Linear Preserver Problems

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Linear preserver problems is an active research area in matrix and operator theory. These problems involve certain linear operators on spaces of matrices or operators. We give a general introduction to the subject in this article. In the first three sections, we discuss motivation, results, and problems. In the last three sections, we describe some techniques, outline a few proofs, and discuss some exceptional results.

1. EXAMPLES AND TYPICAL PROBLEMS. Let $M_{m,n}$ be the set of $m \times n$ complex matrices, and let $M_n = M_{n,n}$. Suppose that $M, N \in M_n$ satisfy $\det(MN) = 1$. Then the mapping $\phi : M_n \to M_n$ given by

$$A \mapsto MAN \tag{1}$$

is linear and satisfies

$$\det(\phi(A)) = \det(A) \text{ for all } A \in M_n. \tag{2}$$

A linear operator ϕ satisfying (2) is called a *linear preserver of the determinant function* or simply a *determinant preserver*. Since $\det(A) = \det(A^t)$, it follows that if $\det(MN) = 1$, then the linear operator given by

$$A \mapsto MA^tN \tag{3}$$

also preserves the determinant. Frobenius [**20**] proved the following somewhat surprising result.

Theorem 1.1. *Every determinant preserver has the form (1) or (3), where $M, N \in M_n$ satisfy* $\det(MN) = 1$.

Note that M_n has dimension n^2. Thus, every linear map on M_n can be identified with an $n^2 \times n^2$ matrix. By Theorem 1.1, if ϕ is a determinant preserver, then ϕ is determined by two matrices $M, N \in M_n$, possibly followed by transposition.

We say that the linear preserver problem for determinants has been solved once a complete description of all linear operators ϕ that preserve determinants is given, as in Theorem 1.1.

We turn to another example of a linear preserver problem. Dieudonné [**15**] proved the following interesting result.

Theorem 1.2. *An invertible linear operator ϕ on M_n mapping the set of singular matrices into itself has the form (1) or (3) for some $M, N \in M_n$ with* $\det(MN) \neq 0$.

Dieudonné's result is valid over any field. We discuss his method in Section 4.

In the study of geometry on matrix spaces, Hua [**30**] studied *coherent* matrix pairs, i.e., a pair of matrices whose difference has rank one. He proved the following.

Theorem 1.3. *An invertible linear operator ϕ on M_n mapping coherent pairs to coherent pairs has the form (1) or (3) for some invertible matrices $M, N \in M_n$.*

These examples illustrate three typical linear preserver problems. To describe them in a general framework, let **V** be a space of matrices over the field **F**. In particular, denote by $M_{m,n}(\mathbf{F})$ (respectively, $M_n(\mathbf{F})$) the linear spaces of $m \times n$ (respectively, $n \times n$) matrices over **F**. We consider the following three types of problems.

Problem A Given a (scalar-valued, vector-valued, or set-valued) function F on **V**, study the linear preservers of F, i.e., those linear operators on **V** satisfying $F(\phi(A)) = F(A)$ for all $A \in \mathbf{V}$.

Problem B Given a subset **S** of **V**, study the linear preservers of **S**, i.e., those linear operators on **V** satisfying $\phi(\mathbf{S}) \subseteq \mathbf{S}$. Sometimes one considers linear operators mapping the set **S** "*onto*" itself when this assumption arises naturally or is implied implicitly in the problem, or when the "into" problem does not have a nice solution. We say that ϕ *strongly preserves* **S** if $\phi(\mathbf{S}) = \mathbf{S}$.

Problem C Given a relation $\sim$ on **V**, study the linear preservers of $\sim$, i.e., those linear operators ϕ on **V** satisfying

$$\phi(A) \sim \phi(B) \text{ whenever } A \sim B.$$

For example, $\sim$ could be commutativity; one would then study those linear operators ϕ satisfying

$$\phi(A)\phi(B) = \phi(B)\phi(A) \text{ whenever } AB = BA.$$

In some cases, one considers the problem of *strongly preserving* a relation, i.e.,

$$\phi(A) \sim \phi(B) \text{ if and only if } A \sim B.$$

2. THE ATTRACTION OF LINEAR PRESERVER PROBLEMS. There has been much research activity on linear preserver problems, especially in the last few decades. While there are many interesting results, there are still many open questions. Here are some reasons why the area is attractive.

The formulation of linear preserver problems is simple and natural. The answer is often very elegant. In addition to the theorems in Section 1, we present a few more results to illustrate these points.

Theorem 2.1. *A linear operator ϕ on M_n maps the set of invertible matrices into itself if and only if it has the form (1) or (3) for some invertible $M, N \in M_n$.*

Theorem 2.2. *Let $F(A)$ denote the set of eigenvalues of $A \in M_n$. A linear operator ϕ on M_n preserves F if and only if it has the form (1) or (3) for some $M, N \in M_n$ satisfying $M = N^{-1}$.*

These two theorems were proved by Marcus and Purves [**54**] for matrices over any algebraically closed field. We discuss results on other fields in Section 3.

Hiai [**25**] proved the following.

Theorem 2.3. *A linear operator ϕ on M_n preserves the relation of similarity if and only if there exist $a, b \in \mathbf{C}$ and an invertible $S \in M_n$ such that ϕ is of the form*

$$A \mapsto aS^{-1}AS + b(\operatorname{tr} A)I \qquad \text{or} \qquad A \mapsto aS^{-1}A^tS + b(\operatorname{tr} A)I, \tag{4}$$

or there exists a fixed $B \in M_n$ such that ϕ is of the form

$$A \mapsto (\operatorname{tr} A)B. \tag{5}$$

We call transformations of the form (1), (3), or (4) *standard transformations*. The image space of a transformation of the form (5) has dimension 1, and some of the transformations of the form (4) may be singular. From these examples, one sees that even if the admissible preservers do not consist entirely of standard transformations, there may not be too many exceptional preservers, and the final answer may still be quite elegant. The process of searching for linear preservers often helps researchers to understand better the matrix invariants, functions, sets, or relations under consideration. This is a nice by-product of studying linear preserver problems.

Applied problems can also lead one to study linear preservers. In elementary matrix and operator theory, one needs to study the properties of a given linear operator. In applications, one often has to construct or search for linear operators with some special properties. For example, in the matrix model in systems theory, what are the linear operators on a matrix space that preserve controllable systems or observable systems? Knowing the answer allows one to transform a complex system to a simpler system by linear maps that do not affect the nature of the system; see [**21**] and its references. In the matrix model of a quantum system, the entropy is related to the determinant of the matrix. Again, one may want to find linear operators that transform systems without affecting their entropy. This naturally leads to the study of linear preservers of determinants.

In addition to numerous matrix or operator invariants that can be studied, the techniques one uses can range from elementary algebraic and basic geometric techniques to deep theory in Lie groups and Lie algebras, completely positive maps, projective or differential geometry, multilinear methods, model theoretic algebra, etc. Because of this, the study of linear preserver problems often leads to interaction of linear algebra with other subjects. Thus one can focus on a single question, or one could use linear preserver results to try to develop general techniques for application to other questions. We elaborate these two directions in the next two sections.

3. SPECIFIC LINEAR PRESERVER PROBLEMS.

In this section we describe several active linear preserver problems not yet completely solved. In each case, we highlight some results, cite some history, and mention the current status of the problem. We try not to repeat the material in the survey [**58**], but it is useful for background. We include some recent topics and developments without making the discussion too technical. The following discussion reinforces our earlier comments that it is easy to generate linear preserver problems (from theory or applications); it is also common to refine known results by weakening the assumptions and to extend existing results to other matrix spaces or algebras.

3.1. Rank and inertia preservers.

Theorem 3.1. *A linear operator ϕ on $M_{m,n}(\mathbf{F})$ satisfies*

$$\operatorname{rank}(\phi(A)) = \operatorname{rank}(A) \qquad \textit{for all } A \in M_{m,n}(\mathbf{F})$$

if and only if there exist invertible matrices $M \in M_m(\mathbf{F})$ and $N \in M_n(\mathbf{F})$ such that

(i) *ϕ satisfies (1), or* (ii) *$m = n$ and ϕ satisfies (3).*

A linear operator on $M_{m,n}(\mathbf{F})$ is a rank k preserver, where $1 \leq k \leq \min\{m, n\}$, if it maps the set of rank k matrices into itself. For algebraically closed fields of characteristic 0, Theorem 3.1 has the following refinement; see [**3**].

Theorem 3.2. *Let $1 \leq k \leq \min\{m, n\}$, and let $\mathbf{F}$ be an algebraically closed field of characteristic 0. A linear operator ϕ on $M_{m,n}(\mathbf{F})$ is a rank k preserver if and only if there exist invertible matrices M and N such that*

(i) *ϕ satisfies (1), or* (ii) *$m = n$ and ϕ satisfies (3).*

Current work is proceeding on rank k preservers when the field is arbitrary. There are also studies of rank k preservers on symmetric matrices, skew-symmetric matrices, or complex Hermitian matrices.

For real symmetric matrices or complex Hermitian matrices, one can refine the concept of rank to *inertia*, i.e., the number of positive and negative eigenvalues. There has been much interest in the preservers of a fixed inertia class. Denote by $G(r, s, t)$ the class of complex Hermitian or real symmetric matrices with r positive, s negative, and t zero eigenvalues. The following result is known; see [**58**, Chapter 3].

Theorem 3.3. *Except for the cases $rs = 0$ or $r = s$, a linear preserver of $G(r, s, t)$ must have the form (1) or (3) with $M = N^*$ for some invertible M.*

There are many nonstandard preservers of $G(n, 0, 0)$, but the collection of all such preservers is not known. We discuss this in Section 5.

For the balanced inertia classes $G(r, r, 0)$, $r > 1$, the preservers are all of the form (1) or (3) with $M = \pm N^*$ and N invertible [**50**]. The preservers of $G(r, r, t)$ are fully known only if $n \geq 5r$ [**49**].

Other examples of rank preserver problems include (a) linear preservers of matrices of rank $\leq k$ for some fixed $k < n$, (b) the preservers of the set of rank k tensors or rank k elements in a given symmetry class of tensors; see [**58**, Chapter 2].

3.2. Functions of eigenvalues, singular values, and entries. Let ϕ be a linear operator on square matrices. If ϕ has the form (4) with $(a, b) = (1, 0)$, then $\phi(A)$ and A have the same eigenvalues. By Theorem 2.2, the converse is also valid. It is interesting to study linear preservers of a certain function of eigenvalues and see whether they are of this form or close to this form. The kth elementary symmetric function and the kth completely symmetric function of $n \geq k$ numbers $\mu_1, \ldots, \mu_n$ are

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} \mu_{i_j} \qquad \text{and} \qquad \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \prod_{j=1}^{k} \mu_{i_j},$$

respectively. When $k = 1$, the two concepts coincide. The study of preservers of the elementary and completely symmetric functions of the eigenvalues is essentially complete over algebraically closed fields. Note that the k^{th} elementary symmetric function is just a coefficient of the characteristic polynomial of A up to a ± 1 factor. We have the following result; see [**1**] and [**58**, Sec. 4.1].

Theorem 3.4. *Suppose $n \geq k \geq 3$. The preservers of the kth symmetric (or completely) symmetric function of eigenvalues on M_n have the form (4) with $b = 0$ and some $a \in \mathbf{F}$ satisfying $a^k = 1$.*

The first elementary (completely) symmetric function is just the trace of a matrix, and many linear maps preserve the trace function. The preservers of the second elementary (completely) symmetric function include many non-standard maps; see [**59**] and [**31**]. These functions are polynomial functions in the entries of the matrix; preserver problems of such functions are related to the study of algebraic sets, see [**58**, Chapter 4] and Section 3.3.

The *singular values* of a real or complex $m \times n$ matrix A are the nonnegative square roots of the eigenvalues of the positive semi-definite matrix A^*A. Unitarily invariant norms are those norms $\|\cdot\|$ that satisfy $\|A\| = \|UAV\|$ for all unitary matrices U and V. In fact, every unitarily invariant norm $\|\cdot\|$ on $m \times n$ matrices corresponds to a norm $|\cdot|$ on $\mathbf{R}^m$ such that $\|A\| = |s(A)|$, where $s(A)$ denotes the vector of singular values of A. For example, consider the ℓ_p norms

$$\ell_p(x) = \begin{cases} \left(\sum_{j=1}^m |x_j|^p\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{|x_j| : 1 \le j \le m\} & \text{if } p = \infty, \end{cases}$$

of $x = (x_1, \ldots, x_m)^t \in \mathbf{R}^m$. The *spectral norm* of A is the l_∞ norm of $s(A)$, the *Frobenius norm* of A is the l_2 norm of $s(A)$, and the *trace norm* of A is the l_1 norm of $s(A)$. Let ϕ be a linear operator on real or complex $m \times n$ matrices. If ϕ has the form (1) or (3) for some unitary matrices M and N, then $\phi(A)$ and A have the same singular values, and hence $\|\phi(A)\| = \|A\|$ for any unitarily invariant norm $\|\cdot\|$.

An interesting problem is to study linear preservers of a certain function of singular values and see whether they are of the standard form. Early results include the study of spectral norm preservers by Kadison [**35**], and preservers of some symmetric functions on singular values by Marcus and Gordon [**52**]. The following general results have been obtained in [**43**] and [**18**].

Theorem 3.5. *Let G_0 be the group of linear operators on $M_{m,n}$ of the form*

$$\text{(i) } A \mapsto UAV, \quad \textit{or} \quad \text{(ii) } A \mapsto UA^tV \textit{ if } m = n,$$

for some unitary matrices $U \in M_m$ and $V \in M_n$. Suppose G is a compact group of linear operators on $M_{m,n}$ containing G_0. Then G is either G_0 or the group of unitary operators, i.e., those linear operators preserving the usual inner product $(A, B) = \operatorname{tr}(AB^*)$ on $M_{m,n}$. Consequently, if the linear preservers of a function of singular values on $M_{m,n}$ form a compact group, then the group is either G_0 or the group of unitary operators on $M_{m,n}$.

Using this theorem, one can solve efficiently many linear preserver problems involving functions of singular values. For example, it is known that the set of linear preservers of a norm is a compact group. By Theorem 3.5, the group of linear preservers of a unitarily invariant norm on $m \times n$ complex matrices is either G_0 or the group of unitary operators on the matrix space. Clearly, the latter case can happen only when the norm is a multiple of the Frobenius norm $\|A\| \equiv \{\operatorname{tr}(AA^*)\}^{1/2} = \ell_2(s(A))$.

For real matrices, we have the following result; see [**43**] and [**18**].

Theorem 3.6. *Let G_0 be the group of linear operators on $M_{m,n}(\mathbf{R})$ of the form*

$$\text{(i) } A \mapsto UAV, \quad \textit{or} \quad \text{(ii) } A \mapsto UA^tV \textit{ if } m = n,$$

for some orthogonal matrices $U \in M_m(\mathbf{R})$ and $V \in M_n(\mathbf{R})$. Suppose G is a compact group of linear operators on $M_{m,n}(\mathbf{R})$ containing G_0. Then one of the following holds:

(a) $G = G_0$.

(b) *G is the group of orthogonal operators on $M_{m,n}(\mathbf{R})$.*

(c) *$m = n = 4$, and G is generated by G_0 and the operator ψ defined by*

$$\psi(A) = (A + B_1AC_1 + B_2AC_2 + B_3AC_3)/2$$

with

$$B_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Consequently, if the linear preservers of a function of singular values on $M_{m,n}(\mathbf{R})$ form a compact group, then the group is given by either (a), (b), *or* (c).

As in the complex case, one can use Theorem 3.6 to solve linear preserver problems involving functions of singular values on $M_{m,n}(\mathbf{R})$. Here is an example of a group of preservers satisfying Theorem 3.6 (c): if G is the group of linear preservers of the Ky Fan 2−norm on 4×4 matrices (the sum of the two largest singular values of A), then G is the group satisfying Theorem 3.6 (c); see [**32**] and [**43**].

3.3. Linear preservers of matrix groups and subsets. An algebraic set in $M_n(\mathbf{F})$ is the zero set of a collection of polynomials in the n^2 entries. An example of an algebraic set is the collection of all singular matrices, the zero set of the polynomial $\det(X)$. Other examples of algebraic sets are: the set of singular matrices, the set of matrices with rank at most k, the special linear group, the isometry groups of a given quadratic form, the nilpotent matrices, and the singular matrices with at least k zero eigenvalues. The following examples are not algebraic sets: the set of invertible matrices, the complex unitary group, the stable matrices, and the controllable matrices.

The linear preservers of the complex unitary group $U(n, \mathbf{C})$, which is not an algebraic set, were determined in [**51**].

Theorem 3.7. *Linear preservers of $U(n, \mathbf{C})$ have the form (1) or (3) for some unitary M and N.*

When we consider algebraic groups and algebraic sets, we prefer problems over an algebraically closed field, because the eigenvalues are accessible. There is a method (see [**17**]) for reducing some preserver problems over a general field (or even a ring) to questions over an algebraically closed field. For example, for the field of rational numbers $\mathbf{Q}$, consider the preservers of

$$SL_n(\mathbf{Q}) = \{A \in M_n(\mathbf{Q}) : \det(A) = 1\}.$$

Any linear preserver of $SL_n(\mathbf{Q})$ naturally extends to a linear preserver of $SL_n(\mathbf{C}) = \{A \in M_n : \det(A) = 1\}$. The preservers of $SL_n(\mathbf{C})$ are known and all have standard form; see Theorem 2.1. Then we restrict to $SL_n(\mathbf{Q})$ and see which ones preserve $SL_n(\mathbf{Q})$. The work of Dixon [**17**] completed much of the investigation for algebraic groups of Lie type over fields of characteristic zero. For other algebraic sets, we know, for example, the preservers of the nilpotent matrices [**9**], the singular matrices [**15**], the matrices with rank $\leq k$, [**3**], and matrices with at least k zero eigenvalues [**40**].

A matrix is D-stable if all of its eigenvalues lie in a given region D in the complex plane. Usually, people consider D to be the open left half plane or the open unit disk. Special cases of this problems were studied in [**34**] and [**39**]. The general result was obtained in [**24**, Corollary 3.4]. One may consider a more general problem: Let D be a region in the complex plane, and let S be the set of all $n \times n$ matrices that have exactly k eigenvalues in D. If D is closed and has more than n points, then the preservers of S are all standard. Without these assumptions on D it is possible to construct exceptions, and the preservers of S have not yet been characterized when D is arbitrary.

3.4. Linear preservers of relations. Linear preservers of commutativity were studied in [**63**], [**59**], and [**36**]. The preservers are standard for $n > 2$, but Watkins noted that there are exceptional maps if $n = 2$. The problem for $n > 2$ used the Fundamental Theorem of Projective Geometry (see Section 4.4). Kunicki [**36**] worked on the case $n = 2$. Notice that commuting pairs form an algebraic set in $M_n(\mathbf{F}) \times M_n(\mathbf{F})$.

Researchers have studied linear preservers of similarity, orthogonal similarity, unitary similarity, t-congruence, $*$-congruence, and relations arising in systems theory; see [**28**], [**26**], [**38**], [**41**], [**21**]. Most of the these results were obtained by treating the set of matrices $O(A) = \{X : X \sim A\}$ as a differentiable manifold, and using differential geometry techniques. Furthermore, the authors of [**28**] pointed out that a linear map ϕ preserves $\sim$ if for each matrix A there is another matrix $\tilde{A}$ such that $\phi(O(A)) \subseteq O(\tilde{A})$. A challenging problem is to consider a given A and determine all $\tilde{A}$ such that there exists a linear map ϕ satisfying $\phi(O(A)) \subseteq O(\tilde{A})$, and characterize those linear maps if they exist. Such problems have been considered for the unitary (real orthogonal) equivalence orbit

$$O(A) = \{UAV : U, V \text{ unitary }\}$$

for a (real) rectangular matrix A, and for the unitary similarity orbit

$$O(A) = \{U^*AU : U \text{ unitary }\}$$

for a square matrix A; see [**45**] and [**46**]. In both cases, the matrix $\tilde{A}$ and A must be closely related. For the unitary equivalence orbit, there exists a linear map ϕ such that $\phi(O(A)) = O(\tilde{A})$ for a pair of matrices $A, \tilde{A} \in M_n$ if and only if $\tilde{A}$ has the form μUAV for some scalar μ and unitary U and V; the mapping ϕ has the form (1.1) or (1.3), where M and N are multiples of unitary matrices. For the unitary similarity orbit, there exists a linear map ϕ such that $\phi(O(A)) = O(\tilde{A})$ if and only if $\tilde{A} - (\operatorname{tr} \tilde{A})I/n$ has the form $\mu U^*(A - (\operatorname{tr} A)I/n)U$ for some unitary U and scalar μ; and the restriction of the mapping ϕ on trace zero matrices has the form (1.1) or (1.3), where M and N are multiple of unitary matrices satisfying $MN = \mu I$, and $\phi(I) = (\operatorname{tr} \tilde{A})I/(\operatorname{tr} A)$ if $\operatorname{tr} A \neq 0$.

We do not know how to characterize the linear maps that send one inertia class of Hermitian matrices to another. Clearly, if ϕ is of the form $A \mapsto -SAS^*$ or $A \mapsto$

$-SA^tS^*$ for some invertible S, then ϕ is invertible and maps $G(r,s,t)$ to $G(s,r,t)$, where $G(r,s,t)$ is defined in Section 3.1. Are there other invertible linear maps that send each inertia class to another one? This problem is far from solved.

3.5. Numerical ranges and norms. The numerical range of an $n \times n$ matrix A is

$$W(A) = \{\langle Ax, x\rangle : x \in \mathbf{C},\ \|x\| = 1\}.$$

It is known (see [**58**, Chapter 6]) that linear preservers of the numerical range have the form $A \mapsto U^*AU$ or $A \mapsto U^*A^tU$ for some unitary matrix U. These are all C*-isomorphisms or C*-anti-isomorphisms.

Motivated by theory as well as applications, researchers have studied generalized numerical ranges on Hilbert spaces, Banach spaces, triangular algebras, and symmetry classes of tensors, and the corresponding linear preservers; see [**27**, Chapter1], [**42**], [**14**], [**47**], and references therein. In most cases, one can show that ϕ is of the standard form (1) and (3) for some unitary M and N such that MN is a scalar matrix. Nonetheless, many techniques have been developed to derive the results.

A related subject is the study of linear preservers of the numerical radius

$$r(A) = \max\{|z| : z \in W(A)\},$$

and preservers of the generalized numerical radii (which is the maximum norm of scalars or vectors in the corresponding generalized numerical ranges). In most cases, numerical radius preservers are just unit multiples of the corresponding numerical range preservers, but the proofs are usually much more involved. It is also worth noting that numerical range preservers are related to norms or semi-norms on square matrices that are invariant under unitary similarity or unitary equivalence; see [**23**] and [**37**].

4. GENERAL TECHNIQUES AND SAMPLE PROOFS. In this section, we mention some methods and ideas that have been successful in solving linear preserver problems. We give sketches of proofs of some results to illustrate the techniques.

4.1. Elementary Linear Algebra and Tensor Product. Basic matrix and operator theoretic techniques can be used. For example, $M_n(\mathbf{F})$ is an n^2-dimensional linear space over $\mathbf{F}$. Denote by $\mathcal{B} = \{E_{11}, E_{12}, \ldots, E_{nn}\}$ the standard basis for $M_n(\mathbf{F})$. Then every matrix A can be regarded as a vector in $\mathbf{F}^r$ with $r = n^2$ and every linear operator on $M_n(\mathbf{F})$ can be regarded as a matrix in $M_r(\mathbf{F})$ with respect to $\mathcal{B}$. One can check that a linear operator ϕ has the standard form (1) for some $M, N \in M_n$ if and only if the matrix representation of ϕ is

$$M \otimes N^t = \left(m_{ij}N^t\right).$$

Therefore, one can try to show that the matrix representation of a certain linear preserver ϕ has a matrix form $M \otimes N^t$ in order to conclude that ϕ has the standard form (1); similarly, if one can show that the modified transformation $A \mapsto \phi(A^t)$ has the matrix representation $M \otimes N^t$ then ϕ has the standard form (3). Early papers such as [**51**] and [**53**] used this method to find the preservers of the complex unitary group (Theorem 3.7) and the set of complex rank one matrices (Theorem 3.2), respectively.

4.2. Reduction and Extreme Points Techniques. Once one has basic linear preserver results in hand, one can try to reduce new linear preserver problems to the known ones, and use the existing results. To illustrate this technique, we give the sketch of the following result.

Theorem 4.1. *A linear operator on M_n preserves the spectral norm $\|\cdot\|$ if and only if it has the form*

$$A \mapsto UAV \qquad or \qquad A \mapsto UA^tV$$

for some unitary matrices $U, V \in M_n$.

Since the set of extreme points of the unit ball of the spectral norm in M_n is the set of unitary matrices, ϕ preserves the spectral norm if and only if ϕ maps the set of unitary matrices onto itself. The result follows form Theorem 3.7.

Similarly, to determine the structure of linear operators ϕ on M_n preserving the trace norm, one can use the fact that the set of extreme points of the unit ball of the trace norm in M_n is the set of rank one matrices with spectral norm equal to one. Hence, if ϕ preserves the spectral norm, then ϕ maps the set of rank one matrices into itself. Theorem 3.2 ensures that ϕ has the standard form (1) or (3) for $M, N \in M_n$. It is then easy to show that M and N can be taken to be unitary.

The extreme point technique is commonly used in studying linear operators preserving norms on matrices. For other problems, one may use other reduction techniques. For example, denote by $E_k(A)$ the kth elementary symmetric function of the eigenvalues of $A \in M_n(\mathbf{F})$ for an algebraically closed field $\mathbf{F}$. For $k \geq 4$, a matrix $A \in M_n(\mathbf{F})$ has rank one if and only if $E_k(xA + B)$ is a polynomial in x with degree at most one for all $B \in M_n(\mathbf{F})$. Hence, if ϕ preserves E_k on $M_n(\mathbf{F})$, then ϕ maps the set of rank one matrices into itself. Again, Theorem 3.2 ensures that ϕ has the standard form (1) or (3) for $M, N \in M_n$. It is then easy to show that if $k < n$ then M and N satisfy $MN = \mu I$ for some $\mu \in \mathbf{F}$ with $\mu^k = 1$.

Another common reduction method is to show that a given linear preserver on $M_n(\mathbf{F})$ is nonsingular and maps the set of nilpotent matrices into itself, and then apply the result on nilpotent preservers [**9**]. We refer the readers to the papers [**39**], [**40**], and [**34**] for more illustrations of this reduction.

4.3. Duality and Group Theory Techniques. The duality technique has been discussed in [**44**]. The basic idea is to obtain information about a linear preserver by studying its dual transformation. Sometimes, the dual transformation is easier to characterize; sometimes, the dual transformation is itself an interesting linear preserver. In any event, studying the linear preserver and its dual transformation often provides useful information. For example, the dual transformation of a linear preserver of the spectral norm on M_n is a trace norm preserver. Thus, knowing the structure of the linear preserver gives complete information of the dual transformation and vice versa. See [**14**], [**46**], and their references for more illustrations of duality techniques.

Next, we turn to a group theory technique. The basic idea is to show that the linear preservers of a certain invariant form a group G of invertible operators. If G contains a fairly "big" subgroup H of "obvious" admissible preservers, then the theory of Lie groups can give us a limited list of groups containing H; thus, there are not too many candidates for G. This group scheme is originally due to Dynkin [**19**], who studied determinant preservers and showed that

(i) the group G of determinant preservers contains the subgroup H of operators of the form (1) and (3) for some matrices M and N satisfying $\det(MN) = 1$, and

(ii) H is a maximal subgroup in the special linear group of operators on square matrices.

Since not all invertible linear operators on square matrices preserve determinant, it follows that $H = G$.

One can use the same idea to treat the spectral norm preserver problem. Using Theorem 3.5, one can readily deduce Theorem 4.1.

The group scheme for linear preserver problems has been exploited in [**17**], [**18**], [**60**], and [**22**].

4.4. Fundamental Theorem of Projective Geometry. Geometric techniques from ordinary Euclidean geometry, projective geometry, and algebraic geometry have been successful in studying linear preserver problems. For an overview, see [**58**, Chapters 2, 4, 8] and their references. In Section 3.4, we briefly described the idea of treating the orbit $O(A) = \{X : X \sim A\}$ of a certain equivalence relation on a matrix space as a differentiable manifold, this permits differential geometry techniques to be used.

Here we give a brief discussion of Dieudonné's method [**15**] to study the invertible preservers of the set of singular matrices.

We first give a statement of the Fundamental Theorem of Projective Geometry in the language of linear algebra. Let V be a vector space of dimension ≥ 3. For our purposes, a *line* in V is a one-dimensional subspace of V. Suppose that ϕ is a bijection of the lines of V such that for any three coplanar lines in V their images under ϕ are also coplanar. Then there is a semi-linear (additive linear) map T on V such that $T(W) = \phi(W)$ for every line W in V.

First, Dieudonné showed that a subspace of $M_n(\mathbf{F})$ of dimension $n^2 - n$ consisting only of singular matrices has a common left or right null space of dimension one. Let L be an invertible linear map that preserves the singular matrices in $M_n(\mathbf{F})$. For each line $W \in \mathbf{F}^n$, let $N(W)$ be the set of all matrices that annihilate W, and let $N^t(W)$ be its transpose. It follows that $L(N(W)) = N(W_1)$ or $N^t(W_1)$ for some line $W_1 \in V$. Another dimension argument shows that $L(N(W)) = N(W_1)$ or $N^t(W_1)$ (uniformly) for every line W. Dieudonné then used this fact to produce a bijective correspondence of the lines of $\mathbf{F}^n$ that satisfies the hypotheses of the Fundamental Theorem of Projective Geometry. Next, he applied this information to show that all invertible linear preservers of singular matrices are standard. One can readily verify the same result when $n = 2$. Other papers using similar arguments along with the Fundamental Theorem of Projective Geometry include [**9**] and [**59**].

4.5. Additional References on General Approaches. Other approaches to linear preserver problems include the use of functional identities and model theoretic algebra; see [**10**], [**24**], and [**55**] for details.

5. EXCEPTIONS. Usually, a well posed linear preserver problem has only standard transformations in its solution. There are, however, exceptions, which arise for various reasons. Among these reasons are

(i) The size n of the matrices is too small.

(ii) The underlying field F of scalars is too small or is not algebraically closed.

(iii) The linear preserver is allowed to be singular.

(iv) The set being preserved contains a special geometric structure, for example, a vector space or convex cone.

Of course, some exceptions fall into more than one of these categories and some are just "mysterious". In this section, we give several examples of such exceptions and briefly review some of the related literature.

Exceptions for small n arise, especially when $n = 2$. One reason for this is that the Fundamental Theorem of Projective Geometry is not valid. Consider the following problem. Let $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ be a non-singular linear transformation that preserves commutativity: if $AB = BA$, then $T(A)T(B) = T(B)T(A)$. Watkins [**63**] showed that if $n \geq 4$ and F has at least 4 elements, then T has the standard forms (4). Later, Pierce and Watkins [**59**] obtained the same result for any field and for any $n \geq 3$. They used the Fundamental Theorem of Projective Geometry, and the proof did not work for $n = 2$. In fact, the following linear map on $M_2(\mathbf{R})$ provides an exceptional case:

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & a-d \\ c & b+d \end{bmatrix}.$$

This T is non-singular, preserves commutativity, and does not have the standard form. See [**36**] for a discussion of the case $n = 2$.

Now let $\mathbf{F}$ be an algebraic number field, that is, a finite extension of the rationals. Given n, there exist polynomials of degree n that are irreducible over $\mathbf{F}$. Let $p(x) \in \mathbf{F}[x]$ be an irreducible monic polynomial of degree n and let C be the companion matrix of $p(x)$. Define a linear map T on $M_n(\mathbf{F})$ by

$$T(A) = \sum_{j=1}^{n-1} a_{1j} C^j.$$

Since $p(x)$ is irreducible, no eigenvalue of $T(A)$ is zero as long as the first row of A is non-zero. Thus T preserves $GL_n(\mathbf{F})$, the collection of invertible matrices over $\mathbf{F}$. Of course, T is singular.

If $F = \mathbf{C}$, then the preservers of $GL_n(\mathbf{C})$ have the standard form (1) or (3) for some invertible matrices M and N; see [**54**]. Thus, all preservers of $GL_n(\mathbf{C})$ must be non-singular. Strikingly, if

$$SL_n(\mathbf{F}) = \{A \in GL_n(\mathbf{F}) : \det(A) = 1\},$$

where $\mathbf{F}$ is a subfield of $\mathbf{C}$, we cannot contrive any similar examples for the preservers of $SL_n(\mathbf{F})$. The difference is that $SL_n(\mathbf{F})$ is an algebraic set, while $GL_n(\mathbf{F})$ is not. By the going-up theorem of Dixon [**17**], any linear preserver of $SL_n(\mathbf{F})$ extends to a linear preserver of $SL_n(\mathbf{C})$ and this extension preserves $GL_n(\mathbf{C})$ as well. Thus a preserver of $SL_n(\mathbf{F})$ must be non-singular.

There are similar exceptions for preservers of the real orthogonal group $O_n(\mathbf{R})$. If T preserves $O_n(\mathbf{R})$, it is possible for T to be singular when $n = 2, 4$, or 8; see [**67**]. This is because $M_n(\mathbf{R})$ admits (respectively) the natural embedding of the complex numbers, quaternions, and Cayley numbers in those dimensions. For example, the map on $M_4(\mathbf{R})$ given by

$$T\begin{bmatrix} a & b & c & d \\ & & & \\ & * & * & \\ & & & \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix}$$

preserves $O_4(\mathbf{R})$. If $n \neq 2, 4$, or 8, then any preserver of $O_n(\mathbf{R})$ must be non-singular and must have the form $A \mapsto UAV$ or $A \mapsto UA^tV$ for some orthogonal matrices U and V.

Let P_n be the convex cone of all positive definite Hermitian matrices in $M_n(\mathbf{F})$, where $\mathbf{F} \subseteq \mathbf{C}$. Any non-singular congruence preserves P_n and thus any sum of non-singular congruences preserves $P_{n.}$ Let $A_1, \ldots, A_r$ be invertible matrices in $M_n(F)$. The linear map T given by

$$T(X) = \sum_{j=1}^{r} A_j^* X A_j$$

is non-singular and preserves P_n, but in general it cannot be reduced to a single congruence. Moreover, the linear map on $M_3(\mathbf{R})$ given by

$$T(A) = \begin{bmatrix} a_{11} + a_{33} & -a_{12} & -a_{13} \\ -a_{12} & a_{11} + a_{22} & -a_{23} \\ -a_{13} & -a_{23} & a_{22} + a_{33} \end{bmatrix}$$

preserves P_3, but is not a sum of congruences. The preservers of P_n are not known although the preservers of any given unbalanced indefinite inertia class have been found; see Section 3.1.

The exceptional situations cited thus far occur with preservers of certain subsets of $M_n(\mathbf{F})$. For preservers of functions on singular values, exceptional cases often occur in $M_4(\mathbf{R})$ because it admits embedding of real quaternions in interesting ways; see [**32**]. For preservers of other functions on singular values of A, exceptional cases occur if the given function is linear or quadratic in the entries of A. For example, consider the second elementary symmetric function $E_2(A)$ of the eigenvalues of the matrix A. Since this is quadratic in the entries of A, the isometries of E_2 must consist of more than similarity and transposition. For example, if $n \geq 3$, a switch of the $(1, 2)$ and $(2, 1)$ entries of A preserves E_2, but is not of the form (1) or (3) for some M and N satisfying $MN = \pm I$. The preservers of E_2 form a group. A discussion of a generating set for this group is found in [**59**] and later in [**31**]. Otherwise, preservers of functions on eigenvalues or singular values of matrices usually have standard form without exception; see Section 3.2.

6. CONCLUSION. We have given a very brief introduction and survey of the subject. There are many other directions that one may explore. For example, one may consider quadratic or bilinear preservers [**65**] and [**66**], one may consider general (nonlinear) preservers of matrix invariants [**5**], and preservers on matrices over rings or integral domains [**4**], [**10**], [**48**], [**58**, Chapter 8], and [**62**]. The subject of linear preservers will continue to prosper and will provide fertile grounds for researchers in many areas to make new discoveries.

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