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Joint matricial range and joint congruence matricial range of operators

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Abstract Let $\mathbf{A} = (A_1, \ldots, A_m)$, where $A_1, \ldots, A_m$ are $n \times n$ real matrices. The real joint (p, q)-matricial range of $\mathbf{A}$, $\Lambda^{\mathbb{R}}_{p,q}(\mathbf{A})$, is the set of m-tuple of $q \times q$ real matrices $(B_1, \ldots, B_m)$ such that $(X^*A_1X, \ldots, X^*A_mX) = (I_p \otimes B_1, \ldots, I_p \otimes B_m)$ for some real $n \times pq$ matrix X satisfying $X^*X = I_{pq}$. It is shown that if n is sufficiently large, then the set $\Lambda^{\mathbb{R}}_{p,q}(\mathbf{A})$ is non-empty and star-shaped. The result is extended to bounded linear operators acting on a real Hilbert space $\mathcal{H}$, and used to show that the joint essential (p, q)-matricial range of $\mathbf{A}$ is always compact, convex, and non-empty. Similar results for the joint congruence matricial ranges on complex operators are also obtained.

Keywords Congruence numerical range · star-shaped · convex · compact perturbation

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1 Introduction

Let $\mathbb{F}$ be the field $\mathbb{R}$ of real numbers and the field $\mathbb{C}$ of complex numbers. Denote by $\mathbf{M}_n(\mathbb{F})$ the set of $n \times n$ matrices with elements in $\mathbb{F}$ and $\mathbf{M}^m_n(\mathbb{F})$ the set of m-tuple of matrices in $\mathbf{M}_n(\mathbb{F})$. The joint numerical range of $\mathbf{A} = (A_1, ..., A_m) \in \mathbf{M}^m_n(\mathbb{C})$ is defined by

$$W(\mathbf{A}) = \{(x^*A_1x, \ldots, x^*A_mx) : x \in \mathbb{C}^n,\ x^*x = 1\}. \tag{1.1}$$

The joint numerical range is useful in studying problems involving a collection of matrices that arise naturally in pure and applied areas, see [9, 16, 30] and

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their references. When $m = 1$, $W(\mathbf{A}) = W(A_1)$ reduces to the classical numerical range of $A_1 \in \mathbf{M}_n(\mathbb{C})$, and the joint numerical range is one of its many generalizations, see [12, Chapter 1]. In particular, researchers also consider the real joint numerical range $W^{\mathbb{R}}(\mathbf{A})$ for $\mathbf{A} = (A_1, \ldots, A_m) \in \mathbf{M}_n^m(\mathbb{R})$, which is defined analogous to (1.1) using real vectors and real matrices; see [2,4,5,8, 20,21,23]. In this case, the conjugate transpose X^* is just the transpose X^t for any $X \in \mathbf{M}_n(\mathbb{R})$.

In theories as well as applications, it is useful to study the geometric properties of the joint numerical ranges such as the convexity and the star-shapedness. The results on the complex case and the real case could be similar or very different. We elaborate this comment in the following.

For complex matrices $A_1, ..., A_m \in \mathbf{M}_n(\mathbb{C})$, by the Hermitian decomposition, $W(A_1, \ldots, A_m) \subseteq \mathbb{C}^m$ can be identified by $W(H_1, K_1, \ldots, H_m, K_m) \subseteq \mathbb{R}^{2m}$, where $A_j = H_j + iK_j$ with $(H_j, K_j) = (H_j^*, K_j^*)$ for $j = 1, ..., m$. Therefore, one can focus on m-tuple of Hermitian matrices in the study of the geometric properties of $W(\mathbf{A})$. The Toeplitz-Hausdorff Theorem asserts that $W(A_1, A_2)$ is always convex for any Hermitian matrices A_1, A_2, see [10,28]. Au-Yeung and Poon [3] showed that $W(A_1, A_2, A_3)$ is always convex for Hermitian matrices $A_1, A_2, A_3 \in \mathbf{M}_n(\mathbb{C})$ if $n \geq 3$. In general, $W(A_1, \ldots, A_m)$ may fail to be convex if $m \geq 4$, for example, see [16]. In fact, it is possible to construct examples such that no three points in $W(\mathbf{A})$ are collinear. For example, one may let $\{A_1, \ldots, A_m\}$ be such that $\{I/\sqrt{n}, A_1, \ldots, A_m\}$ is an orthonormal basis of $\mathbf{M}_n(\mathbb{C})$ under the usual inner product $\langle A, B\rangle = \mathrm{tr}(AB^*)$. Then one can verify that every element $(\mu_1, \ldots, \mu_m) \in W(\mathbf{A})$ satisfies $\sum_{j=1}^n |\mu_j|^2 = 1 - 1/n$. Thus, no three points in $W(\mathbf{A})$ are collinear. In [15], the authors obtained the unexpected result that if n is sufficiently large, then $W(A_1, \ldots, A_m)$ is always star-shaped.

For real matrices $A_1, ..., A_m \in \mathbf{M}_n(\mathbb{R})$, one can decompose $A_j = S_j + G_j$, where S_j is real symmetric and G_j is real skew-symmetric for $j = 1, ..., m$. Since $v^t G_j v = 0$ for all real vectors $v \in \mathbb{R}^n$, we see that $W^{\mathbb{R}}(A_1, \ldots, A_m) = W^{\mathbb{R}}(S_1, \ldots, S_m)$. Brickman [5] showed that $W^{\mathbb{R}}(S_1, S_2)$ is always convex for symmetric matrices $S_1, S_2 \in \mathbf{M}_n(\mathbb{R})$ if $n \geq 3$. However, $W^{\mathbb{R}}(S_1, S_2, S_3)$ may fail to be convex in general. There are examples for which no three points in $W^{\mathbb{R}}(S_1, \ldots, S_m)$ are collinear. One may wonder if $W^{\mathbb{R}}(S_1, \ldots, S_m)$ is star-shaped when n is sufficiently large. We will answer this question in the affirmative by proving more general results.

Let $\mathbf{A} = (A_1, ..., A_m) \in \mathbf{M}_n^m(\mathbb{R})$ be an m-tuple of real matrices. The real joint (p, q)-matricial range of $\mathbf{A}$ is defined by

$$\Lambda_{p,q}^{\mathbb{R}}(\mathbf{A}) = \{(B_1, \ldots, B_m) \in \mathbf{M}_q^m(\mathbb{R}) : \ X^t A_j X = I_p \otimes B_j, \\ X^t X = I_{pq}, \ j = 1, \ldots, m\}.$$

In other words, $I_p \otimes B_1, \ldots, I_p \otimes B_m$ are compressions of $A_1, \ldots, A_m$ to a pq-dimensional subspace of $\mathbb{R}^n$. When $p = q = 1$, $\Lambda_{1,1}^{\mathbb{R}}(\mathbf{A})$ reduces to the real joint numerical range $W^{\mathbb{R}}(A_1, \ldots, A_m)$. The set $\Lambda_{p,q}^{\mathbb{R}}(\mathbf{A})$ is the real analog of the joint (complex) (p, q)-matricial range of $\mathbf{A} \in \mathbf{M}_n^m(\mathbb{C})$ defined as

$$\Lambda_{p,q}(\mathbf{A}) = \{(B_1, \ldots, B_m) \in \mathbf{M}_q^m(\mathbb{C}) : \ X^* A_j X = I_p \otimes B_j, \\ X^* X = I_{pq}, \ j = 1, \ldots, m\}.$$

The study of $\Lambda_{p,q}(\mathbf{A})$ was motivated by the search of quantum error correction code for a given quantum channel, for example, see [13,17]. In [13] it was shown that if the dimension n is sufficiently large, then the complex joint (p, q)-matricial range is non-empty and star-shaped. Consequently, if the underlying Hilbert space of a quantum system has sufficiently high dimension, one can always find a correctable subsystem for a quantum channel; see [6] and [17] more details.

In this paper, we establish the non-emptyness and the star-shapedness result for $\Lambda^{\mathbb{R}}_{p,q}(\mathbf{A})$ when n is sufficiently large. Similar results are obtained for the joint congruence (p,q)-matricial range of $\mathbf{A} \in \mathbf{M}_n^m(\mathbb{C})$ defined by

$$\Lambda^{\mathbb{C}}_{p,q}(\mathbf{A}) = \{(B_1,\dots,B_m) \in \mathbf{M}_q^m(\mathbb{C}) : \ X^t A_j X = I_p \otimes B_j, \\ X^*X = I_{pq},\ j = 1,\dots,m\}.$$

When $p = q = m = 1$, $\Lambda^{p,q}(A_1)$ reduces the congruence numerical range considered in [7,11,18,19,25–27]; the study is related to the action of unitary congruence $A \mapsto U^t AU$ on complex matrices A. Furthermore, we extend the results to infinite dimensional operators.

Our paper is organized as follows. In Section 2, we obtain lower bounds on the dimension n for which $\Lambda^{\mathbb{R}}_{p,q}(\mathbf{A})$ is non-empty and star-shaped. In Section 3, we obtain similar results for the joint congruence (p,q)-matricial range of $\mathbf{A} \in \mathbf{M}_n^m(\mathbb{C})$. In Section 4, we extend the results in Sections 2 and 3 to bounded linear operators acting on infinite dimensional Hilbert space $\mathcal{H}$ over $\mathbb{F}$. In particular, we define the joint (p,q)-matricial range and the joint congruence (p,q)-matricial range for bounded linear operators, respectively, depending on $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. In addition, we use the results to show that the joint essential (p,q)-matricial for real operators and the joint essential congruence (p,q)-matricial range of complex operators are always compact, convex, and non-empty.

2 The Real Joint (p,q)-Matricial Range

Denote by $\mathbf{M}_{n,r}(\mathbb{R})$ the set of all $n \times r$ real matrices and denote by $\mathcal{P}_{n,r}$ the set of $n \times r$ real matrices X such that $X^tX = I_r$. We abbreviate $\mathbf{M}_{n,n}(\mathbb{R})$ and $\mathcal{P}_{n,n}$ to $\mathbf{M}_n(\mathbb{R})$ and $\mathcal{P}_n$ respectively. Note that $\mathcal{P}_n$ is the group of all orthogonal matrices. Recall that for $\mathbf{A} = (A_1,\dots,A_m) \in \mathbf{M}_n^m(\mathbb{R})$, its real joint (p,q)-matricial range is defined by

$$\Lambda^{\mathbb{R}}_{p,q}(\mathbf{A}) = \{(B_1,\dots,B_m) : X^t A_j X = I_p \otimes B_j,\ X \in \mathcal{P}_{n,pq}\}.$$

Let $A_j = S_j + G_j$, S_j be real symmetric and G_j be real skew-symmetric, $j = 1,...,m$. Then $\Lambda^{\mathbb{R}}_{p,q}(\mathbf{A})$ has the same structure as $\Lambda^{\mathbb{R}}_{p,q}(S_1,\dots,S_m,G_1,\dots,G_m)$. Suppose that $\{\tilde{S}_1,\dots,\tilde{S}_u\}$, $\{\tilde{G}_1,\dots,\tilde{G}_v\}$ are bases of $\mathrm{span}\{S_1,\dots,S_m\}$ and $\mathrm{span}\{G_1,\dots,G_m\}$, respectively. Then there are $(t_{ij}) \in \mathbf{M}_{u,m}(\mathbb{R})$ and $(s_{ij}) \in \mathbf{M}_{v,m}(\mathbb{R})$ such that $S_j = \sum_{i=1}^u t_{ij}\tilde{S}_i$ and $G_j = \sum_{i=1}^v s_{ij}\tilde{G}_i$ for $j = 1,\dots,m$. It is clear that $(B_1,\dots,B_m,C_1,\dots,C_m) \in \Lambda^{\mathbb{R}}_{p,q}(S_1,\dots,S_m,G_1,\dots,G_m)$ if and only if $B_j = \sum_{i=1}^u t_{ij}\tilde{B}_i$ and $C_j = \sum_{i=1}^v s_{ij}\tilde{C}_i$, $j = 1,\dots,m$, for some $(\tilde{B}_1,\dots,\tilde{B}_u,\tilde{C}_1,\dots,\tilde{C}_v) \in \Lambda^{\mathbb{R}}_{p,q}(\tilde{S}_1,\dots,\tilde{S}_u,\tilde{G}_1,\dots,\tilde{G}_v)$. Therefore, in order to study the geometrical properties of $\Lambda_{p,q}(S_1,\dots,S_m,G_1,\dots,G_m)$, we may focus on $\Lambda^{\mathbb{R}}_{p,q}(\tilde{S}_1,\dots,\tilde{S}_u,\tilde{G}_1,\dots,\tilde{G}_v)$ where $\{\tilde{S}_1,\dots,\tilde{S}_u,\tilde{G}_1,\dots,\tilde{G}_v\}$ forms a linearly independent set.

For real symmetric matrices, one can adapt the proof in [15, Proposition 2.4] to deduce the following. It is clear that for $\mathbf{A} \in \mathbf{M}_n^m(\mathbb{R})$, $(s_1I_q,\dots,s_mI_q) \in \Lambda^{\mathbb{R}}_{p,q}(\mathbf{A})$ if and only if $(s_1,\dots,s_m) \in \Lambda^{\mathbb{R}}_{pq,1}(\mathbf{A})$.

Proposition 2.1 *Let $S_1,\dots,S_m \in \mathbf{M}_n(\mathbb{R})$ be real symmetric matrices. If $n \geq (m+1)^2(pq-1)$, then there are real numbers $s_1,\dots,s_m$ such that*

$$(s_1,\dots,s_m) \in \Lambda^{\mathbb{R}}_{pq,1}(S_1,\dots,S_m).$$

Clearly, if $G_1, ..., G_m \in \mathbf{M}_n(\mathbb{R})$ are skew-symmetric, then $\Lambda^{\mathbb{R}}_{p,1}(G_1, \dots, G_m) \subseteq \{(0, \dots, 0)\}$. In general, we have the following proposition. In its proof, we will denote by $0_{n,r}$ the $n \times r$ zero matrix and 0_n the $n \times n$ zero matrix.

Proposition 2.2 *Let $G_1, \dots, G_m \in \mathbf{M}_n(\mathbb{R})$ be real skew-symmetric matrices. If $n \geq 2^m pq$, then*

$$\Lambda^{\mathbb{R}}_{pq,1}(G_1, \dots, G_m) = \{(0, ..., 0)\}.$$

Proof. "$\subseteq$" is clear. Now consider "$\supseteq$". We first assume $m = 1$. It suffices to consider the case $n = 2pq$. By spectral decomposition of real skew-symmetric matrices, there is an orthogonal matrix $W \in \mathcal{P}_n$ such that $W^t G_1 W = B_1 \oplus B_2 \oplus \cdots \oplus B_{pq}$ where $B_j = \begin{bmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{bmatrix}$, $\lambda_j \in \mathbb{R}$ and $j = 1, ..., pq$. Then 0_{pq} is the principal submatrix of $W^t G_1 W$ lying in the rows and the columns indexed by $1, 3, \dots, 2pq - 1$. Hence $0 \in \Lambda^{\mathbb{R}}_{pq,1}(G_1)$.

Let $m > 1$ and $n \geq 2^m pq$. We apply an inductive argument. First, there is an orthogonal $U \in \mathcal{P}_n$ such that $U^t G_m U$ has $0_{2^{m-1}pq}$ as the leading principal submatrix. Replace G_j by $U^t G_j U$ for all $j = 1, \dots, m$. Let $\hat{G}_j \in \mathbf{M}_{2^{m-1}pq}(\mathbb{R})$ be the leading principal submatrix of G_j for $j = 1, \dots, m$. Then $\hat{G}_m = 0_{2^{m-1}pq}$. By induction, there is an orthogonal matrix $V \in \mathcal{P}_{2^{m-1}pq}$ such that $V^t \hat{G}_j V$ has leading principal submatrix 0_{pq} for $j = 1, \dots, m-1$. Thus, if $W = V \oplus I \in \mathcal{P}_{2^m pq}$, then $W^t G_j W$ has leading principal submatrix 0_{pq} for $j = 1, \dots, m$. It follows that $(0, ..., 0) \in \Lambda^{\mathbb{R}}_{pq,1}(G_1, \dots, G_m)$ ■

For $\mathbf{A} \in \mathbf{M}^m_n(\mathbb{R})$, it is clear that $(a_1, ..., a_m) \in \Lambda^{\mathbb{R}}_{pq,1}(\mathbf{A})$ for real $a_1, ..., a_m$ if and only if $(a_1 I_q, ..., a_m I_q) \in \Lambda^{\mathbb{R}}_{p,q}(\mathbf{A})$. Combining Proposition 2.1 and Proposition 2.2, we have the following.

Theorem 2.1 *Let $\mathbf{A} = (S_1, \dots, S_{m_1}, G_1, \dots, G_{m_2}) \in \mathbf{M}^{m_1+m_2}_n(\mathbb{R})$ where $S_1, \dots, S_{m_1}$ are real symmetric and $G_1, \dots, G_{m_2}$ are real skew-symmetric. If $n \geq 2^{m_2}(m_1+1)^2(pq-1)$, then there are real numbers $s_1, \dots, s_{m_1+m_2}$ with $s_{m_1+1} = \cdots = s_{m_1+m_2} = 0$ such that*

$$(s_1, \dots, s_{m_1+m_2}) \in \Lambda^{\mathbb{R}}_{pq,1}(\mathbf{A}) \quad \textit{and} \quad (s_1 I_q, \dots, s_{m_1+m_2} I_q) \in \Lambda^{\mathbb{R}}_{p,q}(\mathbf{A}).$$

Proof. If $n \geq 2^{m_2}(m_1 + 1)^2(pq - 1)$, then by Proposition 2.2, there is an orthogonal matrix $U \in \mathcal{P}_n$ such that the leading principal submatrix $U^t G_j U$ is 0_r with $r = (m_1 + 1)^2(pq - 1)$, $j = 1, ..., m_2$. Let $\tilde{S}_j \in \mathbf{M}_r(\mathbb{R})$ be the leading $r \times r$ principal submatrix of $U^t S_j U$. By Proposition 2.1, there are real numbers $s_1, \dots, s_{m_1}$ and $V \in \mathcal{P}_r$ such that $s_j I_{pq}$ is the leading principal submatrix of $V^t \tilde{S}_j V$ for $j = 1, \dots, m_1$. Let $W = U(V \oplus I_{n-r})$ and $s_{m_1+1} = \cdots s_{m_1+m_2} = 0$. Then $s_1 I_{pq}, ..., s_{m_1+m_2} I_{pq}$ are the leading principal submatrices of $W^t S_1 W, ..., W^t S_{m_1} W$, $W^t G_1 W, ..., W^t G_{m_2} W$ respectively. The result follows. ■

Theorem 2.2 *Let $\mathbf{A} = (A_1, \dots, A_m) \in \mathbf{M}^m_n(\mathbb{R})$, where $A_1, \dots, A_m$ are real symmetric or real skew-symmetric. Suppose*

$$\mathbf{C} = (C_1, ..., C_m) \in \Lambda^{\mathbb{R}}_{p,q}(Y^t A_1 Y, ..., Y^t A_m Y) \quad \textit{for all} \quad Y \in \mathcal{P}_{n,n-pq(m+1)}.$$

Then $\mathbf{C}$ is a star-center of $\Lambda^{\mathbb{R}}_{p,q}(\mathbf{A})$.

Proof. Suppose $\mathbf{B} = (B_1, ..., B_m) \in \Lambda^{\mathbb{R}}_{p,q}(\mathbf{A})$. Then there is an $X_1 \in \mathcal{P}_{n,pq}$ such that $X_1^t A_j X_1 = I_p \otimes B_j$ for $j = 1, ..., m$. Extend the X_1 to an orthogonal matrix $U_1 \in \mathcal{P}_n$. Then

$$U_1^t A_j U_1 = \begin{bmatrix} I_p \otimes B_j & R_j \\ \tilde{R}_j & * \end{bmatrix}, \quad \text{where} \quad \tilde{R}_j = R_j^t \ \text{ or } \ \tilde{R}_j = -R_j^t, \quad j = 1, \dots, m.$$

Note that there is a subspace of dimension at most pqm containing the range spaces of $\tilde{R}_1, ..., \tilde{R}_m$. Therefore, there exists an $V \in \mathcal{P}_{n-pq}$ such that $R_jV = [0_{pq,n-pq(m+1)}|S_j]$ where $S_j \in \mathbf{M}_{pq,pqm}(\mathbb{R})$, $j = 1, ..., m$. Then

$$(I_{pq} \oplus V)^t U_1^t A_j U_1 (I_{pq} \oplus V) = \begin{bmatrix} I_p \otimes B_j & R_jV \\ V^t\tilde{R}_j & * \end{bmatrix} = \begin{bmatrix} I_p \otimes B_j & 0 & S_j \\ 0 & \hat{C}_j & * \\ \tilde{S}_j & * & * \end{bmatrix},$$

where $\tilde{S}_j = S_j^t$ or $\tilde{S}_j = -S_j^t$; and $\hat{C}_j \in \mathbf{M}_{n-pq(m+1)}(\mathbb{R})$, $j = 1, \ldots, m$. By assumption, we have $\mathbf{C} \in \Lambda^{\mathbb{R}}_{p,q}(\hat{C}_1, ..., \hat{C}_m)$. Then there is an $X_2 \in \mathcal{P}_{n-pq(m+1),pq}$ such that $X_2^t \hat{C}_j X_2 = I_p \otimes C_j$, $j = 1, ..., m$. Extend X_2 to $U_2 \in \mathcal{P}_{n-pq(m+1)}$. For every $0 \le \alpha \le 1$, let $W = U_1(I_{pq} \oplus V)(I_{pq} \oplus U_2 \oplus I_{pqm})[\alpha I_{pq}|\sqrt{1-\alpha^2} I_{pq}|0] \in \mathcal{P}_{n,pq}$. Then

$$W^t A_j W = I_p \otimes (\alpha B_j + (1-\alpha)C_j), \quad j = 1, ..., m.$$

Hence, $\alpha\mathbf{B} + (1-\alpha)\mathbf{C} \in \Lambda^{\mathbb{R}}_{p,q}(A_1, \ldots, A_m)$ for all $0 \le \alpha \le 1$ and $\mathbf{C}$ is a star-center of $\Lambda^{\mathbb{R}}_{p,q}(\mathbf{A})$. ■

Theorem 2.3 *Let $\mathbf{A} = (A_1, \ldots, A_m) \in \mathbf{M}_n^m(\mathbb{R})$ and p, q, r be positive integers. If $1 \le qr < p \le n$, then*

$$\Lambda^{\mathbb{R}}_{p,q}(\mathbf{A}) \subseteq \bigcap \left\{ \Lambda^{\mathbb{R}}_{p-qr,q}(Y^t\mathbf{A}Y) : Y \in \mathcal{P}_{n,n-r} \right\}$$

where $Y^t\mathbf{A}Y = (Y^tA_1Y, \ldots, Y^tA_mY)$.

Proof. We start with the case when $r = 1$, that is

$$\Lambda^{\mathbb{R}}_{p,q}(\mathbf{A}) \subseteq \bigcap \left\{ \Lambda^{\mathbb{R}}_{p-q,q}(Y^t\mathbf{A}Y) : Y \in \mathcal{P}_{n,n-1} \right\}.$$

Let $Y \in \mathcal{P}_{n,n-1}$. Extend Y to an orthogonal matrix $U = [Y|y] \in \mathcal{P}_n$. Let $\mathbf{C} = (C_1, ..., C_m) \in \Lambda^{\mathbb{R}}_{p,q}(\mathbf{A})$. Then there is an $X = [X_1|...|X_p] \in \mathcal{P}_{n,pq}$ with $X_1, ..., X_p \in \mathcal{P}_{n,q}$ such that $X_\ell^t A_j X_k = \delta_{\ell,k} C_j$, $j = 1, ..., m$ and $\ell, k = 1, ..., p$. Here $\delta_{\ell,k}$ is the Kronecker delta function. Define the $q \times p$ real matrix

$$Q = \left[X_1^t y \; X_2^t y \; \cdots \; X_p^t y \right].$$

Note that the nullity of Q is at least $p - q$. Then there exists a $W = [w_{ij}] \in \mathcal{P}_{p,p-q}$ such that $QW = 0$. Let $V_j = \sum_{i=1}^p w_{ij} X_i$, $j = 1, ..., p-q$. By direct computation, we have $V_i^t V_j = \delta_{i,j} I_q$ and $V_j^t y = 0$; $i, j = 1, ..., p-q$. Therefore, $V = [V_1|\cdots|V_{p-q}] \in \mathcal{P}_{n,(p-q)q}$ and there exists $Z \in \mathcal{P}_{n-1,(p-q)q}$ such that $YZ = V$. Moreover, $V_\ell^t A_j V_k = \sum_{s=1}^p \sum_{t=1}^p w_{s\ell} w_{tk} X_s^t A_j X_t = \delta_{\ell,k} C_j$, $j = 1, ..., m$, and $\ell, k = 1, ..., p-q$. Hence,

$$Z^t Y^t A_j Y Z = V^t A_j V = I_{p-q} \otimes C_j, \quad j = 1, ..., m,$$

and hence $\mathbf{C} \in \Lambda^{\mathbb{R}}_{p-q,q}(Y^t\mathbf{A}Y)$. The general case follows by induction:

$$\bigcap_{Y_1 \in \mathcal{P}_{n,n-k}} \Lambda^{\mathbb{R}}_{p-qk,q}(Y_1^t\mathbf{A}Y_1) \subseteq \bigcap_{Y_1 \in \mathcal{P}_{n,n-k}} \bigcap_{Y_2 \in \mathcal{P}_{n-k,n-k-1}} \Lambda^{\mathbb{R}}_{p-qk-q,q}(Y_2^t Y_1^t \mathbf{A} Y_1 Y_2)$$

$$\subseteq \bigcap_{Y \in \mathcal{P}_{n,n-(k+1)}} \Lambda^{\mathbb{R}}_{p-q(k+1),q}(Y^t\mathbf{A}Y).$$

■

We denote by $\operatorname{conv}(\mathcal{S})$ the convex hull of the set $\mathcal{S}$.

Corollary 2.1 *Let* $\mathbf{A} = (A_1, \dots, A_m) \in \mathbf{M}_n^m(\mathbb{R})$*. If* $n \geq 2^m(m+1)^2(pq(1+q^2(2m+1))-1)$*, then*

(i) $\Lambda^{\mathbb{R}}_{\tilde{p},q}(\mathbf{A})$ *is non-empty where* $\tilde{p} = p(1+q^2(2m+1))$*,*

(ii) $\Lambda^{\mathbb{R}}_{p,q}(\mathbf{A})$ *is star-shaped and* $\mathbf{C}$ *is a star-center of* $\Lambda^{\mathbb{R}}_{p,q}(\mathbf{A})$ *for all* $\mathbf{C} \in \operatorname{conv}(\Lambda^{\mathbb{R}}_{\tilde{p},q}(\mathbf{A}))$*.*

Proof. (i) This follows from Theorem 2.1 by setting $m_1 = m_2 = m$ and identifying $\Lambda^{\mathbb{R}}_{\tilde{p},q}(\mathbf{A})$ with $\Lambda^{\mathbb{R}}_{\tilde{p},q}(S_1, ..., S_m, G_1, ..., G_m)$.

(ii) By Theorem 2.3, we have

$$\begin{aligned} \Lambda^{\mathbb{R}}_{\tilde{p},q}(\mathbf{A}) &\subseteq \bigcap \left\{ \Lambda^{\mathbb{R}}_{\tilde{p}-q^2p(2m+1),q}(Y^t\mathbf{A}Y) : Y \in \mathcal{P}_{n,n-pq(2m+1)} \right\} \\ &= \bigcap \left\{ \Lambda^{\mathbb{R}}_{p,q}(Y^t\mathbf{A}Y) : Y \in \mathcal{P}_{n,n-pq(2m+1)} \right\}. \end{aligned}$$

Note that the set of all star-centers is always convex. Then the result follows from Theorem 2.2. ■

3 Joint Congruence (p,q)-Matricial Range

Let $\mathbf{M}_{n,k}(\mathbb{C})$ be the set of all $n \times k$ complex matrices and let $\mathcal{U}_{n,k}$ be the set of $n \times k$ complex matrices X such that $X^*X = I_k$. We abbreviate $\mathbf{M}_{n,n}(\mathbb{C})$ and $\mathcal{U}_{n,n}$ to $\mathbf{M}_n(\mathbb{C})$ and $\mathcal{U}_n$ respectively. Note that $\mathcal{U}_n$ is the group of all unitary matrices. Define the joint congruence (p,q)-matricial range of $\mathbf{A} = (A_1, \dots, A_m) \in \mathbf{M}_n^m(\mathbb{C})$ by

$$\Lambda^{\mathbb{C}}_{p,q}(\mathbf{A}) = \{(B_1, \dots, B_m) : X^tA_jX = I_p \otimes B_j,\ X \in \mathcal{U}_{n,pq}\}.$$

Evidently, $(B_1, \dots, B_m) \in \Lambda^{\mathbb{C}}_{p,q}(\mathbf{A})$ if and only if there is a unitary $U \in \mathcal{U}_n$ such that $I_p \otimes B_j$ is the leading principal submatrix of U^tA_jU for $j = 1, \dots, m$. The orbits of a matrix A under the group action $(U, A) \mapsto U^tAU$, where U is unitary, have been studied in [7,11,18,19,25–27]. Using the techniques in Section 2, we can obtain similar results for the joint congruence (p,q)-matricial range. To avoid repetitions of arguments, we may omit some details in the proofs unless non-trivial modifications are needed.

For $j = 1, \dots, m$, we can write $A_j = S_j + G_j$ such that $S_j = (A_j + A_j^t)/2$ is complex symmetric and $G_j = (A_j - A_j^t)/2$ is complex skew-symmetric. It is easy to see that the $\Lambda^{\mathbb{C}}_{p,q}(\mathbf{A})$ and $\Lambda^{\mathbb{C}}_{p,q}(S_1, \dots, S_m, G_1, \dots, G_m)$ have the same geometric structure. So we follow the idea in the previous section and consider $(A_1, ..., A_m)$ where A_j is complex symmetric or complex skew-symmetric. After that we can derive similar results for general complex matrices. Moreover, we may remove A_j which is a linear combination of other components until we get a family of linearly independent matrices $A_1, \dots, A_k$. We start by the following observation.

Proposition 3.1 *Suppose* $\mathbf{A} = (A_1, \dots, A_m) \in \mathbf{M}_n^m(\mathbb{C})$*. Then* $(C_1, \dots, C_m)$ *is a star-center of* $\Lambda^{\mathbb{C}}_{p,q}(\mathbf{A})$ *if and only if every element in the set*

$$\operatorname{conv}\{(U^tC_1U, \dots, U^tC_mU) : U \in \mathcal{U}_q\}$$

is a star-center. In particular, if $\Lambda^{\mathbb{C}}_{p,q}(\mathbf{A})$ *is star-shaped, then* $(0_q, \dots, 0_q)$ *is a star-center.*

The proof of the following result is similar to the proof of Theorem 2.2 and will be omitted.

Theorem 3.1 *Let* $\mathbf{A} = (A_1, \ldots, A_m) \in \mathbf{M}_n^m(\mathbb{C})$ *where* $A_1, \ldots, A_m$ *are complex symmetric or complex skew-symmetric. Suppose* $n \geq pq(m+1)$ *and*

$$\mathbf{C} \in \Lambda_{p,q}^{\mathbb{C}}(Y^t A_1 Y, ..., Y^t A_m Y) \quad \text{for all} \quad Y \in \mathcal{U}_{n,n-pq(m+1)}.$$

Then $\mathbf{C}$ *is a star-center of* $\Lambda_{p,q}^{\mathbb{C}}(\mathbf{A})$.

Theorem 3.2 *Suppose* $\mathbf{A} = (A_1, \ldots, A_m) \in \mathbf{M}_n^m(\mathbb{C})$ *where* $A_1, \ldots, A_m$ *are complex symmetric or complex skew-symmetric.*

(i) *If* $n \geq 2^m pq$, *then* $\mathbf{0}_q = (0_q, \ldots, 0_q) \in \Lambda_{p,q}^{\mathbb{C}}(\mathbf{A})$.
(ii) *If* $n \geq pq(2^m + m + 1)$, *then* $\mathbf{0}_q$ *is a star-center of* $\Lambda_{pq}^{\mathbb{C}}(\mathbf{A})$.

Proof. (i) The proof is similar to the proof of Proposition 2.2. If one can show that for $n \geq 2pq$, there exists an $X \in \mathcal{U}_n$ such that $X^* A_1 X$ has 0_{pq} as the leading principal submatrix, then the same inductive argument in the proof of Proposition 2.2 leads to $\mathbf{0}_q \in \Lambda_{p,q}^{\mathbb{C}}(\mathbf{A})$. Therefore, we will show the existence of such X when $n \geq 2pq$. If A_1 is complex skew-symmetric, then X exists by the same construction in the proof of Proposition 2.2.

Now, let A_1 be symmetric. First, we show that A_1 is congruent to a matrix $\tilde{A}_1$ such that the first row has the form $(0, *, 0, \ldots, 0)$ and the first column has the form $(0, *, 0, \ldots, 0)^t$. By Autonne-Takagi factorization there is a unitary $U \in \mathcal{U}_n$ such that $U^t A_1 U = \text{diag}(a_1, \ldots, a_n)$ with $a_1 \geq \cdots \geq a_n \geq 0$ (see [1, 12]). Multiply the second column of U by i to get V and we have $V^t A_1 V = \text{diag}(a_1, -a_2, a_3, a_4, \ldots, a_n)$. Note that there is a real 2×2 orthogonal matrix P such that the $(1,1)$-th entry of $P^t \text{diag}(a_1, -a_2) P$ is zero. Let $W_1 = V(P \oplus I_{n-2})$. Then the first row of $\tilde{A}_1 = W_1^t A_1 W_1$ is $(0, \mu, 0, \ldots, 0)$ for some μ. As $\tilde{A}_1$ is symmetric, we see that the first column of $\tilde{A}_1$ equals $(0, \mu, 0, \ldots, 0)^t$.

We may then replace A_1 by $\tilde{A}_1$. Then consider the principal submatrix of A_1 by deleting the first two rows and the first two columns of A_1 and repeat the above argument on that submatrix. Therefore, there is a unitary matrix $W_2 \in \mathcal{U}_n$ such that the first row and third row of $W_2^t A_1 W_2$ have the form $(0, *, 0, \ldots, 0)$ and $(0, *, 0, *, 0, \ldots, 0)$ respectively. Similar patterns hold for the transposes of the first column and the third column of $W_2^t A_1 W_2$.

As $n \geq 2pq$, one can repeat this argument pq times. Then the principal submatrix of the modified matrix A_1 lying in rows and columns indexed by $1, 3, \ldots, 2pq - 1$ will be 0_{pq}. The result will follow by an inductive argument similar to that in the proof of Proposition 2.2.

(ii) This follows from Theorem 3.1 and (i). ■

Theorem 3.3 *Suppose* $\mathbf{A} = (A_1, \ldots, A_m) \in \mathbf{M}_n^m(\mathbb{C})$. *If* $1 \leq 2qr < p \leq n$, *then*

$$\Lambda_{p,q}^{\mathbb{C}}(\mathbf{A}) \subseteq \bigcap \left\{ \Lambda_{p-2qr,q}^{\mathbb{C}}(Y^t \mathbf{A} Y) : Y \in \mathcal{U}_{n,n-r} \right\}$$

where $Y^t \mathbf{A} Y = (Y^t A_1 Y, \ldots, Y^t A_m Y)$.

Proof. The proof is similar to Theorem 2.3. However, in this case, the corresponding Q will be a $2q \times p$ real matrix in the form

$$Q = \begin{bmatrix} \text{Re}\,(X_1^* y) & \cdots & \text{Re}\,(X_p^* y) \\ \text{Im}\,(X_1^* y) & \cdots & \text{Im}\,(X_p^* y) \end{bmatrix},$$

and the corresponding $W \in \mathcal{P}_{p,p-2q} \subseteq \mathcal{U}_{p,p-2q}$ with $QW = 0$. The result follows by similar argument in Theorem 2.3. ■

Identifying $\Lambda_{p,q}^{\mathbb{C}}(\mathbf{A})$ with $\Lambda_{p,q}^{\mathbb{C}}(S_1, \ldots, S_m, G_1, \ldots, G_m)$, we can apply the preceding results to obtain the following.

Corollary 3.1 *Let* $\mathbf{A} = (A_1, \ldots, A_m) \in \mathbf{M}_n^m(\mathbb{C})$*. If* $n \geq 2^{2m} p(1 + 2q^2(2m + 1))q$*, then*

(i) $\Lambda^{\mathbb{C}}_{\tilde{p},q}(\mathbf{A})$ *is non-empty where* $\tilde{p} = p(1 + 2q^2(2m + 1))$*, and*
(ii) $\Lambda^{\mathbb{C}}_{p,q}(\mathbf{A})$ *is star-shaped and* $\mathbf{C}$ *is a star-center of* $\Lambda^{\mathbb{C}}_{p,q}(\mathbf{A})$ *for all* $\mathbf{C} \in \operatorname{conv}(\Lambda^{\mathbb{C}}_{\tilde{p},q}(\mathbf{A}))$*.*

4 Infinite dimensional operators

Let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators on the Hilbert space $\mathcal{H}$ over $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$). Evidently, when $\mathcal{H}$ has finite dimension n, we can identify $\mathcal{B}(\mathcal{H})$ as $\mathbf{M}_n(\mathbb{F})$. In the following, *we always assume that the dimension of* $\mathcal{H}$ *is infinite.*

We can extend the definition of symmetric matrices and skew-symmetric matrices to infinite dimensional operators. In the following, we will obtain results for infinite dimensional operators as in [13].

First, for every $T \in \mathcal{B}(\mathcal{H})$, the transpose operator of T is the linear operator $T^t : \mathcal{H} \to \mathcal{H}$ such that $\langle T^t x, y\rangle = \overline{\langle T^* x, y\rangle}$ for all unit vectors $x, y \in \mathcal{H}$. We may also define the transpose of T using an arbitrary but fixed orthonormal basis. An operator $A \in \mathcal{B}(\mathcal{H})$ is symmetric if $A = A^t$; it is skew-symmetric if $A^t = -A$. Every operator can be decomposed into $A = S + G$, where $S = (A+A^t)/2$ is symmetric and $G = (A-A^t)/2$ is skew-symmetric. Let $\mathcal{V}_k(\mathcal{H})$ denote the set of operators $X : \mathcal{W} \to \mathcal{H}$ for some k-dimensional subspace $\mathcal{W}$ of $\mathcal{H}$ such that $X^*X = I_{\mathcal{W}}$. Then we can define the joint congruence (p, q)-matricial range of $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{B}(\mathcal{H})^m$ as follows,

$$\Lambda^{\mathbb{F}}_{p,q}(\mathbf{A}) = \{(B_1, \ldots, B_m) \in \mathbf{M}_q^m(\mathbb{F}) : U^t A_j U = I_p \otimes B_j,\ U \in \mathcal{V}_{pq}(\mathcal{H})\}.$$

For simplicity, we name both $\Lambda^{\mathbb{R}}_{p,q}(\mathbf{A})$ and $\Lambda^{\mathbb{C}}_{p,q}(\mathbf{A})$ the joint congruence (p, q)-matricial range. Furthermore, we may define the joint congruence (∞, q)-matricial range by

$$\Lambda^{\mathbb{F}}_{\infty,q}(\mathbf{A}) = \bigcap_{p=1}^{\infty} \Lambda^{\mathbb{F}}_{p,q}(\mathbf{A}).$$

Firstly, we have the following result which is a consequence of Corollary 2.1 and Corollary 3.1.

Proposition 4.1 *Let* $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{B}(\mathcal{H})^m$*. Then* $\Lambda^{\mathbb{F}}_{p,q}(\mathbf{A})$ *is always star-shaped for all positive integers* p, q*. Moreover, if* $\mathbf{C} \in \Lambda^{\mathbb{F}}_{\infty,q}(\mathbf{A})$*, then* $\mathbf{C}$ *is a star-center of* $\Lambda^{\mathbb{F}}_{p,q}(\mathbf{A})$*.*

Denote by $\mathcal{V}_k^{\perp}(\mathcal{H})$ the set of operators $Y : \mathcal{W}^{\perp} \mapsto \mathcal{H}$ such that $Y^*Y = I_{\mathcal{W}^{\perp}}$ where $\mathcal{W}^{\perp}$ is the orthogonal complement of an k-dimensional subspace $\mathcal{W}$ of $\mathcal{H}$. Moreover define $\mathcal{V}^{\perp}(\mathcal{H}) = \bigcup_{k\geq 1} \mathcal{V}_k^{\perp}(\mathcal{H})$ and denote by $\mathcal{F}(\mathcal{H})$ the set of all finite rank operators in $\mathcal{B}(\mathcal{H})$.

We can then use the techniques of [13] to prove the following.

Theorem 4.1 *Let* $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{B}(\mathcal{H})^m$*. Denote by* $\mathcal{S}^{\mathbb{F}}_{p,q}(\mathbf{A})$ *the set of all star-centers of* $\Lambda^{\mathbb{F}}_{p,q}(\mathbf{A})$*. Then*

$$\Lambda^{\mathbb{F}}_{\infty,q}(\mathbf{A}) = \bigcap_{p\geq 1} \Lambda^{\mathbb{F}}_{p,q}(\mathbf{A}) = \bigcap_{p\geq 1} \mathcal{S}^{\mathbb{F}}_{p,q}(\mathbf{A}) \tag{4.2}$$

is a convex set. Moreover, if $Y^t\mathbf{A}Y = (Y^tA_1Y, ..., Y^tA_mY)$ *for* $Y \in \mathcal{V}^{\perp}(\mathcal{H})$*,* $\mathbf{A} + \mathbf{F} = (A_1 + F_1, \ldots, A_m + F_m)$ *for* $\mathbf{F} = (F_1, \ldots, F_m) \in \mathcal{F}(\mathcal{H})^m$*, and* p_0 *is a positive integer, then*

$$\Lambda^{\mathbb{F}}_{\infty,q}(\mathbf{A}) = \bigcap\{\Lambda^{\mathbb{F}}_{p_0,q}(Y^t\mathbf{A}Y) : Y \in \mathcal{V}^{\perp}(\mathcal{H})\} = \bigcap\{\Lambda^{\mathbb{F}}_{p_0,q}(\mathbf{A}{+}\mathbf{F}) : \mathbf{F} \in \mathcal{F}(\mathcal{H})^m\}. \tag{4.3}$$

A key observation is that the set of star-centers $\mathcal{S}^{\mathbb{F}}_{p,q}(\mathbf{A})$ is always convex, and therefore $\cap_{p\geq 1}\mathcal{S}^{\mathbb{F}}_{p,q}(\mathbf{A})$ is a convex set. It is not hard to see that

$$\bigcap_{p\geq 1} \Lambda^{\mathbb{F}}_{p,q}(\mathbf{A}) \subseteq \bigcap_{p\geq 1} \mathcal{S}^{\mathbb{F}}_{p,q}(\mathbf{A}) \subseteq \bigcap_{p\geq 1} \Lambda^{\mathbb{F}}_{p,q}(\mathbf{A}).$$

One can then get the convexity and the set equality (4.2). The equality (4.3) can be established by the following lemma.

Lemma 4.1 *Let $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{B}(\mathcal{H})^m$. Then the following hold.*

(i) $\bigcap_{Y\in\mathcal{V}^\perp} \Lambda^{\mathbb{F}}_{p,q}(Y^t\mathbf{A}Y) \subseteq \bigcap_{Y\in\mathcal{V}^\perp} \Lambda^{\mathbb{F}}_{1,q}(Y^t\mathbf{A}Y) \subseteq \Lambda^{\mathbb{F}}_{\infty,q}(Y^t\mathbf{A}Y)$.

(ii) *For every $\mathbf{F} \in \mathcal{F}(\mathcal{H})^m$, there is a $Y \in \mathcal{V}^\perp(\mathcal{H})$ such that $\Lambda^{\mathbb{F}}_{p,q}(Y^t\mathbf{A}Y) \subseteq \Lambda^{\mathbb{F}}_{p,q}(\mathbf{A}+\mathbf{F})$.*

(iii) *For every $Y \in \mathcal{V}^\perp(\mathcal{H})$, there is an $\mathbf{F} \in \mathcal{F}(\mathcal{H})^m$ such that $\Lambda^{\mathbb{F}}_{p,q}(\mathbf{A}+\mathbf{F}) \subseteq \Lambda^{\mathbb{F}}_{p,q}(Y^t\mathbf{A}Y)$.*

Proof. We may assume without loss of generality that $A_1, ..., A_m$ are either symmetric or skew-symmetric operators.

(i) The first inclusion is obvious. Now given that $\mathbf{C} = (C_1, ..., C_m) \in \bigcap_{Y\in\mathcal{V}^\perp} \Lambda^{\mathbb{F}}_{1,q}(Y^t\mathbf{A}Y)$, we claim that there exists an infinite sequence of operators $\{X_r\}_{r=1}^{\infty} \subseteq \mathcal{V}_q(\mathcal{H})$ with $X_r : \mathcal{H}_r \to \mathcal{H}$ for some q-dimensional subspace $\mathcal{H}_r$ of $\mathcal{H}$ such that for $r \neq s$, $\mathcal{H}_r$ and $\mathcal{H}_s$ are orthogonal and

$$X_r^* X_s = \begin{cases} I_q & r = s, \\ 0_q & r \neq s, \end{cases} \qquad \text{and} \qquad X_r^t A_j X_s = \begin{cases} C_j & r = s, \\ 0_q & r \neq s. \end{cases}$$

Once the claim holds, since $\{\mathcal{H}_r\}_{r=1}^{\infty}$ is an infinite sequence of mutually orthogonal q-dimensional subspaces of $\mathcal{H}$, one can extend $\oplus_{r=1}^{\infty} X_r$ to a unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that $U|_{\mathcal{H}_r} = X_r$ for all r. Then the operator matrix of $U^t A_j U$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \cdots$ has the form

$$\begin{bmatrix} C_j & 0 & 0 & \cdots \\ 0 & C_j & 0 & \cdots \\ 0 & 0 & C_j & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Thus, $\mathbf{C} \in \Lambda^{\mathbb{F}}_{\infty,q}(\mathbf{A})$. Now it remains to prove the claim, which will be done by induction.

Assume $\mathbf{C} = (C_1, ..., C_m) \in \bigcap_{Y\in\mathcal{V}^\perp} \Lambda^{\mathbb{F}}_{1,q}(Y^t\mathbf{A}Y)$. Then $\mathbf{C} = (C_1, ..., C_m) \in \Lambda^{\mathbb{F}}_{1,q}(\mathbf{A})$ and there exists $X_1 \in \mathcal{V}_q(\mathcal{H})$ such that $X_1 : \mathcal{H}_1 \to \mathcal{H}$ with $X_1^* X_1 = I_{\mathcal{H}_1}$ for some q-dimensional subspace $\mathcal{H}_1$ of $\mathcal{H}$ so that $X_1^t A_j X_1 = C_j$ for $j = 1, \ldots, m$. The claim holds for $\{X_1\}$.

Assume the operators $\{X_1, \ldots, X_n\}$ already satisfy the claim. Then $\mathcal{H}_r$ and $\mathcal{H}_s$ are orthogonal for all $1 \leq r < s \leq n$. Since $X_r^* X_s = 0_q$ for $1 \leq r < s \leq n$, $X_r(\mathcal{H}_r)$ is orthogonal to $X_r(\mathcal{H}_s)$ for $r \neq s$. Then one can extend $X_1 \oplus \cdots \oplus X_n$ to a unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that $U|_{\mathcal{H}_r} = X_r$ for $1 \leq r \leq n$, and the operator matrix of $U^t A_j U$ with respect to the decomposition $\mathcal{H} = (\bigoplus_{r=1}^n \mathcal{H}_r) \oplus (\bigoplus_{r=1}^n \mathcal{H}_r)^\perp$ has the form

$$\begin{bmatrix} I_n \otimes C_j & * \\ * & * \end{bmatrix}.$$

Let $\mathcal{L}$ be the subspace spanned by

$$\left\{ (\oplus_{r=1}^n \mathcal{H}_r),\ U^t A_1 U \,(\oplus_{r=1}^n \mathcal{H}_r),\ \ldots,\ U^t A_m U \,(\oplus_{r=1}^n \mathcal{H}_r) \right\}.$$

Then $\mathcal{L}$ has dimension at most $qn(m+1)$. Take $Y = U|_{\mathcal{L}^\perp}$. Then $Y \in \mathcal{V}^\perp(\mathcal{H})$. By the above assumption, we have $\mathbf{C} \in \Lambda^{\mathbb{F}}_{1,q}(Y^t\mathbf{A}Y)$ and there exists $X : \mathcal{H}_{n+1} \to \mathcal{L}^\perp$ with $X^*X = I_{\mathcal{H}_{n+1}}$ for some q-dimensional subspace $\mathcal{H}_{n+1}$ of $\mathcal{L}^\perp$ such that

$$X^t(Y^tA_jY)X = C_j, \quad j = 1, \dots, m.$$

Let $X_{n+1} = YX : \mathcal{H}_{n+1} \to \mathcal{H}$. Clearly, $X^*_{n+1}X_{n+1} = I_q$ and $X^t_{n+1}A_jX_{n+1} = C_j$ for all $j = 1, \dots, m$. Now fix $1 \le r \le n$. For every $u \in \mathcal{H}_{n+1}$ and $v \in \mathcal{H}_r \subseteq \mathcal{L}$, $Xu \in \mathcal{L}^\perp$ and $U^tA_jUv \in \mathcal{L}$. Then

$$\langle u, X^*_{n+1}X_rv\rangle = \langle X_{n+1}u, X_rv\rangle = \langle YXu, Uv\rangle = \langle UXu, Uv\rangle = \langle Xu, v\rangle = 0$$

and

$$\langle u, X^t_{n+1}A_jX_rv\rangle = \overline{\langle X_{n+1}u, A_jX_rv\rangle} = \overline{\langle YXu, A_jUv\rangle} = \langle Xu, U^tA_jUv\rangle = 0.$$

Thus, $X^*_{n+1}X_r = 0$ and $X^t_{n+1}A_jX_r = 0$ for all $1 \le j \le m$ and $1 \le r \le n$. As A_j is symmetric or skew-symmetric, we have $X^t_rA_jX_{n+1} = 0$ for all $1 \le j \le m$ and $1 \le r \le n$. Thus, the operators $\{X_1, \dots, X_{n+1}\}$ satisfy the claim. By induction, the claim holds.

(ii) For every $\mathbf{F} \in \mathcal{F}(\mathcal{H})^m$, there exists $Y \in \mathcal{V}^\perp$ such that $Y^t\mathbf{F}Y = \mathbf{0} = (0, \dots, 0)$. Then

$$\Lambda^{\mathbb{F}}_{p,q}(Y^t\mathbf{A}Y) = \Lambda^{\mathbb{F}}_{p,q}(Y^t(\mathbf{A}+\mathbf{F})Y) \subseteq \Lambda^{\mathbb{F}}_{p,q}(\mathbf{A}+\mathbf{F}).$$

(iii) Suppose $Y \in \mathcal{V}^\perp_r$ for some r. Then $Y : \mathcal{L}^\perp_1 \to \mathcal{H}$ is such that $Y^*Y = I_{\mathcal{L}^\perp_1}$ for some r-dimensional subspace $\mathcal{L}_1$ of $\mathcal{H}$. Since $\mathcal{L}^\perp_1$ is infinite dimensional, by Proposition 4.1, $\Lambda^{\mathbb{F}}_{pq,1}(Y^t\mathbf{A}Y)$ is non-empty. Pick $(b_1, ..., b_m) \in \Lambda^{\mathbb{F}}_{pq,1}(Y^t\mathbf{A}Y)$. Then there exist a pq-dimensional subspace $\mathcal{H}_1$ of $\mathcal{L}^\perp_1$ and $X : \mathcal{H}_1 \to \mathcal{L}^\perp_1$ with $X^*X = I_{\mathcal{H}_1}$ such that $X^t(Y^tA_jY)X = b_jI_{pq}$ for $j = 1, \dots, m$. Extend the operator $YX : \mathcal{H}_1 \to \mathcal{H}$ to a unitary operator $U : \mathcal{H} \to \mathcal{H}$ so that $U|_{\mathcal{H}_1} = YX$. Let $\mathcal{L}_2$ be the subspace spanned by

$$\{\mathcal{L}_1, \mathcal{H}_1, U^tA_1U(\mathcal{H}_1), \dots, U^tA_mU(\mathcal{H}_1)\}.$$

Then $\mathcal{L}_2$ has dimension at most $pqm + r$. Set $W = U|_{\mathcal{L}^\perp_2}$. Then the operator matrix of U^tA_jU with respect to the decomposition $\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{L}^\perp_2 \oplus (\mathcal{K}_1 \oplus \mathcal{L}^\perp_2)^\perp$ has the form

$$\begin{bmatrix} b_jI_{\mathcal{H}_1} & 0 & * \\ 0 & W^tA_jW & * \\ * & * & * \end{bmatrix}.$$

Let B_j be the operator such that the operator matrix of U^tB_jU with respect to the same decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{L}^\perp_2 \oplus (\mathcal{H}_1 \oplus \mathcal{L}^\perp_2)^\perp$ has the form

$$\begin{bmatrix} b_jI_{\mathcal{H}_1} & 0 & 0 \\ 0 & W^tA_jW & 0 \\ 0 & 0 & b_jI_{(\mathcal{H}_1 \oplus \mathcal{L}^\perp_2)^\perp} \end{bmatrix} \quad \text{and} \quad F_j = B_j - A_j.$$

Notice that $\mathcal{H}_1 \oplus (\mathcal{H}_1 \oplus \mathcal{L}^\perp_2)^\perp = \mathcal{L}_2$ is finite dimensional and U^tF_jU has the form $\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{bmatrix}$. Thus, F_j is a finite rank operator. Now denote $\mathbf{F} = (F_1, \dots, F_m)$ $\in \mathcal{F}(\mathcal{H})^m$ and suppose

$$\mathbf{C} = (C_1, ..., C_m) \in \Lambda^{\mathbb{F}}_{p,q}(\mathbf{A}+\mathbf{F}) = \Lambda^{\mathbb{F}}_{p,q}(\mathbf{B}) = \Lambda^{\mathbb{F}}_{p,q}(U^t\mathbf{B}U).$$

Then there exists $Z : \mathcal{H}_2 \to \mathcal{H}$ with $Z^*Z = I_{\mathcal{H}_2}$ for some pq-dimensional subspace $\mathcal{H}_2$ of $\mathcal{H}$ such that $Z^t(U^tB_jU)Z = I_p \otimes C_j$ for $j = 1, \ldots, m$. Write $Z = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}$ according to the same decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{L}_2^\perp \oplus (\mathcal{H}_1 \oplus \mathcal{L}_2^\perp)^\perp$. Then

$$b_j \cdot Z_1^tZ_1 + Z_2^t(W^tA_jW)Z_2 + b_j \cdot Z_3^tZ_3 = Z^t(U^tB_jU)Z = I_p \otimes C_j.$$

Since $\dim \mathcal{H}_2 = pq = \dim \mathcal{H}_1$, one can always find an operator $\hat{Z}_1 : \mathcal{H}_1 \to \mathcal{H}_1$ such that $\hat{Z}_1^*\hat{Z}_1 = Z_1^*Z_1 + Z_3^*Z_3$. Define $\hat{Z} : \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{L}_2^\perp$ by $\hat{Z} = \begin{bmatrix} \hat{Z}_1 \\ Z_2 \end{bmatrix}$ with respect to the decomposition $\mathcal{H}_1 \oplus \mathcal{L}_2^\perp$. Then $\hat{Z}^*\hat{Z} = I_{\mathcal{H}_2}$ and hence, $\hat{Z} \in \mathcal{V}_{pq}(\mathcal{H})$. Furthermore,

$$\begin{aligned} \hat{Z}^t \begin{bmatrix} b_jI_{\mathcal{H}_1} & 0 \\ 0 & W^tA_jW \end{bmatrix} \hat{Z} &= b_j \cdot \hat{Z}_1^t\hat{Z}_1 + Z_2^t(W^tA_jW)Z_2 \\ &= b_j \cdot (Z_1^tZ_1 + Z_3^tZ_3) + Z_2^t(W^tA_jW)Z_2 \\ &= Z^t(U^tB_jU)Z = I_p \otimes C_j. \end{aligned}$$

Recall that $\mathcal{H}_1 \subseteq \mathcal{L}_1^\perp$ and $\mathcal{L}_2^\perp \subseteq \mathcal{L}_1^\perp$, and hence, $\mathcal{H}_1 \oplus \mathcal{L}_2^\perp \subseteq \mathcal{L}_1^\perp$. Thus, the operator $b_jI_{\mathcal{H}_1} \oplus W^tA_jW$ is a compression of Y^tA_jY to $\mathcal{H}_1 \oplus \mathcal{L}_2^\perp$. Thus,

$$\mathbf{C} \in \Lambda_{p,q}^{\mathbb{F}}\left(b_1I_{\mathcal{H}_1} \oplus W^tA_1W, \ldots, b_mI_{\mathcal{H}_1} \oplus W^tA_mW\right) \subseteq \Lambda_{p,q}^{\mathbb{F}}(Y^t\mathbf{A}Y).$$

Hence, the proof is complete. ■

Proof of Theorem 4.1. By Theorem 2.3, Theorem 3.3, Proposition 4.1 and Lemma 4.1 (i),

$$\begin{aligned} \Lambda_{\infty,q}^{\mathbb{F}}(\mathbf{A}) &\subseteq \bigcap_{p\geq 1} S_{p,q}^{\mathbb{F}}(\mathbf{A}) \subseteq \bigcap_{p\geq 1} \Lambda_{p,q}^{\mathbb{F}}(\mathbf{A}) \subseteq \bigcap_{p\geq 1} \Lambda_{p_0+2pq,q}^{\mathbb{F}}(\mathbf{A}) \\ &\subseteq \bigcap_{p\geq 1}\left(\bigcap_{Y\in\mathcal{V}_p^\perp(\mathcal{H})} \Lambda_{p_0,q}^{\mathbb{F}}(Y^t\mathbf{A}Y)\right) = \bigcap\{\Lambda_{p_0,q}^{\mathbb{F}}(Y^t\mathbf{A}Y) : Y \in \mathcal{V}^\perp(\mathcal{H})\} \\ &\subseteq \Lambda_{\infty,q}^{\mathbb{F}}(\mathbf{A}). \end{aligned}$$

Moreover, by Lemma 4.1 (ii) and (iii), we have

$$\bigcap\{\Lambda_{p_0,q}^{\mathbb{F}}(Y^t\mathbf{A}Y) : Y \in \mathcal{V}^\perp(\mathcal{H})\} = \bigcap\{\Lambda_{p_0,q}^{\mathbb{F}}(\mathbf{A}+\mathbf{F}) : \mathbf{F} \in \mathcal{F}(\mathcal{H})^m\}.$$

Note that $S_{p,q}^{\mathbb{F}}(\mathbf{A})$ is convex for all positive integer p. Hence, the result follows. ■

Let $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{B}(\mathcal{H})^m$. Define the joint essential congruence (p, q)-matricial range of $\mathbf{A}$ by

$$\Lambda_{p,q}^{e,\mathbb{F}}(\mathbf{A}) = \bigcap\{\mathbf{cl}(\Lambda_{p,q}^{\mathbb{F}}(\mathbf{A}+\mathbf{F})) : \mathbf{F} \in \mathcal{F}(\mathcal{H})^m\}.$$

One will see later that $\mathcal{F}(\mathcal{H})$ can be replaced by the set of compact operators $\mathcal{C}(\mathcal{H})$. The set $\Lambda_{p,q}^{e,\mathbb{F}}(\mathbf{A})$ can be viewed as the "core" of the joint (p, q)-matricial range of $\mathbf{A}$ under compact perturbation. The definition is motivated from the study of the Calkin algebra, see [14,22,24]. We have the following.

Theorem 4.2 *Let $\mathbf{A} \in \mathcal{B}(\mathcal{H})^m$ and q be a positive integer. Then $\Lambda^{e,\mathbb{F}}_{p,q}(\mathbf{A}) = \Lambda^{e,\mathbb{F}}_{1,q}(\mathbf{A})$ for every positive integer p, and they are compact, convex and equal to*

$$\begin{aligned}\bigcap\{\tilde{\mathcal{S}}^{\mathbb{F}}_{r,q}(\mathbf{A}) : r = 1, 2, \dots\} &= \bigcap\{\mathbf{cl}(\Lambda^{\mathbb{F}}_{r,q}(\mathbf{A})) : r = 1, 2, \dots\} \\ &= \bigcap\{\mathbf{cl}\left(\Lambda^{\mathbb{F}}_{p,q}(Y^t\mathbf{A}Y)\right) : Y \in \mathcal{V}^{\perp}(\mathcal{H})\},\end{aligned}$$

where $\tilde{\mathcal{S}}^{\mathbb{F}}_{r,q}(\mathbf{A})$ is the set of all star-centers of $\mathbf{cl}(\Lambda^{\mathbb{F}}_{r,q}(\mathbf{A}))$.

Proof. We first show that for every integers p_0 and p

$$\begin{aligned}\bigcap\{\mathbf{cl}(\Lambda^{\mathbb{F}}_{p_0,q}(Y^t\mathbf{A}Y)) : Y \in \mathcal{V}^{\perp}(\mathcal{H})\} &\subseteq \bigcap\{\mathbf{cl}(\Lambda^{\mathbb{F}}_{1,q}(Y^t\mathbf{A}Y) : Y \in \mathcal{V}^{\perp}(\mathcal{H})\} \\ &\subseteq \mathbf{cl}(\Lambda^{\mathbb{F}}_{p,q}(\mathbf{A})). \end{aligned} \tag{4.4}$$

The first inclusion is trivial. Let $\mathbf{C} = (C_1, \dots, C_m) \in \bigcap\{\mathbf{cl}(\Lambda^{\mathbb{F}}_{1,q}(Y^t\mathbf{A}Y) : Y \in \mathcal{V}^{\perp}(\mathcal{H})\}$. Fix a positive integer n. As $\mathbf{C} \in \mathbf{cl}(\Lambda^{\mathbb{F}}_{1,q}(\mathbf{A}))$, there exists $X_1 \in \mathcal{V}_q(\mathcal{H})$ such that $X_1 : \mathcal{H}_1 \to \mathcal{H}$ with $X_1^*X_1 = I_{\mathcal{H}_1}$ so that $X_1^tA_jX_1 = B_j^{(1)}$ and $\|B_j^{(1)} - C_j\| \le \frac{1}{n}$, $j = 1, ..., m$. By an inductive argument similar to that in Lemma 4.1, one can construct two infinite sequences $\{X_r\}_{r=1}^{\infty}$ and $\{\mathcal{H}_r\}_{r=1}^{\infty}$ with $X_r : \mathcal{H}_r \to \mathcal{H}$ for some q-dimensional subspace $\mathcal{H}_r$ such that any two distinct subspaces $\mathcal{H}_r$ and $\mathcal{H}_s$ are orthogonal,

$$X_r^*X_s = \begin{cases} I_q & r = s, \\ 0_q & r \ne s, \end{cases} \text{ and } X_r^tA_jX_s = \begin{cases} B_j^{(r)} & r = s, \\ 0_q & r \ne s, \end{cases} \text{with } \left\|B_j^{(r)} - C_j\right\| \le \frac{1}{n},$$

for $j = 1, \dots, m$. Take $d \ge (p-1)(2q^2m + 1) + 1$ and set $X = \oplus_{r=1}^d X_r : \oplus_{r=1}^d \mathcal{H}_r \to \mathcal{H}$. Then $X^*X = I_{\oplus_{r=1}^d \mathcal{H}_r}$ and

$$X^tA_jX = B_j^{(1)} \oplus B_j^{(2)} \oplus \cdots \oplus B_j^{(d)} \quad \text{and} \quad \left\|B_j^{(r)} - C_j\right\| \le \frac{1}{n},$$

for $j = 1, \dots, m$ and $r = 1, \dots, d$. Denote $\mathbf{B}_r = (B_1^{(r)}, \dots, B_m^{(r)})$ for $r = 1, \dots, d$. Identifying $\mathbf{B}_1, \dots, \mathbf{B}_d$ as d points in $\mathbb{R}^{2q^2m}$, then by Tverberg's Theorem (see [29]), one can partition $\{\mathbf{B}_r : r = 1, \dots, d\}$ into p sets

$$\mathcal{B}_1 = \{\mathbf{B}_r : r \in \mathcal{I}_1\}, \quad \mathcal{B}_2 = \{\mathbf{B}_r : r \in \mathcal{I}_2\}, \quad \dots \quad \mathcal{B}_p = \{\mathbf{B}_r : r \in \mathcal{I}_p\}$$

such that $\operatorname{conv}(\mathcal{B}_1) \cap \cdots \cap \operatorname{conv}(\mathcal{B}_p) \ne \emptyset$. Pick $\mathbf{C}^{(n)} = (C_1^{(n)}, \dots, C_m^{(n)}) \in \operatorname{conv}(\mathcal{B}_1) \cap \cdots \cap \operatorname{conv}(\mathcal{B}_p)$. Then $\mathbf{C}^{(n)} \in \Lambda^{\mathbb{F}}_{p,q}(\oplus_{r\in\mathcal{I}_\ell}\mathbf{B}_j)$ for $\ell = 1, \dots, p$ and hence,

$$\mathbf{C}^{(n)} \in \Lambda^{\mathbb{F}}_{p,q}(\oplus_{t=1}^p \oplus_{r\in\mathcal{I}_\ell} \mathbf{B}_j) = \Lambda^{\mathbb{F}}_{p,q}\left(\oplus_{t=1}^d \mathbf{B}_j\right) = \Lambda^{\mathbb{F}}_{p,q}(X^t\mathbf{A}X) \subseteq \Lambda^{\mathbb{F}}_{p,q}(\mathbf{A}).$$

Now as $C_j^{(n)}$ is a convex combination of $B_j^{(r)}$'s, $\|B_j^{(r)} - C_j\| \le \frac{1}{n}$ implies $\|C_j^{(n)} - C_j\| \le \frac{1}{n}$. We have $\mathbf{C}^{(n)} \in \Lambda^{\mathbb{F}}_{p,q}(\mathbf{A})$ and $\left\|C_j^{(n)} - C_j\right\| < \frac{1}{n}$. Therefore, there exists a sequence $\{\mathbf{C}^{(n)}\}_{n=1}^{\infty} \subseteq \Lambda^{\mathbb{F}}_{p,q}(\mathbf{A})$ converging to $\mathbf{C}$. Hence, $\mathbf{C} \in \mathbf{cl}(\Lambda^{\mathbb{F}}_{p,q}(\mathbf{A}))$. Note that by Lemma 4.1, we obtain

$$\begin{aligned}\Lambda^{e,\mathbb{F}}_{p,q}(\mathbf{A}) &= \bigcap\{\mathbf{cl}\left(\Lambda^{\mathbb{F}}_{p,q}(\mathbf{A} + \mathbf{F})\right) : \mathbf{F} \in \mathcal{F}(\mathcal{H})^m\} \\ &= \bigcap\{\mathbf{cl}\left(\Lambda^{\mathbb{F}}_{p,q}(Y^t\mathbf{A}Y)\right) : Y \in \mathcal{V}^{\perp}(\mathcal{H})\}.\end{aligned}$$

Now by Corollary 3.1 and Corollary 2.1,

$$\bigcap_{r\geq 1} \tilde{S}^{\mathbb{F}}_{r,q}(\mathbf{A}) \subseteq \bigcap_{r\geq 1} \mathbf{cl}(\Lambda^{\mathbb{F}}_{r,q}(\mathbf{A})) \subseteq \bigcap_{r\geq 1} \mathbf{cl}(\Lambda^{\mathbb{F}}_{r(1+2q^2(2m+1)),q}(\mathbf{A}))$$
$$\subseteq \bigcap_{r\geq 1} \mathbf{cl}(S^{\mathbb{F}}_{r,q}(\mathbf{A})) \subseteq \bigcap_{r\geq 1} \tilde{S}^{\mathbb{F}}_{r,q}(\mathbf{A}).$$

Moreover, by Theorem 2.3, Theorem 3.3 and (4.4), we have

$$\bigcap_{r\geq 1} \mathbf{cl}(\Lambda^{\mathbb{F}}_{r,q}(\mathbf{A})) \subseteq \bigcap_{r\geq 1} \mathbf{cl}(\Lambda^{\mathbb{F}}_{p_0+2rq,q}(\mathbf{A})) \subseteq \bigcap_{r\geq 1} \left(\bigcap_{Y\in\mathcal{V}_r^{\perp}(\mathcal{H})} \mathbf{cl}(\Lambda^{\mathbb{F}}_{p_0,q}(Y^t\mathbf{A}Y)) \right)$$
$$= \bigcap \{\mathbf{cl}(\Lambda^{\mathbb{F}}_{p_0,q}(Y^t\mathbf{A}Y)) : Y \in \mathcal{V}^{\perp}(\mathcal{H})\} \subseteq \bigcap_{r\geq 1} \mathbf{cl}\left(\Lambda^{\mathbb{F}}_{r,q}(\mathbf{A})\right).$$

Setting $p_0 = p$, we have $\Lambda^{e,\mathbb{F}}_{p,q}(\mathbf{A}) = \bigcap_{r\geq 1} \mathbf{cl}\left(\Lambda^{\mathbb{F}}_{r,q}(\mathbf{A})\right) = \bigcap_{r\geq 1} \tilde{S}^{\mathbb{F}}_{r,q}(\mathbf{A})$. As $\tilde{S}^{\mathbb{F}}_{r,q}(\mathbf{A})$ is a compact convex set for all positive integers r, their intersection is also compact and convex. Thus, $\Lambda^{e,\mathbb{F}}_{p,q}(\mathbf{A})$ is compact and convex.

Note that $\Lambda^{e,\mathbb{F}}_{p,q}(\mathbf{A}) = \bigcap_{r\geq 1} \tilde{S}^{\mathbb{F}}_{r,q}(\mathbf{A})$, which is independent on p. We see that

$$\Lambda^{e,\mathbb{F}}_{p,q}(\mathbf{A}) = \Lambda^{e,\mathbb{F}}_{1,q}(\mathbf{A}).$$

■

We close the paper with the following result showing that one may define $\Lambda^{e,\mathbb{F}}_{p,q}(\mathbf{A})$ using compact operators instead of finite rank operators as commented before Theorem 4.2.

Theorem 4.3 *Let* $\mathbf{A} \in \mathcal{B}(\mathcal{H})^m$ *and* p, q *be any positive integers. Then*

$$\Lambda^{e,\mathbb{F}}_{p,q}(\mathbf{A}) = \bigcap \{\mathbf{cl}(\Lambda^{\mathbb{F}}_{p,q}(\mathbf{A}+\mathbf{G})) : \mathbf{G} \in \mathcal{C}(\mathcal{H})^m\}.$$

Proof. Let

$$S_1(\mathbf{A}) = \bigcap \{\mathbf{cl}(\Lambda^{\mathbb{F}}_{1,q}(\mathbf{A}+\mathbf{G})) : \mathbf{G} \in \mathcal{C}(\mathcal{H})^m\}.$$

We first show that $\Lambda^{e,\mathbb{F}}_{1,q}(\mathbf{A}) = S_1(\mathbf{A})$. Evidently, $S_1(\mathbf{A}) \subseteq \Lambda^{e,\mathbb{F}}_{1,q}(\mathbf{A})$ as $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{C}(\mathcal{H})$. We focus on the reverse inclusion. Suppose $\mathbf{C} = (C_1, \ldots, C_m) \in \Lambda^{e,\mathbb{F}}_{1,q}(\mathbf{A})$. We will show that $\mathbf{C} \in \mathbf{cl}(\Lambda^{\mathbb{F}}_{1,q}(\mathbf{A}+\mathbf{G}))$ for every $\mathbf{G} \in \mathcal{C}(\mathcal{H})^m$.

Suppose $\mathbf{G} = (G_1, \ldots, G_m) \in \mathcal{C}(\mathcal{H})^m$. For every given $\varepsilon > 0$, there exists $Y \in \mathcal{V}^{\perp}$ such that $\|Y^tG_iY\| < \varepsilon/2$ for all $1 \leq i \leq m$. By Theorem 4.2, we can find $X \in \mathcal{V}_q$ such that $\|C_i - X^tY^tA_iYX\| < \varepsilon/2$ for all $1 \leq i \leq m$. Therefore, $\|C_i - X^tY^t(A_i+G_i)YX\| < \varepsilon$ for all $1 \leq i \leq m$. Hence, $\mathbf{C} \in \mathbf{cl}(\Lambda^{\mathbb{F}}_{1,q}(\mathbf{A}+\mathbf{G}))$. As $\mathbf{G}$ is arbitrary, we have $\mathbf{C} \in S_1(\mathbf{A})$.

Now let

$$S_p(\mathbf{A}) = \bigcap \{\mathbf{cl}(\Lambda^{\mathbb{F}}_{p,q}(\mathbf{A}+\mathbf{G}) : \mathbf{G} \in \mathcal{C}(\mathcal{H})^m\}.$$

Clearly, $S_p(\mathbf{A}) \subseteq \Lambda^{e,\mathbb{F}}_{p,q}(\mathbf{A})$. We consider the reverse inclusion. By Theorem 4.2, for a fixed $\mathbf{G} \in \mathcal{C}(\mathcal{H})^m$,

$$\Lambda^{e,\mathbb{F}}_{p,q}(\mathbf{A}) = \Lambda^{e,\mathbb{F}}_{1,q}(\mathbf{A}) = S_1(\mathbf{A}) = S_1(\mathbf{A}+\mathbf{G}) = \Lambda^{e,\mathbb{F}}_{p,q}(\mathbf{A}+\mathbf{G}) \subseteq \mathbf{cl}(\Lambda^{\mathbb{F}}_{p,q}(\mathbf{A}+\mathbf{G})).$$

Thus, $\Lambda^{e,\mathbb{F}}_{p,q}(\mathbf{A}) \subseteq \bigcap \{\mathbf{cl}(\Lambda^{\mathbb{F}}_{p,q}(\mathbf{A}+\mathbf{G}) : \mathbf{G} \in \mathcal{C}(\mathcal{H})^m\} = S_p(\mathbf{A})$. ■

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