Determinants of Certain Classes of Zero-One Matrices with Equal Line Sums

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Abstract

We study the possible determinant values of various classes of $n \times n$ zero-one matrices with fixed row and column sums. Some new results, open problems, and conjectures are presented.

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1 Introduction

Let k, n be positive integers with $k \leq n$. Denote by $S(n,k)$ the set of zero-one $n \times n$ matrices with row sums and column sums equal to k.

There has been considerable interest in studying the determinant values of matrices in $S(n,k)$ and various its subsets. This interest is motivated, among other things, by many interesting connections with graph theory and combinatorics (designs and configurations). So far the research in this area focused on the minimal positive value of determinants of matrices in $S(n,k)$ (see, e.g., [13, 7, 8, 11]) and on the maximal value of determinants for

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matrices in certain subsets of $S(n,k)$ and for certain values of n and k (see, e.g., [15, 4, 6]) and see also the books [16, 2]. The main focus of the present paper is to describe in some cases the complete set of determinantal values of matrices in $S(n,k)$. We also consider the subset of symmetric matrices in $S(n,k)$ and the subset of $S(n,k)$ which is generated by powers of the standard circulant. Both subsets are of considerable interest in combinatorics.

Note that if $A \in S(n,k)$ with $\det(A) = t$, then one can interchange the first two rows of A to obtain a matrix in $S(n,k)$ with determinant $-t$. Thus we can focus on the set

$$D(n,k) = \{|\det(A)| : \; A \in S(n,k)\}.$$

The problem of determining the set $D(n,k)$ remains generally open. In particular, there is no general information about the quantity

$$M(n,k) = \max\{|\det(A)| : \; A \in S(n,k)\}.$$

We consider here also two subsets in $S(n,k)$: the set of symmetric zero-one matrices with constant row and column sums:

$$\mathrm{Sym}(n,k) = \{A \in S(n,k) : \; A = A^T\},$$

and the set of polynomials with zero-one coefficients of the standard circulant $n \times n$ matrix $P_n = E_{12} + \cdots + E_{n-1,n} + E_{n1}$, where E_{ij} are the standard matrix units:

$$\mathrm{Cir}(n,k) = \left\{ \sum_{q=1}^{k} P_n^{i_q} : \; 0 \leq i_1 < i_2 < \cdots < i_k < n \right\}.$$

The possible values of determinants of matrices in $\mathrm{Sym}(n,k)$ and $\mathrm{Cir}(n,k)$ are of particular interest. Thus, we introduce the following notions analogously to those introduced for the set $S(n,k)$:

$$D_{\mathrm{Sym}}(n,k) = \{|\det A| : \; A \in \mathrm{Sym}(n,k)\},$$

and

$$D_{\mathrm{Cir}}(n,k) = \{|\det A| : \; A \in \mathrm{Cir}(n,k)\}.$$

We emphasize that the problem under consideration and the several related subjects are well known to be difficult, and researchers have invested a lot of effort to them in the last few decades. This purpose of this paper is to add some more results as well as useful techniques to the study of these problems. In particular, we shall present results, open problems and conjectures concerning the sets $D(n,k)$, $D_{\mathrm{sym}}(n,k)$, $D_{\mathrm{Cir}}(n,k)$, and the maximum values in these sets, and explore connections between this topic and other areas such as designs and graph theory.

Throughout the paper we denote by P_n the standard circulant $n \times n$ matrix, and by F_n the symmetric $n \times n$ matrix defined by $F_n = E_{1n} + E_{2,n-1} + \cdots + E_{n1}$.

2 Upper Bounds for $M(n,k)$

In this section we present some known information concerning the quantity $M(n,k)$.

Denote by J_n, or simply J, the unique matrix in $S(n,n)$. It is clear that $\det(J_n) = 0$, if $n \geq 2$. It is also easy to see that $D(n,1) = \{1\}$ and $D(n,n-1) = \{n-1\}$. One may therefore focus on those k satisfying $1 < k < n-1$. We have the following general results (e.g., see [13]):

(2.1) Let $A \in S(n,k)$. Then $\tilde{A} = J - A \in S(n, n-k)$ and $k \cdot \det(\tilde{A}) = (n-k)\det(A)$.

(2.2) If $A \in S(n,k)$, then $\det(A)$ is a multiple of $k \cdot \gcd(n,k)$.

(2.3) Let n, k be integers such that $n \geq k \geq 1$. Then $S(n,k)$ always contains a non-singular matrix, except when $n = k > 1$, and when $n = 4,\ k = 2$.

It is easy to verify that $D(4,2) = \{0\}$. Newman [13] conjectured that:

(2.4) If $1 \leq k \leq n-1$ and $(n,k) \neq (4,2)$, then

$$m(n,k) := \min\{|\det(A)| > 0 : \ A \in S(n,k)\} = k \cdot \gcd(n,k).$$

This conjecture was confirmed in [11]. The number $M(n,k)$ is unknown in general. However, several upper bounds exist in the literature:

Lemma 2.1 (i) *If n is divisible by 4, then $M(n-1,k) \leq n^{n/2}/2^{n-1}$.*

(ii) *If n is odd, then $M(n-1,k) \leq (2n-1)^{1/2}(n-1)^{(n-1)/2}/2^{n-1}$.*

(iii) *If $n \equiv 2 \pmod 4$, then $M(n-1,k) \leq (2n-2)(n-2)^{(n/2)-1}/2^{n-1}$.*

Lemma 2.1 is presented in [12] (the part (ii) is attributed there to [1]). For small values of n and k, the following table provides the upper bounds given by Lemma 2.1:

n	4	5	6	7	8	9	10	11	12	13	14
upper bound on $M(n,k)$	3	5	12	32	65	144	447	1458	3645	9477	34648

Notice that the bounds in Lemma 2.1 do not make use of the value k. For $k \geq n/2$, one may use (2.2) to improve the bounds. Then one can use (2.1) to get bounds for $M(n, n-k)$. Here are some examples. (Again, we focus on $1 < k < n-1$.)

	n=4	5	6	7	8	9	10
k=2	0	2	4	12	20	40	108
3		3	9	24	39	72	189
4			8	32	64	112	296
5				30	65	140	425
6					60	144	444
7						140	441
8							432

Table 1. Upper bounds for $M(n,k)$

Ryser [15] obtained a bound for the determinant of a zero-one matrix in terms of the size and the number of one's in the matrix. The result is certainly applicable to our study. We give a short proof of the result for our special case in the following.

Lemma 2.2 *If $A \in S(n,k)$, then*

$$|\det(A)| \leq |(xn+k)^{-1}k|[k((1+x)^2+(n-k)x^2]^{\frac{n}{2}} \tag{2.5}$$

for any $x \in \mathbf{R}$. Consequently, we have

$$M(n,k) \leq k(k-\lambda)^{(n-1)/2} \quad \textit{with} \quad \lambda = \frac{k(k-1)}{n-1}.$$

Proof. Suppose $A \in S(n,k)$. The Hadamard Bound for determinants shows that

$$|\det(xJ+A)| \leq [k(1+x)^2+(n-k)x^2]^{\frac{n}{2}} \text{ for every } x \in \mathbf{R}.$$

By [13, Lemma1], one can write $|\det(A)|$ in terms of $|\det(xJ+A)|$:

$$|\det(A)| = k|(xn+k)|^{-1}|\det(xJ+A)|,$$

and (2.5) follows. Let $f(x)$ be the right-hand side of (2.5). It is easy to see that $f(x)$ has its minimum value at

$$x^* = \frac{-k(n-1) \pm \sqrt{k(n-1)(n-k)}}{n(n-1)}.$$

Substituting x^* in (2.5) and simplifying the expression, we get the last assertion. □

Lemma 2.2 together with (2.2) give better upper bounds for $M(n,k)$ when k or $n-k$ is small. For example, we have the following improvement of Table 1 (improved values are underlined):

	n=4	5	6	7	8	9	10
k=2	0	2	4	$\underline{8}$	$\underline{12}$	$\underline{18}$	$\underline{24}$
3		3	9	24	39	72	$\underline{135}$
4			8	32	64	112	296
5				$\underline{20}$	65	140	425
6					$\underline{36}$	144	444
7						$\underline{63}$	$\underline{315}$
8							$\underline{96}$

Table 2. Improved upper bounds for $M(n,k)$

For certain values of n, k, one can use the theory of *symmetric* (n,k,λ) *designs* (also known as (n,k,λ)-*configurations*) to get the exact value of $M(n,k)$. We refer the readers to [2] and [17] for the basic definitions and results on this subject. In connection to our problem, every (n,k,λ) symmetric design can be represented by a matrix $A \in S(n,k)$, the *incidence matrix* of the symmetric design, such that $AA^T = A^T A = B$, where the matrix B has k on the main diagonal and λ in all the other positions, i.e.,

$$B = \lambda J + (k - \lambda)I.$$

The eigenvalues of B are $(k-\lambda) + n\lambda, \overbrace{k-\lambda, \ldots, k-\lambda}^{n-1}$. Therefore,

$$\det(B) = (k - \lambda + n\lambda)(k - \lambda)^{n-1}.$$

Because of the equality $k - \lambda = k^2 - \lambda n$ (see, e.g., [15]), we have $\det(B) = k^2(k-\lambda)^{n-1}$, and hence

$$\det(A) = k(k - \lambda)^{(1/2)(n-1)}.$$

The following result was proved in [15]:

Lemma 2.3 *If an (n, k, λ) symmetric design exists, then*

$$M(n, k) = k(k - \lambda)^{(n-1)/2}.$$

It is known (e.g., see [2]) that if an (n, k, λ) symmetric design exists, then $(n-1)\lambda = k(k-1)$. However, the converse does not hold in general. The existence problem for symmetric (n, k, λ) designs is open in general. In the following, we list all the $M(n, k)$ for $n \leq 20$ determined by symmetric (n, k, λ) designs.

$$\begin{array}{ll}
M(7,3) = 24; & M(7,4) = 32; \\
M(11,5) = 1215; & M(11,6) = 1458; \\
M(13,4) = 2916; & M(13,9) = 6561; \\
M(15,7) = 114688; & M(15,8) = 131072; \\
M(16,6) = 196608; & M(16,10) = 327680; \\
M(19,9) = 17578125; & M(19,10) = 19531250.
\end{array}$$

By the above discussion, one sees that determining $M(n, k)$ and $D(n, k)$ is indeed a difficult problem. Some partial results and techniques are presented in the following sections.

3 Results for $D(n, 2)$

We start with the following theorem:

Theorem 3.1 *Suppose $n = 3k + t > 3$ with $k \geq 1$ and $t = 0, 2,$ or 4. Then*

$$D(n, 2) = \{0\} \cup \left\{2^{k-2i} : \ 0 \leq i < k/2\right\},$$

and

$$D(n, n-2) = \{0\} \cup \left\{(n-2)2^{k-2i-1} : \ 0 \leq i < k/2\right\}.$$

Proof. Let P_m be the standard circulant $m \times m$ matrix. Then $\det(\lambda I - P_m) = \lambda^m - 1$, and hence

$$|\det(I + P_m)| = \begin{cases} 0 & \text{if } m \text{ is even,} \\ 2 & \text{otherwise.} \end{cases} \tag{3.1}$$

Now suppose $A \in S(n,2)$ and $\det A \neq 0$. Then (e.g., see [3, Corollary 1.2.5]) $A = \tilde{Q} + \tilde{R}$ for some permutation matrices $\tilde{Q}$ and $\tilde{R}$. So $|\det A| = |\det(I + R)|$ where $R = \tilde{Q}^{-1}\tilde{R}$. By Cycle Decomposition, R can be written as the following:

$$\tilde{P} R \tilde{P}^T = P_{m_1} \oplus P_{m_2} \oplus \cdots \oplus P_{m_j},$$

where $\tilde{P}$ is a certain permutation matrix. By (3.1), it is easy to see that $|\det(A)| = 0$ if m_i is even for some i; and $|\det(A)| = 2^j$ if m_i is odd for $i = 1, \ldots, j$. Since $\det A \neq 0$, the numbers $m_1, \ldots, m_j$ are odd; moreover, since $P_{m_1} \oplus \cdots \oplus P_{m_j} \in S(n,2)$, we must have $m_r \geq 3$ for $r = 1, \ldots, j$. Also,

$$n = m_1 + \cdots + m_j,$$

and therefore j and k have the same parity. Now it is easy to see that $j = k - 2i$ for some i, $0 \leq i < \frac{k}{2}$. This proves that

$$D(n,2) \subseteq \{0\} \cup \left\{ 2^{k-2i} : \ 0 \leq i < \frac{k}{2} \right\}.$$

It is not difficult to construct $A \in S(n,2)$ such that $|\det A| = 2^{k-2i}$ $(0 \leq i < \frac{k}{2})$. Namely, let

$$m_1 = m_2 = \cdots = m_{k-2i-2} = 3; \ m_{k-2i-1} = 3 + t; \ m_{k-2i} = 6i + 3$$

(it is assumed there that $k - 2i > 2$; if $k - 2i = 2$, we let $m_1 = 3 + t$; $m_2 = 6i + 3$; and if $k - 2i = 1$, we let $m_1 = n$). In any case,

$$\left| \det \left((I_{m_1} + P_{m_1}) \oplus \cdots \oplus (I_{m_{k-2i}} + P_{m_{k-2i}}) \right) \right| = 2^{k-2i},$$

as required. Finally, the second formula in Theorem 3.1 follows from (2.1). $\square$

We note that a very similar proof of Theorem 3.1 was obtained independently in [6, Section 3] with emphasis on finding $M(n,2)$.

Theorem 3.1, together with the results (2.1)–(2.4) and the bound $M(6,k) \leq 9$ (see Table 1) allows us to determine $D(n,k)$ for $n \leq 6$.

Corollary 3.2 $D(5,2) = \{0,2\}$; $D(5,3) = \{0,3\}$;
$D(6,2) = \{0,4\}$; $D(6,3) = \{0,9\}$; $D(6,4) = \{0,8\}$.

One may try to extend the technique in the proof of Theorem 3.1 to $D(n,k)$ for $k \geq 2$. In fact, it is true that every $A \in S(n,k)$ can be written as

$$A = P_1 + \cdots + P_k$$

for k different permutation matrices such that no two of them have a nonzero entry at the same position. As a result, we have

$$|\det(A)| = |\det(P_1)| \cdot |\det(I + Q_2 \cdots + Q_k)| = |\det(I + Q_2 \cdots + Q_k)|$$

with $Q_j = P_1^{-1}P_j$ for $j = 2, \ldots, k$. Unfortunately, unlike the case when $k = 2$, there does not seem to have an easy way to determine $\det(I + Q_2 + \cdots + Q_k)$ if $k > 2$. Even for $n = 7, 8$ and $k = 3, 4$, the problems are highly nontrivial and we need to develop some new techniques to determine $S(n, k)$ as shown in the next section.

4 Partial Results on $D(n, k)$ and Some Techniques

In this section, we determine $D(n, k)$ for $n = 7, 8$. By Theorem 3.1 and (2.1), we need only to consider case $2 < k \leq n/2$.

Theorem 4.1 $D(7, 3) = \{0, 3, 6, 24\}$; $D(8, 3) = \{0, 3, 6, 9, 15, 27\}$; $D(8, 4) = \{0, 16, 32\}$.

Observe that by (2.4) and Lemma 2.3, we have

$$\{0, 3, 24\} \subseteq D(7, 3) \subseteq \{0, 3, 6, 9, 12, 15, 18, 21, 24\}.$$

Thus, by Theorem 4.1, there is just one additional non-zero value of $|\det A|$, $A \in S(7, 3)$, in addition to $m(7, 3)$ and $M(7, 3)$. Also, Table 1 only guarantees that $M(8, 3) \leq 39$; $M(8, 4) \leq 64$. Thus, already for relatively small numbers n, such as $n = 8$, there is a significant gap between the upper bounds and the actual values of $D(n, k)$.

One may wonder if, for some values of n and k, a simple computer search can be done to determine $D(n, k)$. However, even for small (n, k) pairs such as $(7, 3), (8, 3), (8, 4)$, it seems difficult to write an efficient computer program to generate all the matrices in $S(n, k)$ and compute the determinants. We therefore develop some techniques to study the problem so as to obtain the result directly, or reduce the computer work to a manageable level. Hopefully, our techniques can be further developed to obtain more results on the topic. In the following we discuss several ideas and lemmas that are useful to prove Theorem 4.1. A sketch of the proof will be given without details on the computer work. One may consult [9] for the full details.

A. Permutation of rows and columns.

Clearly, the value $\det(A)$ is invariant under permutation of rows and columns. Such operations are used frequently in our study.

B. Using the structure of $A^T A$.

Sometimes, one can use the structure of $A^T A$ to get information about $\det(A^T A) = |\det(A)|^2$. (Likewise, one can use the structure of AA^T by replacing A with A^T.) For example, we have the following observation.

Lemma 4.2 *Suppose $A \in S(n, k)$. Then* $\det(A) = 0$ *if*

(a) *$A^T A$ has an off-diagonal entry equal to k, or*

(b) *$n = 2k$ and A^TA has an off-diagonal entry equal to 0.*

Proof. (a) Suppose the (i,j) entry of A^TA is k. Then the ith and the jth columns of A must be identical. Thus $\det(A) = 0$.

(b) Suppose $n = 2k$ and the (i,j) entry of A^TA is 0. Then the entries of the ith and the jth columns of A have disjoint support, and the sum of the two columns equal to e, the vector of all entries equal to one. Now the sum of all other columns equals $(k-1)e$. Thus A has linearly dependent columns, and hence $\det(A) = 0$. $\square$

There are other results one can prove using the structure of A^TA. For example, the following result was developed in the study of $D(8,3)$.

Lemma 4.3 *Let $A \in S(8,3)$. If all the off-diagonal entries of A^TA are not larger than* 1*, then $|\det(A)| = 27$.*

Proof. By the hypothesis, each row and column of A^TA has a diagonal entry equal to 3, six off-diagonal entries equal to 1, and one off-diagonal entry equal to 0. Thus $A^TA - J = 2I - Q$ for some symmetric permutation matrix Q with all diagonal entries equal to 0. Since all eigenvalues of $2I - Q$ is real, the cycle decomposition of Q has only cycles of length 2. Thus Q is permutationally similar to $\sum_{i+j=9} E_{ij}$, and hence $|\det(2I - Q)| = 3^4$. Now A^TA has all line sums equal to 9. By [13, Lemma 1], $|\det(A^TA)| = 3^2|\det(J - A^TA)| = 3^6$, and hence $|\det(A)| = 27$. $\square$

By Lemmas 4.2 and 4.3, we have the following corollary.

Corollary 4.4 *If $A \in S(8,3)$ is such that $\det(A) \neq 0,\ \pm 27$, then A has two columns b_i and b_j satisfying $b_i^Tb_j = 2$.*

C. Use of graph theory.

Note that if $k^2 = nm$ for some nonnegative integer m and if $A \in S(n,k)$, then $B = A^TA - mJ$ is a symmetric matrix with zero line sums. Under certain additional assumption on A, the matrix B can be viewed as the Laplacian of a graph G (e.g., see [3] for the basic definitions and theory). Then one may use some graph theory to determine $\det(B)$, and hence $|\det(A)|^2 = \det(A^TA)$. We illustrate this idea by the following lemma.

Lemma 4.5 *Let $A \in S(n,k)$ with $(n,k) = (8,4)$ or $(9,3)$. If A^TA has no off-diagonal entries equal to $k-1$, then $|\det(A)| = 0$ or nk.*

Proof. We first consider the case when $(n,k) = (8,4)$. If A^TA has an off-diagonal entry equal to 0 or 4, then $\det(A) = 0$ by Lemma 4.2. Suppose A^TA has no off-diagonal entries equal to 0, 3 or 4. Then each row and each column of A^TA has a diagonal entry equal to 4, five off-diagonal entries equal to 2, and two off-diagonal entries equal to 1. Thus $B = A^TA - 2J$ has diagonal entries all equal to 2 and all line sums equal to zero. In particular, B can be viewed as the Laplacian of a 2-regular graph G. Suppose B has eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$, so that $Be = \lambda_n e$, where e is the vector of all ones. Then $A^TA = B + 2J$ has eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$, and 16 (eigenvalue of $2J$). If G is disconnected, then $\lambda_{n-1} = 0$. Thus $\det(A^TA) = 0$. If G is connected, then G is a cycle and

$\lambda_1 \cdots \lambda_{n-1}$ = sum of 7×7 principal minors of $B = 8 \times t = 64$, where $t = 8$ is the number of spanning trees of B. (See [3, Theorem 2.5.3].) We have $|\det(A^T A)| = 16 \times 64$, and hence $|\det(A)| = 32$.

Now for $(n, k) = (9, 3)$, if $A^T A$ has an off-diagonal entry equal to 3, then $\det(A) = 0$ by Lemma 4.2. Suppose $A^T A$ has no off-diagonal entries equal to 2 or 3. Then $B = A^T A - J$ can be viewed as the Laplacian of a 2-regular graph, and one can get the conclusion by arguments similar to those in the preceding paragraph. □

D. Schur complement.

One may use Schur complement to reduce the problem. For example, in many cases a matrix $A \in S(7, 3)$ or $S(8, 3)$ can be reduced (by row and column permutations) to the situation

$$\begin{bmatrix} B & C \\ D & E \end{bmatrix} \quad \text{with} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Since $|\det A| = |\det(E - DB^{-1}C)|$, one only need to study the determinants of the matrix $E - DB^{-1}C$.

In the following, we give a

Sketch of the proof of Theorem 4.1. It is relatively easy to construct determinant values in $D(n, k)$ as proposed in the theorem. The difficult part is to show that those are the only possible values. We shall focus on this part of the proof.

If $A \in S(7, 3)$, one can consider three cases:
(a) $A^T A$ has an off-diagonal entry equal to 3, and hence $\det(A) = 0$ by Lemma 4.2,
(b) $A^T A$ has all off-diagonal entries equal to 1, then $A^T A = 2I + J$ and hence $|\det(A)| = 24$ by the result of Ryser [15],
(c) $A^T A$ has all off-diagonal entries equal to 2 or 0, then by a suitable permutation of rows and columns the first four rows of A may be put in the form:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

and hence the remaining three rows must be identical, contradicting the fact that $A^T A$ has no off-diagonal entries equal to 3,
(d) $A^T A$ has off-diagonal entries equal to 2 and 1, respectively, but not 3, then by a suitable permutation of rows and columns, one may assume that A is of the block form mentioned in D, and use the Schur complement technique. The proof is then finished by a computer search for $\det(E - DB^{-1}C)$.

If $A \in S(8, 3)$, then by Lemma 4.2 and Corollary 4.4, either
(a) $\det(A) = 0$ or ± 27, or
(b) $A^T A$ has off-diagonal entries equal to 2, but not 3. If there is a row in $A^T A$ containing entries 2 and 1, then by a suitable permutation of rows and columns, one may assume

that A is of the block form mentioned in D. and use the Schur complement technique. The proof is then finished by a computer search for $\det(E - DB^{-1}C)$. If all off-diagonal entries are either 0 or 2, then by a suitable permutation of the rows and columns of A, we may assume that the first 4 rows of A are of the form

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus A is essentially the direct sum of two matrices A_1 and A_2 in $S(4,3)$ and hence $\det(A) = \det(A_1)\det(A_2) = \pm 9$.

If $A \in S(8,4)$, then by Lemmas 4.2 and 4.5, either

(a) $\det(A) = 0$ or ± 32, or

(b) A^TA has off-diagonal entries equal to 3, but not 0. By a suitable permutation of rows and columns of A we may assume that the first row of A^TA is of the form

$$(4,3,3,2,1,1,1,1) \quad \text{or} \quad (4,3,2,2,2,1,1,1).$$

One can then use computer search to finish the proof. □

The general problem of identifying the sets $D(n,k)$, or even the maximal number $M(n,k)$ in $D(n,k)$, remains open. There are a few other techniques one may use.

E. Direct sum.

The following result is clear.

Lemma 4.6 *Suppose $d_1 \in D(n_1,k)$ and $d_2 \in D(n_2,k)$. Then $d_1d_2 \in D(n_1+n_2,k)$. Alternatively, we can write $D(n_1,k)D(n_2,k) \subseteq D(n_1+n_2,k)$.*

F. Reversing the Schur Complement.

Lemma 4.7 *Suppose $A = [A_{rs}] \in S(n,k)$ is such that $A_{ij} = 1$ for all those $i+j \le k$. Define the $(n+k)\times(n+k)$ matrix $\tilde{A} = \begin{bmatrix} B & C \\ C^T & A' \end{bmatrix}$, where $B = [B_{rs}]$ is a $k\times k$ matrix and $C = [C_{rs}]$ is a $k \times n$ matrix satisfying*

$$B_{rs} = \begin{cases} 1 & \text{if } r+s \ge k+1, \\ 0 & \text{otherwise,} \end{cases} \qquad C_{rs} = \begin{cases} 1 & \text{if } r+s \le k, \\ 0 & \text{otherwise,} \end{cases}$$

and A' is obtained from A by setting A_{rs} to 0 for all those $r+s \le k$. Then $\tilde{A} \in S(n+k,k)$ and $|\det(\tilde{A})| = |\det(A)|$.

Proof. Apply the Schur complement to $\tilde{A}$, which gives the equality

$$\det(\tilde{A}) = \det(B)\det(A' - C^TB^{-1}C).$$

It is easy to see that $|\det(B)| = 1$, and that the entries of $B^{-1} = [B^{-1}_{rs}]$ are given by $B^{-1}_{rs} = 1$ if $r + s = k + 1$, $B^{-1}_{rs} = -1$ if $r + s = k$, and $B^{-1}_{rs} = 0$ otherwise. Now a computation shows that $A' - C^T B^{-1} C = A$. $\square$

For example, using Lemma 4.7, one can show that $D(n, 3) \subseteq D(n + 3, 3)$ for $n \geq 4$. The next unsolved case is $D(9, 3)$. By (2.2), (2.4), and Lemma 2.3 we know that

$$\{0, 9\} \subseteq D(9, 3) \subseteq \{0, 9, 18, 27, 36, 45, 54, 63, 72\}.$$

Our experience shows that if $A \in S(n, k)$ satisfies $\det(A) = M(n, k)$, then the off-diagonal entries of $A^T A$ and AA^T are as uniform as possible. Motivated by Lemmas 4.3 and 4.5, we formulate the following conjecture:

Conjecture 4.8 *Suppose $A \in S(n, k)$ is non-singular such that the off-diagonal entries x of $A^T A$ and of AA^T satisfy*

$$\left| x - \frac{k^2 - k}{n - 1} \right| < 1.$$

Then $|\det(A)| = M(n, k)$.

5 Results for Sym(n, k)

In this section we focus on the set $\mathrm{Sym}(n, k)$ of all symmetric zero-one $n \times n$ matrices having row sums and column sums equal to k. Here $1 \leq k \leq n$ are integers. Let

$$D_{\mathrm{Sym}}(n, k) = \{|\det A| : \ A \in \mathrm{Sym}(n, k)\}.$$

Since $J = J^T$, the result of (2.1) holds with $S(n, k)$ (resp. $S(n, n-k)$) replaced by $\mathrm{Sym}(n, k)$ (resp. $\mathrm{Sym}(n, n - k)$). Also, (2.2) trivially holds for $\mathrm{Sym}(n, k)$ (just because $\mathrm{Sym}(n, k) \subseteq S(n, k)$). Clearly, $D_{\mathrm{Sym}}(n, 1) = \{1\}$, $D_{\mathrm{Sym}}(n, n - 1) = \{n - 1\}$. Further, observe that if $F_n = \sum_{i+j=n+1} E_{ij}$, then $F_n f(P_n) \in \mathrm{Sym}(n, k)$ for any polynomial f. It follows that $D_{\mathrm{Cir}}(n, k) \subseteq D_{\mathrm{Sym}}(n, k)$.

Proposition 5.1 *If $1 < k < n - 1$, then there is a singular matrix in* $\mathrm{Sym}(n, k)$.

Proof. If $k \leq n/2$, then $A = J_k \oplus A_0 \in \mathrm{Sym}(n, k)$ is singular, where

$$A_0 = F_{n-k}(\sum_{j=1}^{k} P^j_{n-k}).$$

If $n < 2k$, let $A \in \mathrm{Sym}(n, n - k)$ be a singular matrix. Then $J - A \in \mathrm{Sym}(n, k)$ is singular. $\square$

The "symmetric" analog of (2.3) and (2.4) is also valid as shown in the following result.

Theorem 5.2 *If $1 \leq k \leq n - 1$, and $(n, k) \neq (4, 2)$, then there is $A \in \mathrm{Sym}(n, k)$ such that*

$$|\det A| = k \cdot \gcd(n, k).$$

Proof. It suffices to consider the case when $n \geq 2k$. Let $Q = Q_1 \oplus 0_{n-2k}$, where

$$Q_1 = \begin{bmatrix} -I_{k-1} & 0_{k-1,2} & I_{k-1} \\ 0_{2,k-1} & 0_2 & 0_{2,k-1} \\ I_{k-1} & 0_{k-1,2} & -I_{k-1} \end{bmatrix}.$$

If $n = 2k > 4$, then (see [11]) $A = Q + \sum_{i=-1}^{k-2} P_n^i \in S(n,k)$ satisfies $|\det(A)| = k^2$. Then $F_n A \in \mathrm{Sym}(n,k)$ satisfies $|\det(F_n A)| = k^2$. If $n > 2k$, then (see [8]) $A = Q + \sum_{i=0}^{k-1} P_n^i \in S(n,k)$ satisfies $|\det(A)| = k \cdot \gcd(n,k)$. One easily checks that $\tilde{A} = F_n P_n^k A P_n^{-k} \in \mathrm{Sym}(n,k)$ satisfies $|\det(F_n A)| = k \cdot \gcd(n,k)$. $\square$

Thus, the minimal absolute value of determinants of non-singular matrices in $S(n,k)$ is achieved actually in the smaller set $\mathrm{Sym}(n,k)$.

We obtain the exact values for $D_{\mathrm{Sym}}(n,k)$ in some cases:

Theorem 5.3 *We have*

$$D_{\mathrm{Sym}}(n,k) = D(n,k) \tag{5.1}$$

for $n \leq 8$ *and* $1 \leq k \leq n$. *Also*

$$D_{\mathrm{Sym}}(n,2) = D(n,2) \tag{5.2}$$

for $n \geq 2$.

Proof. It suffices to consider the cases when $1 < k \leq n/2$. The equalities (5.2) follow from the proof of Theorem 3.1 and the fact that $F_m(I + P_m) \in \mathrm{Sym}(m,2)$. Furthermore, by Corollary 3.2, Theorem 4.1 and Theorem 5.2, it remains to consider the following cases:

$$(n,k) \in \{(7,3), (8,3), (8,4)\}.$$

For the case $(n,k) = (7,3)$ we need only to exhibit matrices A_1 and A_2 in $\mathrm{Sym}(7,3)$ having absolute values of determinants 6 and 24:

$$A_1 = \begin{bmatrix} 1&1&1&0&0&0&0 \\ 1&1&0&1&0&0&0 \\ 1&0&0&0&1&1&0 \\ 0&1&0&0&1&0&1 \\ 0&0&1&1&0&1&0 \\ 0&0&1&0&1&0&1 \\ 0&0&0&1&0&1&1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1&1&1&0&0&0&0 \\ 1&0&0&1&1&0&0 \\ 1&0&0&0&0&1&1 \\ 0&1&0&1&0&1&0 \\ 0&1&0&0&1&0&1 \\ 0&0&1&1&0&0&1 \\ 0&0&1&0&1&1&0 \end{bmatrix}.$$

For the case $(n,k) = (8,3)$, one obtains (using Matlab) 3, 9, 15, 27 as the absolute values of determinants of matrices of the form $I_8 + P_8^{i_1} + P_8^{i_2}$, where $0 < i_1 < i_2 \leq 7$. Multiplying such matrices on the left by $F_8 = \sum_{i+j=9} E_{ij}$, we get matrices in $\mathrm{Sym}(8,3)$ having the same absolute values of determinants. Thus $D_{\mathrm{Sym}}(8,3) \supseteq \{0, 3, 9, 15, 27\}$. On the other hand,

$A_3 \in \mathrm{Sym}(8,3)$ satisfies $|\det(A_3)| = 16$, where

$$A_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

So we are done for $(n,k) = (8,3)$ by Theorem 4.1.

Finally, assume $(n,k) = (8,4)$. By Theorem 5.2, there exists $A \in \mathrm{Sym}(8,4)$ with $|\det A| = 16$. On the other hand, using Matlab we have verified that $\{0, 32\}$ are the absolute values of determinants of matrices of the form $I + P_8^{i_1} + P_8^{i_2} + P_8^{i_3}$, where $0 < i_1 < i_2 < i_3 \leq 7$. By Theorem 4.1, we are done. $\square$

We do not think that it is true that $D(n,k) = D_{\mathrm{Sym}}(n,k)$ in general. It is interesting to consider the following problem.

Problem 5.4 *Determine those positive integers $k \leq n$ so that*

(a) $D(n,k) = D_{\mathrm{Sym}}(n,k)$.

(b) $M(n,k) = \max D_{\mathrm{Sym}}(n,k)$.

6 Results for Cir(n,k)

Another interesting class of matrices in $S(n,k)$ are polynomials with zero-one coefficients of the standard circulant.

$$\mathrm{Cir}(n,k) = \left\{ \sum_{q=1}^{k} P_n^{i_q} : \ 0 \leq i_1 < i_2 < \cdots < i_q < n \right\}.$$

Proposition 6.1 *Let $1 \leq k \leq n-1$. Then all matrices* Cir(n,k) *are singular if and only if n is a power of 2 and either $k = 2$ or $k = n-2$.*

Proof. The proof of Theorem 1 in [13] shows that Cir(n,k) contains a non-singular matrix if $3 \leq k \leq n-3$. In view of (2.1) we have to consider only the case $k = 2$. Assume that n is not a power of 2. Let q be a divisor of n such that $s = n/q$ is an odd prime. It is then easy to see that $1 + \lambda^q \neq 0$ for any n^{th} root of unity λ. Thus $I + P_n^q$ is non-singular. Assume now that n is a power of 2: $n = 2^x$. Given any integer i, $1 \leq i \leq n-1$, write: $i = 2^y \ell$, where $0 \leq y \leq x-1$ and $\ell \geq 1$ is odd. Now let $q = \frac{n}{2^{y+1}}$. Clearly, q is a positive integer, and, denoting by w a primitive n^{th} root of unity, we have $1 + (w^q)^i = 0$. Since w^q is also an n^{th} root of unity, it follows that $I + P_n^i$ has zero as one of its eigenvalues, and hence $I + P_n^i$ is singular. Thus Cir$(2^x, 2)$ consists of singular matrices only. $\square$

Note that if w is an primitive nth root of unity, and if $0 = i_1 < i_2 < \cdots < i_k < n$, then $A = P_n^{i_1} + \cdots + P_n^{i_k} \in \mathrm{Cir}(n, k)$ has eigenvalues $(w^r)^{i_1} + \cdots + (w^r)^{i_k}$, $r = 1, \ldots, n$. Thus there exists a singular matrix in $\mathrm{Cir}(n, k)$ if and only if there exist integers $0 = i_1 < i_2 < \cdots < i_k < n$ such that $(w^r)^{i_1} + \cdots + (w^r)^{i_k} = 0$ for some $1 \le r < n$. (In fact, it suffices to consider only the factors r of n.) Unfortunately, the condition mentioned above is not easy to check. We shall describe several readily computable criteria on some special cases in the next proposition.

Proposition 6.2 *Assume* $1 < k < n - 1$.

(a) *If* $\gcd(n, k) > 1$, *then there exists a singular matrix in* $\mathrm{Cir}(n, k)$.

(b) *The converse of* (a) *holds if at least one of the following is true:*
(b.i) $k \le 4$; (b.ii) n *is a power of a prime;* (b.iii) $n = 2p$, *where* p *is an odd prime.*

Proof. For the part (a), let $n = n_1 q$, $k = k_1 q$, where $q = \gcd(n, k) > 1$. Let w be a q^{th} primitive root of unity. Since w is also a k^{th} root of unity, we have $1 + w + \cdots + w^{k-1} = 0$. But w is also an n^{th} root of unity, so $I + P_n + \cdots + P_n^{k-1}$ is singular.

For part (b), we first consider the case when (b.i) is true. Let μ be an nth primitive root of unity. Then $\mathrm{Cir}(n, 2)$ contains a singular matrix if and only if $1 + (\mu^r)^{i_2} = 0$ for some $r < n$ and $0 < i_2 < n$. Thus $-1 = \mu^{ri_2}$, and hence n is even. Suppose $\mathrm{Cir}(n, 3)$ contains a singular matrix. Then $1 + (\mu^r)^{i_2} + (\mu^r)^{i_3} = 0$ for some $r < n$ and $0 < i_2 < i_3 < n$. Taking complex conjugates in this equality, and multiplying the resulting equality by $(\mu^r)^{i_3}$, we obtain $1 + (\mu^r)^{(i_3 - i_2)} + (\mu^r)^{i_3} = 0$. It follows that $i_3 = 2i_2$, and therefore μ^{ri_2} has to be a primitive cube root of unity. Thus $\gcd(n, 3) = 3$. Next, suppose $\mathrm{Cir}(n, 4)$ contains a singular matrix. Then $1 + (\mu^r)^{i_2} + (\mu^r)^{i_3} + (\mu^r)^{i_4} = 0$ for some $r < n$ and $0 < i_2 < i_3 < i_4 < n$. We need to show that n is even. Apply the relation $\mu^n = 1$, and relabel μ^{ri_2}, μ^{ri_3}, and μ^{ri_4} as μ^a, μ^b, and μ^c, respectively, so that $0 \le a \le b \le c < n$. If $a, b - a, c - b, n - c$ are all odd, then n is even. Otherwise, one of the above integers is even. One can then multiply $1, \mu^a, \mu^b, \mu^c$ by a suitable μ^d for some integer d so that the resulting four numbers are of the form $\nu, \bar{\nu}, z_1, z_2$ (for example, if a is even, we let $d = -\frac{a}{2}$). Since $\nu + \bar{\nu} + z_1 + z_2 = 0$, we have that either $\nu + \bar{\nu} = 0$, i.e., $\nu = \pm i$, or $\{z_1, z_2\} = \{-\nu, -\bar{\nu}\}$. In both cases, $\mu^s = -1$ for some integer s, and hence n is even.

For part (b.ii) and (b.iii), we use the cyclotomic polynomial $\Phi_n(x) = \prod(x - \zeta)$, where the product is taken over all primitive n^{th} roots of unity ζ. It is well known that $\Phi_n(x)$ is irreducible over the field $\mathbf{Q}$ of rational numbers; $\Phi_n(x)$ has integer coefficients; and $\Phi_n(x)$ is the minimal polynomial of the primitive n^{th} root of unity over $\mathbf{Q}$. We will use the equality

$$x^n + x^{n-1} + \cdots + 1 = \prod \Phi_d(x), \tag{6.1}$$

where the product is taken over all divisors d of n, excluding $d = 1$ (see, e.g., [5]).

Assume first that $n = p^\alpha$, where p is a prime and α is a positive integer. The equality (6.1) shows easily (using induction on α), that $\Phi_n(1) = p$. Assume that there is a singular matrix in $\mathrm{Cir}(n, k)$. Then a primitive n^{th} root of unity ζ is a root of some polynomial $p(x)$ of the form $p(x) = \sum_{j=1}^{k} x^{i_j}$. By the minimality of $\Phi_n(x)$ we have $p(x) = \Phi_n(x) q(x)$ for some polynomial $q(x)$ with integer coefficients. Therefore, $k = p(1) = \Phi_n(1) q(1) = p \cdot q(1)$, which contradicts the fact that $\gcd(n, k) = 1$.

Assume now $n = 2p$, where p is an odd prime, and let k be relatively prime to n. Using (6.1), one verifies that $\Phi_n(x) = x^{p-1} - x^{p-2} + \cdots - x + 1$. We show that if $w^n = 1$, then w cannot be a root of any polynomial of the form $f(x) = x^{i_1} + x^{i_2} + \cdots + x^{i_k}$, where $0 \leq i_1 < \cdots < i_k < n$. Suppose it is. Without loss of generality, we can assume $k \leq p$. Then in fact, because of the relative primeness of n and k, we have that $k < p$ and k is odd. Three cases can occur: (1) $w^2 = 1$; (2) $w^p = 1$; (3) w is a primitive n^{th} root of unity. In case (1) we clearly obtain a contradiction, because $w = \pm 1$, and therefore $f(w) \neq 0$. In case (2), $f(x)$ is divisible by $\Phi_p(x) = 1 + x + \cdots + x^{p-1}$: $f(x) = \Phi_p(x)q(x)$ for some polynomial with integer coefficients $q(x)$. Evaluating both sides for $x = 1$, a contradiction follows: $k = p \cdot q(1)$. In case (3),

$$f(x) = \Phi_n(x)q(x) = (x^{p-1} - x^{p-2} + \cdots - x + 1)q(x)$$

for some polynomial $q(x)$ (with integer coefficients). Evaluating for $x = -1$, we have $f(-1) = p \cdot q(-1)$, which is clearly impossible, because $0 < |f(-1)| < p$, in view of $k < p$ and k being odd. $\square$

The following example shows that the converse of the first assertion of (a) may not be true if $k \geq 5$.

Example 6.3 *Let* $A = I + P_{12}^3 + P_{12}^6 + P_{12}^8 + P_{12}^{10} \in \mathrm{Cir}(12, 5)$. *Then* $\det(A) = 0$.

It is worthwhile to mention the idea behind the construction of the above example that can be viewed as a generalization of Proposition 6.2 (a). Observe that to construct (n, k) so that $\mathrm{Cir}(n, k)$ contains a singular matrix, one may consider $k = k_1 + \cdots + k_t$ so that the mth primitive root of unity $\eta = \exp(2\pi i/m)$ satisfies

$$\eta^{j_{s1}} + \cdots + \eta^{j_{sk_s}} = 0, \quad s = 1, \ldots, t,$$

for some integer sequences $0 = j_{s1} < j_{s2} < \cdots < j_{sk_s} < m$. Then for $n = mr$ with $r \geq t$, and $\mu = \exp(2\pi i/n)$, each matrix

$$A_s = P_n^{(s-1)m}(P_n^{j_{s1}} + \cdots + P_n^{j_{sk_s}}) \qquad (s = 1, \ldots, t)$$

has an eigenvalue 0 with $v = (1, \mu^r, \mu^{2r}, \ldots, \mu^{(n-1)r})^T$ as a corresponding eigenvector. Thus the matrix $A = A_1 + \cdots + A_t \in S(n, k)$ is singular. In Example 6.3, we have $k = 5 = 3 + 2$, $m = 6$, $n = 2m = 12$, $(j_{11}, j_{12}) = (0, 3)$ and $(j_{21}, j_{22}, j_{23}) = (0, 2, 4)$.

It is interesting to point out the connection of the problem of existence of a singular matrix in $\mathrm{Cir}(n, k)$ and some other subjects. First, the same property on (n, k) appears in the study of stability of invariant subspaces (see [14]). Second, very recently, it was shown in [9] that if n has prime factor decomposition $n = p_1^{a_1} \cdots p_m^{a_m}$, then there exists positive integers $i_1, \ldots, i_k$ not necessarily distinct such that $w^{i_1} + \cdots + w^{i_k} = 0$ if and only if $k = b_1 p_1 + \cdots + b_m p_m$ for some nonnegative integers $b_1, \ldots, b_m$. Unfortunately, this latter condition is necessary but not sufficient to ensure the existence of a singular matrix in $D_{\mathrm{Cir}}(n, k)$. For instance, consider $(n, k) = (10, 7)$. Then $7 = 2 + 5$, is a sum of the prime factors of 10, but there is no singular $A \in D_{\mathrm{Cir}}(10, 7)$ by Proposition 6.2 (b).

The identification of the set $D_{\mathrm{Cir}}(n,k)$ is an open problem in general. Calculations using Matlab show that

$$\begin{array}{rl} D_{\mathrm{Cir}}(7,3)=\{3,24\}; & \\ D_{\mathrm{Cir}}(8,3)=\{3,9,15,27\}; & D_{\mathrm{Cir}}(8,4)=\{0,32\}; \\ D_{\mathrm{Cir}}(9,3)=\{0,27\}; & D_{\mathrm{Cir}}(9,4)=\{4,16,28,76\}; \\ D_{\mathrm{Cir}}(10,3)=\{3,9,33\}; & D_{\mathrm{Cir}}(10,4)=\{0,8,16,88\}; \\ D_{\mathrm{Cir}}(11,3)=\{3,69\}. & \end{array}$$

By the remark before Proposition 5.1 we have

$$D_{\mathrm{Cir}}(n,k)\subseteq D_{\mathrm{Sym}}(n,k). \tag{6.2}$$

However, the above calculations show that the proper containment is possible in (6.2). Moreover, in contrast with Theorem 5.2,

$$m(8,4)=16\notin D_{\mathrm{Cir}}(8,4).$$

Concerning the maximal value of the determinant, we have

$$M(7,3)\in D_{\mathrm{Cir}}(7,3);\ M(8,3)\in D_{\mathrm{Cir}}(8,3);\ M(8,4)\in D_{\mathrm{Cir}}(8,4),$$

but $M(11,3)\notin D_{\mathrm{Cir}}(11,3)$. Indeed, we can construct a matrix $A\in S(11,3)$ having the form $A=A_1\oplus A_2$, where $A_1\in S(7,3)$ with $\det A_1=24$ and $A_2\in S(4,3)$ with $\det A_2=3$. Thus, $\det A=72$, and therefore $M(11,3)\geq 72$.

Another open problem involves the symmetric (n,k,λ) designs for which the exact value of $M(n,k)$ is known (see Section 2). One easily verifies that the exact values $M(7,3)=24$ and $M(11,5)=1215$ are achieved on the set $\mathrm{Cir}(7,3)$ and $\mathrm{Cir}(11,5)$, respectively. For example,

$$\det(I+P_{11}+P_{11}^2+P_{11}^4+P_{11}^7)=\det(I+P_{11}+P_{11}^2+P_{11}^6+P_{11}^9)=1215.$$

In fact, it is known (e.g., see [2]) that $M(n,k)=\max D_{\mathrm{Cir}}(n,k)$ if there exists a symmetric (n,k,λ) design arising from a cyclic difference set. Nonetheless, it is interesting to study the following problem:

Problem 6.4 *Determine those positive integers $k\leq n$ such that*

(a) $D_{\mathrm{Sym}}(n,k)=D_{\mathrm{Cir}}(n,k)$,

(b) $D(n,k)=D_{\mathrm{Cir}}(n,k)$,

(c) $M(n,k)=\max D_{\mathrm{Cir}}(n,k)$.

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