - A - \nu_t E_{21}).$$

Hence,

$$t = r(A + \nu_t E_{21}) = r(f(A) + \nu_t E_{21})).$$

Similarly, if

$$\widetilde{B}_t := Z_{12}(tI_{n-2} - Z_{22})^{-1}Z_{21} = [\widetilde{b}_{ij}]_{i,j=1}^2 \in M_2^+ \quad \text{and} \quad \widetilde{C}_t := Z_{11} + \nu_t E_{21} + \widetilde{B}_t,$$

then

$$\det(tI_n - f(A) - \nu_t E_{21}) = \det(tI_2 - \widetilde{C}_t) \det(tI_{n-2} - Z_{22}),$$

and there exists $\widetilde{T} > 0$ such that every entry of $\widetilde{B}_t$ is smaller than $\varepsilon/3$ whenever $t > \widetilde{T}$. Observe that $\widetilde{C}_t$ has eigenvalues

$$\left[(z_{11} + z_{22} + \widetilde{b}_{11} + \widetilde{b}_{22}) \pm \sqrt{(z_{11} + \widetilde{b}_{11} - z_{22} - \widetilde{b}_{22})^2 + 4(z_{12} + \widetilde{b}_{12})(z_{21} + \widetilde{b}_{21} + \nu_t)}\right] /2.$$

So, $2r(A+\nu_t E_{21})$ and $2r(f(A)+\nu_t E_{21})$ are equal to the following quantities, respectively:

$$a_{11} + a_{22} + b_{11} + b_{22} + \sqrt{(a_{11} + b_{11} - a_{22} - b_{22})^2 + 4(a_{12} + b_{12})(a_{21} + b_{21} + \nu_t)} \tag{2.16}$$

and

$$z_{11} + z_{22} + \widetilde{b}_{11} + \widetilde{b}_{22} + \sqrt{(z_{11} + \widetilde{b}_{11} - z_{22} - \widetilde{b}_{22})^2 + 4(z_{12} + \widetilde{b}_{12})(z_{21} + \widetilde{b}_{21} + \nu_t)}. \quad (2.17)$$

Evidently, $\nu_t \to \infty$ as $t \to \infty$. Since

$$a_{12} + b_{12} - (z_{12} + \widetilde{b}_{12}) > \varepsilon - 2(\varepsilon/3) > 0 \quad \text{for} \quad t > \max\{T, \widetilde{T}\},$$

we have

$$r(A + \nu_t E_{21}) > r(f(A) + \nu_t E_{21})$$

for sufficiently large $t$, which is the desired contradiction. □

# 3 Spectral radius preservers on $S_n^+$

An adaptation of the proof of Theorem 2.1 yields the following preserver result on the set $S_n^+$ of $n \times n$ symmetric nonnegative matrices.

**Theorem 3.1** *The following statements* (1) - (4) *are equivalent for a function* $f : S_n^+ \longrightarrow S_n^+$.

(1)
$$r(A + B) = r(f(A) + f(B)), \qquad \forall \ A, B \in S_n^+. \quad (3.1)$$

(2)
$$\sigma_p(A + B) = \sigma_p(f(A) + f(B)), \qquad \forall \ A, B \in S_n^+. \quad (3.2)$$

(3)
$$\sigma(A + B) = \sigma(f(A) + f(B)), \qquad \forall \ A, B \in S_n^+. \quad (3.3)$$

(4) *There exists a matrix* $Q \in \mathcal{P}$ *such that* $f(A) = Q^{-1}AQ, \qquad \forall \ A \in S_n^+$.

*Proof.* We only need to deal with the non-trivial implication (1) ⇒ (4). So assume that $f$ satisfies (3.1). Then $r(A) = r(f(A))$ for every $A \in S_n^+$, and in particular $f(A) = 0$ if and only if $A = 0$.

We divide the rest of the proof into several steps.

**Step 1.** Assertion 2.6, together with its proof, remains valid. Thus, there exists a permutation $Q$ such that for any $\mu > 0$ the diagonal of the matrix $Qf(\mu E_{ii})Q^{\text{tr}}$ is the same as that of $\mu E_{ii}$ for $i = 1, \ldots, n$.

**Step 2.** Let $Q$ be the matrix in Step 1. We show, by following the proof of Assertion 2.7 that for $i \neq j$, the $2 \times 2$ submatrix in $Qf(E_{ij} + E_{ji})Q^{-1}$ with row and column indices $i, j$ has the form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

**Step 3.** Assuming the matrix $Q$ in Step 1 equal $I_n$, we prove that $f(X)$ is a nonzero multiple of $X$, for $X = \mu(E_{ij} + E_{ji})$ with $\mu > 0$ and $i \neq j$.

*Proof of Step 3.* By Step 2, for any $i \neq j$, the matrix $f(E_{ij} + E_{ji})$ has a submatrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with row and column indices in $\{i, j\}$. Let $f(X) = [z_{rs}]_{r,s=1}^n$. As in the proof of Assertion 2.8 (but using $E_{ij} + E_{ji}$ in place of $E_{ij}$) we show that $z_{kk} = 0$ for all $k$, and that $z_{pq} = 0$ for all pairs $\{p, q\}$, $p \neq q$ such that at least one of $p$ and $q$ does not belong to $\{i, j\}$. Since $f(X)$ is symmetric and $f(X) \neq 0$, the result of Step 3 follows. $\square$

**Step 4.** Again assuming $Q = I$, we prove the symmetric analog of Assertion 2.10: The equality

$$f(\mu(E_{ij} + E_{ji})) = \mu(E_{ij} + E_{ji}) \tag{3.4}$$

holds for all $\mu > 0$ and all pairs $(i, j)$.

*Proof of Step 4.* For $i \neq j$, the result follows easily from Step 3: $f(\mu(E_{ij} + E_{ji})) = \mu'(E_{ij} + E_{ji})$ for some $\mu' > 0$, but the equality

$$r(f(\mu(E_{ij} + E_{ji}))) = r(\mu(E_{ij} + E_{ji}))$$

yields $\mu' = \mu$, as claimed.

Next, consider $f(\mu E_{ii}) = [z_{rs}]_{r,s=1}^n \in S_n^+$ for fixed $\mu > 0$ and fixed $i \in \{1, \ldots, n\}$. Then $z_{ii} = \mu$ and $z_{jj} = 0$ for $j \neq i$ by Step 1. Also, $z_{pq} = 0$ for any $p \neq q$. Suppose it is not true and $z_{pq} = z_{qp} \neq 0$ for some $p \neq q$. Then using the already proved part of (3.4), we can choose $\nu > \mu$ so that

$$\begin{aligned} & r(f(\mu E_{ii}) + f(\nu(E_{qp} + E_{pq}))) \\ \geq\ & r(\mu E_{ii} + z_{pq}(E_{pq} + E_{qp}) + f(\nu(E_{qp} + E_{pq}))) \\ =\ & r(\mu E_{ii} + z_{pq}(E_{pq} + E_{qp}) + \nu(E_{qp} + E_{pq})) \\ =\ & \begin{cases} z_{pq} + \nu & \text{if } p \neq i,\ q \neq i, \\ \frac{1}{2}\left(\mu + \sqrt{\mu^2 + 4(z_{pq} + \nu)^2}\right) & \text{if } p = i \text{ or } q = i, \end{cases} \end{aligned} \tag{3.5}$$

and

$$r(\mu E_{ii} + \nu(E_{pq} + E_{qp})) = \begin{cases} \nu & \text{if } p \neq i,\ q \neq i, \\ \frac{1}{2}\left(\mu + \sqrt{\mu^2 + 4\nu^2}\right) & \text{if } p = i \text{ or } q = i. \end{cases} \tag{3.6}$$

But then the right hand side of (3.6) is smaller than that of (3.5), a contradiction with (3.1). Hence $f(\mu E_{ii}) = \mu E_{ii}$. □

**Step 5.** Conclusion of the proof that (assuming $Q = I$) $f(A) = A$ for all $A \in S_n^+$.

*Proof of Step 5.* Suppose $A = [a_{ij}]_{i,j=1}^n \in S_n^+$ and $f(A) = [z_{ij}]_{i,j=1}^n$. As in the proof of Assertion 2.11, we show that $z_{jj} = a_{jj}$ for each $j$.

Next, we will prove the equalities $a_{ij} = z_{ij}$ for $i \neq j$, following the (suitably modified) arguments of the proof of Assertion 2.11. We use the notation introduced in the proof of Assertion 2.11, with obvious additional properties that follow from symmetry; thus $A_{12}^{\mathrm{tr}} = A_{21}$, $A_{22}^{\mathrm{tr}} = A_{22}$, etc. For simplicity, consider $z_{12}$. First, suppose $n = 2$. Since

$$r(A + t(E_{21} + E_{12})) = [(a_{11} + a_{22}) + \sqrt{(a_{11} - a_{22})^2 + 4(a_{12} + t)(a_{21} + t)}]/2$$

and

$$r(f(A) + t(E_{21} + E_{12})) = [(z_{11} + z_{22}) + \sqrt{(z_{11} - z_{22})^2 + 4(z_{12} + t)(z_{21} + t)}]/2$$

are equal for all $t > 0$, and using $a_{11} = z_{11}$, $a_{22} = z_{22}$, $a_{12} = a_{21}$, $z_{12} = z_{21}$, we see that $a_{12} = z_{12}$.

Now suppose $n > 2$. We argue as in the proof of Assertion 2.11, replacing everywhere $E_{21}$ with $E_{21} + E_{12}$, and using the partitions

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad f(A) = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},$$

where $A_{21} = A_{12}^{\mathrm{tr}}$, $Z_{21} = Z_{12}^{\mathrm{tr}}$, and $A_{jj}$, $Z_{jj}$ are symmetric for $j = 1, 2$,

$$B_t := A_{12}(tI_{n-2} - A_{22})^{-1}A_{21} = [b_{ij}]_{i,j=1}^2 \in S_2^+,$$

$$\widetilde{B}_t := Z_{12}(tI_{n-2} - Z_{22})^{-1}Z_{21} = [\widetilde{b}_{ij}]_{i,j=1}^2 \in S_2^+.$$

Then

$$r(A + \nu_t(E_{21} + E_{12})) = r(f(A) + \nu_t(E_{21} + E_{12})), \tag{3.7}$$

on the other hand, $r(A + \nu_t(E_{21} + E_{12}))$ and $r(f(A) + \nu_t(E_{21} + E_{12}))$ are equal to the quantities (2.16) and (2.17), respectively, with $b_{12}$ replaced by $b_{12} + \nu_t$, and with $\widetilde{b}_{12}$ replaced by $\widetilde{b}_{12} + \nu_t$. Let $\varepsilon = a_{12} - z_{12} = a_{21} - z_{21} > 0$. Then

$$a_{12} + b_{12} + a_{21} + b_{21} - (z_{12} + \widetilde{b}_{12} + z_{21} + \widetilde{b}_{21}) > \frac{1}{3}\varepsilon > 0$$

for large $t$, a contradiction with (3.7). □

# 4 Numerical radius and numerical range preservers

It turns out that preservers of the numerical radius of the sum of nonnegative matrices have more complicated form than the "standard" maps as in other results of this paper.

To state and prove the result, we need to work with the set $K_n$ of $n \times n$ real skew-symmetric matrices.

**Theorem 4.1** *Let $f : M_n^+ \to M_n^+$. Then*

$$w(A+B) = w(f(A)+f(B)), \qquad \forall \ \ A,B \in M_n^+ \tag{4.1}$$

*if and only if there is a permutation matrix $P$ and a function $g : M_n^+ \to K_n$ satisfying $A + A^{\rm tr} - g(A) \in M_n^+$ for each $A$ such that*

$$f(A) = P(A + A^{\rm tr} + g(A))P^{\rm tr}/2 \qquad \textit{for all } A \in M_n^+. \tag{4.2}$$

*Proof.* Observe that we have

$$w(A) = r(A + A^{\rm tr})/2, \quad A \in M_n, \tag{4.3}$$

because for any unit length vector $x$ we can let $|x|$ be obtained from $x$ by replacing all its entries by their absolute values so that

$$|x^*Ax| \le |x|^{\rm tr} A|x| = |x|^{\rm tr}(A + A^{\rm tr})|x|/2 \le r(A + A^{\rm tr})/2, \tag{4.4}$$

and for $x$ a nonnegative eigenvector corresponding to the largest eigenvalue of the symmetric matrix $A + A^{\rm tr}$ the equality prevails in (4.4). Thus, (4.1) reads

$$r(A + A^{\rm tr} + B + B^{\rm tr}) = r(f(A) + f(A)^{\rm tr} + f(B) + f(B)^{\rm tr}), \quad \forall \ A,B \in M_n^+. \tag{4.5}$$

With this observation, the "if" part of Theorem 4.1 is clear.

We focus on the "only if" part. First, note that $w(f(A)+f(A)) = w(A+A)$ implies that $w(A) = w(f(A))$ for all $A \in M_n^+$. Also, $w(A) = r(A)$ for all $A \in S_n^+$.

**Assertion 4.2** *There is a permutation matrix $P$ such that for any $A \in S_n^+$ we have*

$$f(A) = P(A + A_K)P^{\rm tr}$$

*with $A_K \in K_n$ such that $A + A_K \in M_n^+$.*

*Proof.* Consider the map $f_0 : S_n^+ \to S_n^+$ defined by $f_0(A) = [f(A) + f(A)^{\rm tr}]/2$. Then

$$r(f_0(A) + f_0(B)) = w(f(A) + f(B)) = w(A+B) = r(A+B) \qquad \forall A,B \in S_n^+.$$

By Theorem 3.1, we see that $f_0$ has the form $A \mapsto PAP^{\rm tr}$ for some permutation matrix $P$, and Assertion 4.2 follows. □

**Assertion 4.3** *Let $P$ be as in Assertion 4.2. For any $A \in M_n^+$, we have $f(A) = P(A + A^{\rm tr} + A_K)P^{\rm tr}/2$ with $A_K \in K_n$ such that $A + A^{\rm tr} + A_K \in M_n^+$.*

*Proof.* For simplicity, we may assume that $P = I$. Suppose $A = A_1 + A_2$ and $f(A) = Z_1 + Z_2$ with $(A_1, A_2), (Z_1, Z_2) \in S_n^+ \times K_n$. Then for any $B \in S_n^+$, we have

$$r(A_1 + B) = w(A + B) = w(f(A) + f(B)) = r(Z_1 + \frac{1}{2}(f(B) + f(B)^{\rm tr})) = r(Z_1 + B), \tag{4.6}$$

where the last but one equality follows from (4.3), and the last equality holds by Assertion 4.2.

We now prove that

$$A_1 = [a_{ij}]_{i,j=1}^n = [z_{ij}]_{i,j=1}^n = Z_1.$$

First we prove $a_{ii} = z_{ii}$, and for simplicity assume $i = 1$. Then we argue as in the proof of Assertion 2.11, using the partitions

$$A_1 = A = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and } Z_1 = \begin{bmatrix} z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},$$

and the property (which follows from (4.6)) that

$$r(A_1 + \mu E_{11}) = r(Z_1 + \mu E_{11}), \quad \forall\ \mu > 0.$$

For the proof that $a_{ij} = z_{ij}$, $i \neq j$, and assume for simplicity $(i, j) = (1, 2)$, proceed in the same way as in Step 5 of the proof of Theorem 3.1; here, we use the partitions

$$A_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad Z_1 = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}, \quad A_{11}, Z_{11} \in S_2^+,$$

and the property that

$$r(A_1 + \nu(E_{12} + E_{21})) = r(Z_1 + \nu(E_{12} + E_{21})), \quad \forall\ \nu > 0.$$

□

Now, define $g : M_n^+ \to K_n$ by $g(A) = 2f(A) - P(A + A^{\rm tr})P^{\rm tr}$. In view of Assertion 4.3 we see that $f$ has the desired form (4.2). □

**Theorem 4.4** *Let $f : M_n^+ \to M_n^+$. Then*

$$W(A + B) = W(f(A) + f(B)), \qquad \forall\ \ A, B \in M_n^+ \tag{4.7}$$

*if and only if there is a permutation matrix $Q$ such that either*

$$f(A) = Q^{-1}AQ, \qquad \forall \ \ A \in M_n^+,$$

*or*

$$f(A) = Q^{-1}A^{\rm tr}Q, \qquad \forall \ \ A \in M_n^+.$$

In the proof the following well known facts will be used. See, for example, Theorem 1.3.6 and Theorem 1.5.2 in [9].

**Lemma 4.5** (a) *For $n \times n$ complex matrices $X$ and $Y$, the equality $W(X) = W(Y)$ holds if and only if the largest eigenvalues of the two matrices $e^{it}X + e^{-it}X^*$ and $e^{it}Y + e^{-it}Y^*$ are always the same for every $t \in [0, 2\pi)$.*

(b) *For a complex $2 \times 2$ matrix $X$, $W(X)$ is an elliptical disk with foci at the eigenvalues of $X$.*

*Proof of Theorem* 4.4. The implication "if" is clear. (Note that $W(X) = W(X^{\rm tr})$ for any $n \times n$ complex matrix $X$.) We focus on the converse. Thus, suppose $f$ satisfies (4.7). Note that $W(f(A) + f(A)) = W(A + A)$ implies that $W(f(A)) = W(A)$ for all $A \in M_n^+$.

Clearly, since (4.7) holds, then (4.1) holds as well. By Theorem 4.1, $f(A)$ has symmetric part $P(A + A^{\rm tr})P^{\rm tr}/2$ for each $A \in M_n^+$. For simplicity, we may assume that $P = I_n$. If $A \in S_n^+$, then $W(f(A)) = W(A) \subseteq \mathbb{R}$ and hence $f(A) = f(A)^{\rm tr} \in S_n^+$. It follows that

$$f(A) = A, \quad \forall \ \ A \in S_n^+. \tag{4.8}$$

We divide the rest of the proof into two steps.

**Step 1.** One of the following holds:

(a) $f(\mu E_{ij}) = \mu E_{ij}$ for all $i \neq j$ and $\mu > 0$, or
(b) $f(\mu E_{ij}) = \mu E_{ji}$ for all $i \neq j$ and $\mu > 0$.

*Proof of Step 1.* Let $f(E_{12}) = X + Y$ with $(X, Y) \in S_n^+ \times K_n$. Then $X = (E_{12} + E_{21})/2$ and $X + Y \in M_n^+$, only the $(1, 2)$ and $(2, 1)$ entries of $Y$ can be nonzero. Since $W(E_{12}) = W(X + Y)$, by Lemma 4.5(b) we see that $X + Y$ is nilpotent. Thus, $Y = (E_{12} - E_{21})/2$ or $(E_{21} - E_{12})/2$. Hence $f(E_{12}) = E_{12}$ or $E_{21}$. We may assume that the former case holds. Otherwise, replace $f$ by the map $X \mapsto f(X)^{\rm tr}$.

Now, we will show that (a) holds. First, we can use the argument in the preceding paragraph to show that for $\mu > 0$, either $f(\mu E_{21}) = \mu E_{21}$ or $f(\mu E_{21}) = \mu E_{12}$ holds. Since

$$W(\mu E_{21} + E_{12}) = W(f(\mu E_{21}) + f(E_{12})) = W(f(\mu E_{21}) + E_{12}),$$

Lemma 4.5(b) yields $f(\mu E_{21}) = \mu E_{21}$. Now, change the roles of $E_{12}$ and $E_{21}$ in the above argument. We see that $f(\mu E_{12}) = \mu E_{12}$ for any $\mu > 0$. We are done if $n = 2$.

Suppose $n \geq 3$. We can show (as in the preceding paragraph) that for $\mu > 0$ and $j > 2$, either $f(\mu E_{1j}) = \mu E_{1j}$ or $f(\mu E_{1j}) = \mu E_{j1}$. For simplicity, assume that $j = 3$. Suppose $f(\mu E_{13}) = \mu E_{31}$. Let $A = \mu(E_{23} + E_{32})$ and $B = \mu(E_{12} + E_{13})$. Then

$$f(A) = A, \qquad f(B) + f(B)^{\mathrm{tr}} = B + B^{\mathrm{tr}}.$$

Since

$$B + B^{\mathrm{tr}} + f(B) - f(B)^{\mathrm{tr}} = f(B) + f(B)^{\mathrm{tr}} + f(B) - f(B)^{\mathrm{tr}} = 2f(B) \in M_n^+,$$

the skew-symmetric matrix $f(B) - f(B)^{\mathrm{tr}}$ can have nonzero entries only in $(1,2)$, $(1,3)$, $(2,1)$, $(3,1)$ positions, and the absolute value of these entries cannot exceed $\mu$. On the other hand, $W(f(B)) = W(B)$, which is known to be the circular disk centered at zero with radius $\mu/\sqrt{2}$ (see [16] or [11, Theorem 4.1], for example), and therefore $\pm \mathrm{i}\mu\sqrt{2}$ are eigenvalues of the matrix $f(B) - f(B)^{\mathrm{tr}}$. It follows that the $(1,2)$, $(2,1)$, $(1,3)$, $(3,1)$ entries of $f(B) - f(B)^{\mathrm{tr}}$ have absolute values equal to $\mu$, and $f(B)$ must be one of the following four matrices:

$$\mu(E_{12} + E_{13}),\ \mu(E_{21} + E_{31}),\ \mu(E_{12} + E_{31}),\ \mu(E_{21} + E_{13}).$$

Suppose the third or the fourth case holds. Then for $X = A + B$ and $Y = f(A) + f(B)$, the largest eigenvalues of $e^{\mathrm{i}\pi/3}X + e^{-\mathrm{i}\pi/3}X^{\mathrm{tr}}$ and $e^{\mathrm{i}\pi/3}Y + e^{-\mathrm{i}\pi/3}Y^{\mathrm{tr}}$ are $1.6861\mu$ and $1.6007\mu$, respectively, by a Matlab computation. Thus, $W(X) \neq W(Y)$, which is a contradiction with (4.7). Now, if $f(B) = \mu(E_{21} + E_{31})$, then for $X = \mu E_{12} + B$ and $Y = \mu E_{12} + f(B)$, the largest eigenvalues of $\mathrm{i}(X - X^{\mathrm{tr}})$ and $\mathrm{i}(Y - Y^{\mathrm{tr}})$ are $\mu$ and $\sqrt{5}\mu$, respectively. Thus, $W(\mu E_{12} + B) \neq W(\mu E_{12} + f(B))$, a contradiction again. So, we must have $f(B) = \mu(E_{12} + E_{13}) = B$. Now, consider $X = \mu E_{13} + B$ and $Y = f(\mu E_{13}) + B = \mu E_{31} + B$. But then $W(X) \neq W(Y)$ (indeed, $W(X)$ is a circular disk but $W(Y)$ is not because $Y$ has 3 distinct eigenvalues [11, Corollary 2.5]), a contradiction. So $f(\mu E_{13}) = \mu E_{31}$ is impossible, and we see that $f(\mu E_{13}) = \mu E_{13}$ holds. Analogously we show that $f(\mu E_{1j}) = \mu E_{1j}$ and $f(\mu E_{j1}) = \mu E_{j1}$ for all $j > 2$.

Now, consider $i, j \geq 2$ and $i \neq j$. Repeat the arguments of the preceding paragraph with $E_{12}$, $E_{21}$, $E_{13}$, $E_{31}$ replaced by $E_{i1}$, $E_{1i}$, $E_{ij}$, $E_{ji}$, respectively, thereby proving the equalities $f(\mu E_{ij}) = \mu E_{ij}$, $\mu > 0$. □

**Step 2.** Assume that condition (a) of Step 1 holds. Then $f(A) = A$ for all $A \in M_n^+$.

*Proof of Step 2.* Suppose $f(A) = [z_{ij}]_{i,j=1}^n$. Since $f(A) + f(A)^{\mathrm{tr}} = A + A^{\mathrm{tr}}$, we see that $z_{jj} = a_{jj}$ for all $j = 1, \dots, n$. Suppose

$$A - A^{\mathrm{tr}} = [x_{ij}]_{i,j=1}^n \quad \text{and} \quad f(A) - f(A)^{\mathrm{tr}} = [y_{ij}]_{i,j=1}^n.$$

Suppose there is $x_{ij} > y_{ij} \geq 0$ for some $i \neq j$. (We may interchange the roles of $A$ and $f(A)$ in the following if $0 \leq x_{ij} < y_{ij}$.) Say,

$$x_{12} > y_{12} \geq 0. \tag{4.9}$$

Then for sufficiently large $\mu > 0$, one can use a similar argument in the proof of Assertion 2.11 to show that

$$r\left((A + \mu E_{12}) - (A + \mu E_{12})^{\mathrm{tr}}\right) \neq r\left(f(A) + \mu E_{12}) - (f(A) + \mu E_{12})^{\mathrm{tr}}\right). \tag{4.10}$$

For the reader's benefit, we provide details.

By Step 1 and (4.8) we know that $f(\mu E_{ij}) = \mu E_{ij}$ for all pairs $(i, j)$ and all $\mu > 0$. If $n = 2$, inequality (4.10) is immediate. So assume $n \geq 3$. Partition:

$$A - A^{\mathrm{tr}} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad f(A) - f(A)^{\mathrm{tr}} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},$$

with

$$A_{11} = -A_{11}^{\mathrm{tr}} \in M_2, \quad Z_{11} = -Z_{11}^{\mathrm{tr}} \in M_2, \quad A_{21} = -A_{12}^{\mathrm{tr}}, \tag{4.11}$$

$$Z_{21} = -Z_{12}^{\mathrm{tr}}, \quad A_{22} = -A_{22}^{\mathrm{tr}} \in M_{n-2}, \quad Z_{22} = -Z_{22}^{\mathrm{tr}} \in M_{n-2}. \tag{4.12}$$

For sufficiently large $t \in \mathbb{R}$ and for $\mu > 0$, consider

$$B_t := A_{12}(\mathrm{i}tI - A_{22})^{-1}A_{21} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad \text{and} \quad C_{t,\mu} := A_{11} + (\mu E_{12} - \mu E_{21}) + B_t.$$

Note that $B_t$ and $C_{t,\mu}$ are complex skew-Hermitian matrices. Then $C_{t,\mu}$ has eigenvalues (note that $x_{11} = x_{22} = 0$)

$$\left[b_{11} + b_{22} \pm \sqrt{(b_{11} - b_{22})^2 + 4(x_{12} + b_{12} + \mu)(x_{21} + b_{21} - \mu)}\right]/2. \tag{4.13}$$

(Here and in the rest of the proof, for a negative number $w$, we denote $\sqrt{w} = \mathrm{i}|\sqrt{-w}|$.) Also, since Trace $(B_t)$ is obviously an analytic function of $t$ in a neighborhood of infinity, we have

$$\mathrm{Trace}\,(B_t) = \mathrm{i}q(t),$$

where $q(t) \in \mathbb{R}$ has a fixed sign for all sufficiently large values of $t$; say $q(t) \geq 0$. We note also the formula

$$B_t = \sum_{k=0}^{\infty} (\mathrm{i}t)^{-1} A_{12}((\mathrm{i}t)^{-1} A_{22})^k A_{21}. \tag{4.14}$$

Formula (4.13) shows that there is (unique) $\nu_t > 0$ such that the eigenvalue of $C_{t,\nu_t}$ with the larger absolute value equals $\mathrm{i}t$; here we use the inequality $q(t) \geq 0$. Moreover, for any $|\lambda| > t$, $\lambda \in \mathbb{C}$, we have

$$\det(\lambda I - (A - A^{\rm tr}) - \nu_t(E_{12} - E_{21})) \neq 0 = \det(\mathrm{i}tI - (A - A^{\rm tr}) - \nu_t(E_{12} - E_{21}))$$

(this follows from a Schur complement equality analogous to (2.15)). Hence,

$$t = r((A - A^{\rm tr}) + \nu_t(E_{12} - E_{21}))$$

and

$$\begin{aligned} & -\mathrm{i}2r(A - A^{\rm tr} + \nu_t(E_{12} - E_{21})) \\ = \; & b_{11} + b_{22} + \sqrt{(b_{11} - b_{22})^2 + 4(x_{12} + b_{12} + \nu_t)(x_{21} + b_{21} - \nu_t)}. \end{aligned}$$

Similarly, if

$$\widetilde{B}_t := Z_{12}(\mathrm{i}tI_{n-2} - Z_{22})^{-1}Z_{21} = [\widetilde{b}_{ij}]_{i,j=1}^2 \in M_2 \text{ and } \widetilde{C}_t := Z_{11} + \nu_t(E_{12} - E_{21}) + \widetilde{B}_t,$$

then $\widetilde{C}_t$ has eigenvalues

$$\left[(\widetilde{b}_{11} + \widetilde{b}_{22}) \pm \sqrt{(\widetilde{b}_{11} - \widetilde{b}_{22})^2 + 4(y_{12} + \widetilde{b}_{12} + \nu_t)(y_{21} + \widetilde{b}_{21} - \nu_t)}\right] /2.$$

So, for sufficiently large $t$, and hence for sufficiently large $\nu_t$, we have

$$\begin{aligned} & -\mathrm{i}2r(f(A) - f(A)^{\rm tr} + \nu_t(E_{12} - E_{21})) \\ = \; & \widetilde{b}_{11} + \widetilde{b}_{22} \pm \sqrt{(\widetilde{b}_{11} - \widetilde{b}_{22})^2 + 4(y_{12} + \widetilde{b}_{12} + \nu_t)(y_{21} + \widetilde{b}_{21} - \nu_t)}. \end{aligned} \tag{4.15}$$

Since $x_{12} > y_{12}$, $x_{21} = -x_{12}$, $y_{21} = -y_{12}$, and since the absolute values of $b_{ij}$ and $\widetilde{b}_{ij}$, $i, j = 1, 2$, are small in view of (4.14) and an analogous formula for $\widetilde{B}_t$, we see that the right hand sides of (4.15) and (4.15) are not equal for sufficiently large $\nu_t$. This proves (4.10).

Now, in view of (4.10), we have

$$W(A + \mu E_{12}) \neq W(f(A) + \mu E_{12}) = W(f(A) + f(\mu E_{12})),$$

where the equality follows from Step 1. This is a contradiction with (4.7). So, $A - A^{\rm tr} = f(A) - f(A)^{\rm tr}$, and we conclude that $f(A) = A$. $\square$

## 5 Spectral norm preservers

In this section, we consider spectral norm preservers on nonnegative matrices. In contrast with other sections in the paper, here it is natural to prove the result in the framework of the set $M_{m,n}^+$ of $m \times n$ entrywise nonnegative matrices.

**Theorem 5.1** *Let* $f : M_{m,n}^+ \longrightarrow M_{m,n}^+$. *Then*

$$\|A + B\| = \|f(A) + f(B)\|, \qquad \forall \;\; A, B \in M_{m,n}^+ \tag{5.1}$$

*if and only if there exist permutation matrices* $P \in M_m^+$ *and* $Q \in M_n^+$ *such that one of the following holds.*

(a) $f(A) = PAQ$ *for all* $A \in M_{m,n}^+$.

(b) $m = n$ *and* $f(A) = PA^{\rm tr}Q$ *for all* $A \in M_{m,n}^+$.

*Proof.* We focus on the non-trivial "only if" part. Thus, assume (5.1) holds. We may assume that $m \le n$ and $n \ge 2$ (The case $m > n$ can treated similarly, and the case $m = n = 1$ is trivial.) The following easy observation will be used repeatedly:

**Observation 5.2** (a) *For* $\mu > 0$, *we have*

$$1 + \mu = \|E_{ij} + \mu E_{pq}\|$$

*if and only if* $(i, j) = (p, q)$.

(b) *The equality* $\sqrt{2} = \|E_{ij} + E_{pq}\|$ *holds if and only if either* $i = p$, $j \neq q$, *or* $i \neq p$, $j = q$.

We divide the proof into several steps.

**Step 1** For every $\mu > 0$, there exist permutation matrices $P \in M_m^+$ and $Q \in M_n^+$ (which a priori may depend on $\mu$) such that

(a) $Pf(\mu E_{ii})Q = \mu E_{ii}$ for $i = 1, \ldots, m$, and

(b) for $j = 1, \ldots, n - m$, the equalities $Pf(\mu E_{1,m+j})Q = \mu E_{1,m+j}$ hold, in case $n > m$.

*Proof of Step 1.* Fix $\mu > 0$, and let $f(\mu E_{ii}) = F_{ii}$ for $1 \le i \le m$. The condition (5.1) implies that $\|f(A)\| = \|A\|$ for every $A \in M_{m,n}^+$; in particular,

$$\|F_{ii}\| = \mu. \tag{5.2}$$

Since $F_{ii}$ is entrywise nonnegative, there exist entrywise nonnegative vectors of unit length $x_i \in \mathbb{R}^m$ such that

$$\|x_i^{\rm tr} F_{ii}\| = \mu, \qquad i = 1, ..., m.$$

For any $i \neq j$, since

$$\begin{aligned}\mu^2 &= \|\mu E_{ii} + \mu E_{jj}\|^2 = \|F_{ii} + F_{jj}\|^2 \\ &\geq x_i^{\rm tr}\,(F_{ii} + F_{jj})(F_{ii} + F_{jj})^{\rm tr}\,x_i \\ &\geq x^{\rm tr}\,F_{ii}F_{ii}^{\rm tr}\,x_i = \mu^2,\end{aligned}$$

we see that $x_i^{\rm tr}\,(F_{jj}F_{jj}^{\rm tr}\,)x_i = 0$. So, $x_i$ is an eigenvector of $F_{jj}F_{jj}^{\rm tr}$ corresponding to the (smallest) eigenvalue 0. Recall that $x_j$ is the eigenvector of $F_{jj}F_{jj}^{\rm tr}$ corresponding to the (largest) eigenvalue $\mu^2$. So, $x_i$ and $x_j$ are orthogonal. As a result, $\{x_1, \ldots, x_m\}$ is an orthonormal basis of $\mathbb{R}^m$. Since $x_1, \ldots, x_m$ are nonnegative, we can conclude that $x_1, \ldots, x_m$ is a permutation of $e_1, ...., e_m$. We may replace $f$ by a map of the form $A \mapsto P(f(A))$ for a suitable permutation matrix $P$ and assume that $x_j = e_j$. Then $e_i^{\rm tr}\,F_{jj}F_{jj}^{\rm tr}\,e_i = 0$ $(i \neq j)$. It follows that the $(i, i)$ entry of $F_{jj}F_{jj}^{\rm tr}$ is zero for all $i \neq j$. Since $F_{jj}F_{jj}^{\rm tr}$ is positive semidefinite of norm $\mu^2$, we see that $F_{jj}F_{jj}^{\rm tr} = \mu^2 E_{jj}$. As a result, $F_{jj} = \mu e_j v_j^{\rm tr}$ for some nonnegative vector $v_j \in \mathbb{R}^n$ of unit length, for $j = 1, 2, \ldots, m$. Moreover, the equation $\|F_{ii} + F_{jj}\| = \mu$ for $i \neq j$ implies that $v_i^{\rm tr}\,v_j = 0$ for $i \neq j$, i.e., the vectors $v_1, \ldots, v_m$ have positive entries at different positions.

If $m = n$, then the vectors $v_1, \ldots, v_m$ are a permutation of $e_1, \ldots, e_m$, and the proof of Step 1 is complete. Suppose $n > m$. Consider $F(\mu E_{1j}) = F_{1j}$ for $j > m$. Applying the preceding argument to $F_{1j}, F_{22}, \ldots, F_{mm}$, we see that $F_{1j} = \mu e_1 w_j^{\rm tr}$ for some nonnegative unit length vector $w_j \in \mathbb{R}^n$ such that $w_j$ and $v_k$ has positive entries at different positions for any $k = 2, \ldots, m$. Note that (for $j > m$)

$$\begin{aligned}2\mu^2 &= \|\mu E_{11} + \mu E_{1j}\|^2 = \|F_{11} + F_{1j}\|^2 = \|e_1(\mu v_1^{\rm tr} + \mu w_j^{\rm tr}\,)\|^2 \\ &= \|\mu v_1^{\rm tr} + \mu w_j^{\rm tr}\,\|^2 = (\mu v_1 + \mu w_j)^{\rm tr}\,(\mu v_1 + \mu w_j) = 2\mu^2 + 2w_j^{\rm tr}\,v_1,\end{aligned}$$

hence $w_j$ and $v_1$ also have positive entries at different positions. Applying the same reasoning to $\mu E_{1,j_1}$ and $\mu E_{1,j_2}$, $(j_1, j_2 > m)$, we see that also $w_{j_1}$ and $w_{j_2}$ have positive entries at different positions. Now it follows that each of the vectors

$$v_1, \ldots, v_m, w_1, \ldots, w_{n-m} \tag{5.3}$$

has exactly one positive entry and the positions of these positive entries are different for different vectors in the set (5.3). Thus, the set (5.3) is a permutation of $e_1, \ldots, e_n$, and the results of Step 1 follows. □

**Step 2** There exist permutation matrices $P \in M_m^+$ and $Q \in M_n^+$ such that

(a) $Pf(\mu E_{ii})Q = \mu E_{ii}$ for $i = 1, \ldots, m$ and all $\mu > 0$, and

(b) for $j = 1, \dots, n-m$, the equalities $Pf(\mu E_{1,m+j})Q = \mu E_{1,m+j}$ hold for all $\mu > 0$, in case $n > m$.

*Proof of Step 2.* By Step 1, there exist permutations $P(\mu)$ and $Q(\mu)$ such that

$$P(\mu)f(\mu E_{ii})Q(\mu) = \mu E_{ii}, \quad i = 1, \dots, m,$$

and

$$P(\mu)f(\mu E_{1,m+j})Q(\mu) = \mu E_{1,m+j}, \quad j = 1, \dots, n-m$$

(if $m < n$).

We may assume that $P(1) = I_m$ and $Q(1) = I_n$. Otherwise, replace $f$ by the map of the form $X \mapsto P(1)^{-1}f(X)Q(1)^{-1}$. Hence, if

$$\mathcal{S} = \{E_{jj} : 1 \le j \le n\} \cup \{E_{1j} : m < j \le n\},$$

then $f(X) = X$ for any $X \in \mathcal{S}$. Moreover, for any $\mu > 0$ and $X \in \mathcal{S}$, $f(\mu X) = P(\mu)\mu XQ(\mu) = \mu E_{pq}$ for some $(p,q)$ pair. Since

$$1 + \mu = \|X + \mu X\| = \|f(X) + f(\mu X)\| = \|X + f(\mu X)\|,$$

we see (using Observation 5.2) that $f(\mu X) = \mu X$.

□

**Step 3.** Assume that $P = I$ and $Q = I$ in Step 2. Then one of the two following possibilities holds:

(a) $f(\mu E_{ij}) = \mu E_{ij}$ for all $\mu > 0$ and $(i,j)$ pairs,

(b) $m = n$ and $f(\mu E_{ij}) = \mu E_{ji}$ for all $\mu > 0$ and $(i,j)$ pairs.

*Proof of Step 3.* We may suppose $i \ne j$ (the cases when $i = j$ are taken care of in Step 2). Here $1 \le i \le m; 1 \le j \le n$.

First, we prove Step 3 for the case $m = 1$. By Step 2, we have $f(\mu E_{11}) = \mu E_{11}$ for all $\mu > 0$. If $f([x_1, x_2, \dots, x_n]) = [z_1, \dots, z_n]$, then

$$(\mu+x_1)^2+\sum_{j=2}^{n} x_i^2 = \|\mu E_{11}+[x_1, x_2, \dots, x_n]\|^2 = \|\mu E_{11}+[z_1, \dots, z_n]\|^2 = (\mu+z_1)^2+\sum_{j=2}^{n} z_i^2$$

for all $\mu > 0$ which implies $x_1 = z_1$. In particular, $f$ maps the set $\{[a_1, \dots, a_n] \in M_{1,n}^+ : a_1 = 0\}$ to itself, and using the induction on $n$, we obtain the equalities $f(\mu E_{1j}) = \mu E_{1j}$ for all $\mu > 0$ and $j = 2, 3, \dots, n$. From now on in the proof of Step 3 we assume $m \ge 2$.

Next, for any pair $(i,j)$, $1 \le i \le m$, $1 \le j \le n$, we can find permutation matrices $R$ (of size $m \times m$) and $S$ (of size $n \times n$) such that $E_{ij} = RE_{11}S$. Then, applying the result of Step 2 to the map $\widehat{f}(X) = f(RXS)$, $X \in M_{m,n}^+$, we see that $f(\mu E_{ij}) = \mu E_{pq}$ for some index pair $(p, q)$ which is independent of $\mu$. It remains to show that

(a) $(p, q)$ always equals $(i, j)$, or
(b) $m = n$ and $(p, q)$ always equals $(j, i)$.

To this end, consider $f(E_{ij})$ with $i \neq j$. By Step 1,

$$\|f(E_{ij}) + E_{kk}\| = \|f(E_{ij}) + f(E_{kk})\| = \|E_{ij} + E_{kk}\| = \sqrt{2}$$

for $k \in \{i, j\}$, $j \leq m$. By Observation 5.2 (b), we see that

$$f(E_{ij}) = E_{ij} \qquad \text{or} \qquad f(E_{ij}) = E_{ji}. \tag{5.4}$$

Consider $f(E_{12})$. If $n > m$, then

$$\sqrt{2} = \|E_{12} + E_{1,m+1}\| = \|f(E_{12}) + E_{1,m+1}\|.$$

Thus, $f(E_{12}) = E_{12}$. Suppose $m = n$ and $f(E_{12}) = E_{21}$. We may replace $f$ by the map $A \mapsto f(A)^{\mathrm{tr}}$ and assume that $f(E_{12}) = E_{12}$.

Assuming that $f(E_{12}) = E_{12}$, we can easily shows that $f(E_{1j}) = E_{1j}$ for all $j = 3, \ldots, m$, because

$$\sqrt{2} = \|E_{12} + E_{1j}\| = \|f(E_{12}) + f(E_{1j})\| = \|E_{12} + f(E_{1j})\|,$$

where (5.4) was used. Note that $f(E_{1j}) = E_{1j}$ for $j > m$ by Step 1. Recall that for $i = 3, \ldots, m$, we have $f(E_{i1}) = E_{i1}$ or $f(E_{i1}) = E_{1i}$. Since

$$1 = \|E_{1i} + E_{i1}\| = \|f(E_{1i}) + f(E_{i1})\| = \|E_{1i} + f(E_{i1})\|,$$

we see that $f(E_{i1}) = E_{i1}$ for $i = 2, \ldots, m$.

Now, for any $E_{ij}$ for $2 \leq i, j \leq m$, since

$$\sqrt{2} = \|E_{1j} + E_{ij}\| = \|f(E_{1j}) + f(E_{ij})\| = \|E_{1j} + f(E_{ij})\|,$$

and using Observation 5.2, we see that $f(E_{ij}) = E_{ij}$. For $E_{ij}$ with $i \geq 2$ and $j > m$, we have

$$\sqrt{2} = \|E_{1j} + E_{ij}\| = \|f(E_{1j}) + f(E_{ij})\| = \|E_{1j} + E_{pq}\|,$$

where the pair $(p, q)$ is such that $f(E_{ij}) = E_{pq}$, and

$$\sqrt{2} = \|E_{ir} + E_{ij}\| = \|E_{ir} + E_{pq}\|, \quad r = 1, 2, \ldots m,$$

so by Observation 5.2 we must have $(p,q) = (i,j)$. So, for any $(r,s)$ pair, we have $f(E_{rs}) = E_{rs}$. □

**Step 4.** Assume that (a) in Step 3 holds, and assume also $P = I$, $Q = I$. Then $f(A) = A$ for all $A \in M_{m,n}^+$.

*Proof of Step 4.* Let $A = [a_{ij}]$ and $f(A) = [z_{ij}]$ in $M_{m,n}^+$. We show that $a_{ij} = z_{ij}$ for each $(i,j)$ pair. Arguing by contradiction, assume $a_{ij} \neq z_{ij}$ for some pair $(i,j)$. Say, $(i,j) = (1,1)$ (for other pairs $(i,j)$ the proof is exactly the same). Suppose $a_{11} > z_{11}$ (if the opposite inequality holds, interchange the roles of $A$ and $f(A)$ in the subsequent argument). Then for $\mu > 0$,

$$(A + \mu E_{11})^*(A + \mu E_{11}) = \mu^2 E_{11} + \mu(A^* E_{11} + E_{11} A) + A^* A.$$

Note that the largest eigenvalue of $\widetilde{A} = \mu^2 E_{11} + \mu(A^* E_{11} + E_{11} A)$ equals

$$[\mu(\mu + 2a_{11}) + \sqrt{\mu^2(\mu + 2a_{11})^2 + \mu^2\alpha}]/2 \qquad \text{with } \alpha = 4\sum_{j=2}^{n} a_{1j}^2.$$

Similarly, we have

$$(f(A) + \mu E_{11})^*(f(A) + \mu E_{11}) = \mu^2 E_{11} + \mu(f(A)^* E_{11} + E_{11} f(A)) + f(A)^* f(A)$$

and the largest eigenvalue of $\widetilde{Z} = \mu^2 E_{11} + \mu(f(A)^* E_{11} + E_{11} f(A))$ equals

$$[\mu(\mu + 2z_{11}) + \sqrt{\mu^2(\mu + 2z_{11})^2 + \mu^2\beta}]/2 \qquad \text{with } \beta = 4\sum_{j=2}^{n} z_{1j}^2.$$

Denote by $\lambda_1(X)$ the largest eigenvalue of $X \in S_n^+$. Since $a_{11} > z_{11}$, there is a sufficiently large $\mu > 0$ such that

$$\mu(\mu + 2a_{11}) > \mu(\mu + 2z_{11}) + 2\lambda_1(f(A)^* f(A))$$

and

$$(\mu + 2a_{11})^2 + \alpha \geq (\mu + 2z_{11})^2 + \beta.$$

Consequently,

$$\lambda_1((\mu E_{11} + A)^*(\mu E_{11} + A)) \geq \lambda_1(\widetilde{A})$$

$$> \lambda_1(\widetilde{Z}) + \lambda_1(f(A)^* f(A)) \geq \lambda_1((\mu E_{11} + f(A))^*(\mu E_{11} + f(A))).$$

It follows that $\|A + \mu E_{11}\| > \|f(A) + \mu E_{11}\|$, which is the desired contradiction, because by Step 3 we have $f(\mu E_{11}) = \mu E_{11}$. □

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