Lecture notes on Numerical Range

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The numerical range $W(A)$ of an $n \times n$ matrix A is the collection of complex numbers of the form x^*Ax, where $x \in \mathbb{C}^n$ is a unit vector. It can be viewed as a "picture" of A containing useful information of A. Even if the matrix A is not known explicitly, the "picture" $W(A)$ would allow one to "see" many properties of the matrix. For example, the numerical range can be used to locate eigenvalues, deduce algebraic and analytic properties, obtain norm bounds, help find dilations with simple structure, etc. Related to the numerical range are the numerical radius of A defined by $w(A) = \max_{\mu \in W(A)} |\mu|$ and the distance of $W(A)$ to the origin denoted by $\widetilde{w}(A) = \min_{\mu \in W(A)} |\mu|$. The quantities $w(A)$ and $\widetilde{w}(A)$ are useful in studying perturbation, convergence, stability, approximation problems.

Basic results in our discussion can be found in [23], [29, Chapter 22], [32, Chapter 1], and [34, Chapter 6]. In addition, we will mention some immediately related papers and books containing more related work and references that readers can further pursue.

1 Basic examples and properties

Let $A \in \mathbb{C}^{n\times n}$. The *numerical range* (also known as *the field of values*) of A is defined and denoted by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

We begin with some simple examples and properties, which can be easily verified.

Example 1.1 (a) Let $A = \mathrm{diag}\,(1, 0)$. Then $W(A) = [0, 1]$.

(b) Let $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. Then $W(A)$ is the closed unit disk $\mathbf{D} = \{\mu \in \mathbb{C} : |\mu| \le 1\}$.

Theorem 1.2 *Let $A \in \mathbb{C}^{n\times n}$, $\alpha, \beta \in \mathbb{C}$.*

(a) $W(\alpha A + \beta I) = \alpha W(A) + \beta$.

(b) $W(U^*AU) = W(A)$ *for any unitary* $U \in \mathbb{C}^{n\times n}$.

(c) *Suppose $k \in \{1, \ldots, n-1\}$ and $X \in \mathbb{C}^{n\times k}$ satisfies $X^*X = I_k$. Then*

$$W(X^*AX) \subseteq W(A).$$

A set $\mathcal{S}$ in $\mathbb{C}$ is compact if it is closed and bounded; it is convex if a line segment $\mathcal{L}$ joining two points in $\mathcal{S}$ satisfies $\mathcal{L} \subseteq \mathcal{S}$. We have the following.

Theorem 1.3 *Let $A \in \mathbb{C}^{n\times n}$. Then $W(A)$ is a compact convex set in $\mathbb{C}$.*

Observe that $W(A)$ is the range of the unit sphere $\{x \in \mathbb{C}^n : x^*x = 1\}$ of $\mathbb{C}^n$ under the continuous map $x \mapsto x^*Ax$. So, $W(A)$ is compact.

The convexity of numerical range was proved by Toeplitz and Hausdorff. Toeplitz [64] first showed that the outer boundary curve of $W(A)$ is convex, and Hausdorff [30] showed that the set $W(A)$ is simply connected. There are many proofs of the convexity result after Toeplitz and Hausdorff. Many of the proofs reduce the problem to the 2×2 case by the following argument: Suppose we know the result is true for the 2×2 case. Then for any $A \in \mathbb{C}^{n\times n}$ with $n > 2$ and any two unit vectors $x, y \in \mathbb{C}^n$, we can let $X \in \mathbb{C}^{n\times 2}$ with column space containing x and y such that $X^*X = I_2$. Then $x^*Ax, y^*Ay \in W(X^*AX)$, and thus the line segment $\mathcal{L}$ joining x^*Ax and y^*Ay satisfies $\mathcal{L} \subseteq W(X^*AX) \subseteq W(A)$ by the convexity result in the 2×2 case and Proposition 1.2 (b).

Here is a description of the numerical ranges of matrices in $\mathbb{C}^{2\times 2}$.

Theorem 1.4 *Suppose $A \in \mathbb{C}^{2\times 2}$ has eigenvalues λ_1, λ_2. Then $W(A)$ is an elliptical disk with foci λ_1, λ_2 and minor axis with length $\{\mathrm{tr}\,(A^*A) - |\lambda_1|^2 - |\lambda_2|^2\}^{1/2}$.*

One can use the above fact to determine $W(A)$ for the matrices A in Example 1.1 without doing any calculation. On the other hand, one can verifies Example 1.1 directly and then reduce the general case to these examples using Theorem 1.1 (a); see [39].

Let $\mathcal{P}(\mathbb{C})$ be the set of subsets of $\mathbb{C}$. The following statement, which can be viewed as a functional characterization of the numerical range as a function from $\mathbb{C}^{n\times n}$ to $\mathcal{P}(\mathbb{C})$.

Theorem 1.5 *Suppose a function $F : \mathbb{C}^{n\times n} \to \mathcal{P}(\mathbb{C})$ satisfies the following three conditions.*

(i) *$F(A)$ is compact and convex for every $A \in \mathbb{C}^{n\times n}$.*

(ii) *$F(\mu A + \nu I) = \mu F(A) + \nu$ for any $\mu, \nu \in \mathbb{C}$ and $A \in \mathbb{C}^{n\times n}$.*

(iii) *$F(A) \subseteq \{\mu \in \mathbb{C} : \mu + \bar{\mu} \geq 0\}$ if and only if $A + A^*$ is positive semidefinite.*

Then $F(A) = W(A)$ for all $A \in \mathbb{C}^{n\times n}$.

Let $\mathcal{K}(\mathbb{C})$ be the set of nonempty compact subsets in $\mathbb{C}$ equipped with the Hausdorff metric

$$d(\mathcal{A}, \mathcal{B}) = \max\left\{\max_{a\in\mathcal{A}} \min_{b\in\mathcal{B}} |a - b|, \max_{b\in\mathcal{B}} \min_{a\in\mathcal{A}} |a - b|\right\}$$

for any $\mathcal{A}, \mathcal{B} \in \mathcal{K}(\mathbb{C})$. Using the usual topology on $\mathbb{C}^{n\times n}$ and the Hausdorff metric on $\mathcal{K}(\mathbb{C})$, we have the following.

Theorem 1.6 *The mapping $A \mapsto W(A)$ is continuous.*

2 The spectrum

Let $A \in \mathbb{C}^{n\times n}$. Its spectrum $\sigma(A)$ can be viewed as another useful "picture" of the matrix A. For instance, it is known that A is invertible if and only if $0 \notin \sigma(A)$; $\lim_{k\to\infty} A^k$ exists if and only if $\sigma(A) \subseteq \{\mu \in \mathbb{C} : |\mu| < 1\}$; see [31]. Here are some relations between $\sigma(A)$ and $W(A)$.

Theorem 2.1 *Let $A \in \mathbb{C}^{n\times n}$. Then*

$$\sigma(A) \subseteq W(A) \subseteq \{\mu \in \mathbb{C} : |\mu| \le \|A\|\}.$$

Consequently, for any $E \in \mathbb{C}^{n\times n}$, we have

$$\begin{aligned}\sigma(A+E) &\subseteq W(A+E) \subseteq W(A) + W(E) \\ &\subseteq \{\xi + \mu \in \mathbb{C} : \xi \in W(A), \mu \in \mathbb{C} \text{ with } |\mu| \le \|E\|\}.\end{aligned}$$

While $W(A)$ does not give a very tight containment region for $\sigma(A)$ as shown in Example 1.1, Theorem 2.1 shows that the numerical range can be used to estimated the spectrum of the resulting matrix when A is under a perturbation E. In contrast, $\sigma(A)$ and $\sigma(E)$ usually do not carry much information about $\sigma(A+E)$ in general as shown in the following.

Example 2.2 Let $A = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ and $E = \begin{pmatrix} 0 & 0 \\ \varepsilon & 0 \end{pmatrix}$. Then $\sigma(A) = \sigma(E) = \{0\}$, $\sigma(A+E) = \{\pm\sqrt{M\varepsilon}\} \subseteq W(A+E)$, which is the elliptical disk with foci $\pm\sqrt{M\varepsilon}$ and length of minor axis equal to $|\,|M| - |\varepsilon|\,|$.

Let $\partial\mathcal{S}$ and $\mathrm{Int}\,(\mathcal{S})$ be the boundary and the interior of a subset $\mathcal{S}$ of $\mathbb{C}$. A boundary point μ of the convex set $\mathcal{S}$ of $\mathbb{C}$ is non-differentiable if there are more than one support lines of $\mathcal{S}$ passing through μ. We have the following.

Theorem 2.3 *Let $A \in \mathbb{C}^{n\times n}$ and $\mu \in \mathbb{C}$. Then $\mu \in \sigma(A) \cap \partial W(A)$ if and only if A is unitarily similar to $\mu I_k \oplus B$ such that $\mu \notin \sigma(B) \cup \mathrm{Int}\,(W(B))$.*

Theorem 2.4 *Let $A \in \mathbb{C}^{n\times n}$ and $\mu \in \mathbb{C}$. Then μ is a non-differentiable boundary point of $W(A)$ if and only if A is unitarily similar to $\mu I_k \oplus B$ such that $\mu \notin W(B)$. Consequently, non-differentiable boundary points of $W(A)$ are eigenvalues of A and there can be at most n of them. If $W(A)$ has at least $n-1$ non-differentiable boundary points, then A is normal.*

3 Special classes of matrices

The following facts illustrate the interesting interplay between the geometric properties of $W(A)$ and the algebraic properties of $A \in \mathbb{C}^{n\times n}$.

Theorem 3.1 *Let $A \in \mathbb{C}^{n\times n}$.*

(a) $A = \lambda I$ *if and only if* $W(A) = \{\lambda\}$.

(b) $A = A^*$ *if and only if* $W(A) \subseteq \mathbb{R}$.

(c) $A = A^*$ *is positive definite if and only if* $W(A) \subseteq (0, \infty)$.

(d) $A = A^*$ *is positive semidefinite if and only if* $W(A) \subseteq [0, \infty)$.

Theorem 3.2 *If* $A \in \mathbb{C}^{n\times n}$ *is unitarily similar to* $A_1 \oplus A_2$, *then* $W(A) = \operatorname{conv}\{W(A_1) \cup W(A_2)\}$. *Consequently, if* A *is normal, then* $W(A) = \operatorname{conv}\sigma(A)$ *is a convex polygon.*

A matrix $A \in \mathbb{C}^{n\times n}$ is a convexoid if $W(A) = \operatorname{conv}\sigma(A)$. For $n \leq 4$, a matrix is a convexoid if and only if A is normal. In general, we have the following.

Theorem 3.3 *Let* $A \in \mathbb{C}^{n\times n}$. *Then* $W(A)$ *is a convex polygon with vertices* $\mu_1, \ldots, \mu_k$ *if and only if* A *is unitarily similar to* $\operatorname{diag}(\mu_1, \ldots, \mu_k) \oplus B$ *such that* $W(B) \subseteq \operatorname{conv}\{\mu_1, \ldots, \mu_k\}$. *In particular, these conditions hold if and only if* A *is a convexoid.*

Here is a consequence of the above fact.

Theorem 3.4 *Let* $A \in \mathbb{C}^{n\times n}$. *Then* A *is unitary if and only if all eigenvalues of* A *have modulus one and* $W(A) = \operatorname{conv}\sigma(A)$.

Here are some recent results concerning the description of the numerical ranges of some special classes of matrices; see [46, 66].

Theorem 3.5 *Let* $A \in \mathbb{C}^{n\times n}$. *Suppose there are* $\alpha, \beta \in \mathbb{C}$ *such that* $(A - \alpha I)(A - \beta I) = 0_n$. *Then* $W(A)$ *is the elliptical disk with foci* α, β *and minor axis of length*

$$\{(\|A\|^2 - |\alpha|^2)(\|A\|^2 - |\beta|^2)\}^{1/2}/\|A\|.$$

Consequently, the matrix A *is normal if and only if* $\|A\| = \max\{|\alpha|, |\beta|\}$.

A matrix $A \in \mathbb{C}^{n\times n}$ satisfying the hypothesis of Theorem 3.5 is called a quadratic operator. When $(\alpha, \beta) = (1, 0)$, we get the idempotent operator.

Theorem 3.6 *Suppose* $A = (A_{ij})_{1\leq i,j\leq m}$ *such that* $A_{11}, \ldots, A_{mm}$ *are square matrices and* $A_{ij} = 0$ *whenever* $(i, j) \notin \{(1, 2), \ldots, (m-1, m)\}$. *Then* $W(A)$ *is a circular disk centered at* 0 *with radius* $\lambda_1(A + A^*)/2$.

There has been some recent study of the numerical range of a tridiagonal matrix, a weighted shift matrix, and a companion matrix of a polynomial; see [54, 26]. Also, there has been other results concerning when the numerical range is a circular disk or an elliptical disk; see [12, 15, 21].

A closed smooth convex curve in $\mathbb{C}$ satisfies the m-Poncelet property if there are infinitely many m-side convex polygons with vertices on the unit circle containing the curve such that each edge of the polygon touches the curve at exactly one point. For example, the circle centered at the origin with diameter 1 is a Poncelet curve. More generally, any ellipse with foci α, β each has modulus less than one, and minor axis of length $\{(1-|\alpha|^2)(1-|\beta|^2)\}^{1/2}$ is a Poncelet curve. There are infinitely many triangles with vertices on the unit circle containing the ellipse such that each edge of the triangle touches the ellipse at exactly one point. In connection to the numerical range we have the following.

Theorem 3.7 *Suppose $A \in \mathbb{C}^{n\times n}$. Then $I_n - A^*A$ is a rank one positive semidefinite matrix if and only if there are $x, y \in \mathbb{C}^n$ and $\gamma \in \mathbb{C}$ such that $\begin{pmatrix} A & x \\ y^* & \gamma \end{pmatrix} \in \mathbb{C}^{(n+1)\times(n+1)}$ is unitary. If these conditions hold, and all eigenvalues of A have moduli less than one, then $\partial W(A)$ has the $(n+1)$-Poncelet property.*

For recent study of $\partial W(A)$, see [13, 54] and their references. Also, note that the last fact asserts that $W(A)$ has some special property if A is a certain $n \times n$ principal submatrix of a unitary matrix in $\mathbb{C}^{(n+1)\times(n+1)}$. One may see [24] for further discussion about this topic, and see [27] for results on $W(A)$ if A is a certain principal submatrix of a normal matrix.

4 Location of the numerical range

A compact convex set in $\mathbb{C}$ can be written as the intersection of closed half-space and the convex hull of its boundary points. Denote by $\lambda_1(H) \geq \cdots \geq \lambda_n(H)$ the eigenvalues of a Hermitian matrix $H \in \mathbb{C}^{n\times n}$.

Let $A \in \mathbb{C}^{n\times n}$. For each $t \in [0, 2\pi)$, let

$$\mathcal{P}_t = \{\mu \in \mathbb{C} : e^{it}\mu + e^{-it}\bar{\mu} \leq \lambda_1(e^{it}A + e^{-it}A^*)\},$$

and let $x_t \in \mathbb{C}^n$ be a unit eigenvector corresponding to the largest eigenvalue of $e^{it}A + e^{-it}A^*$.

Theorem 4.1 *Let $A \in \mathbb{C}^{n\times n}$. Define P_t and μ_t as above. For each $t \in [0, 2\pi)$,*

$$e^{it}W(A) \subseteq \mathcal{P}_t \quad \text{and} \quad \mu_t = x_t^* A x_t \in \partial W(A) \cap \partial \mathcal{P}_t.$$

We have

$$W(A) = \cap_{t\in[0,2\pi)} \mathcal{P}_t = \text{conv}\,\{\mu_t : t \in [0, 2\pi)\}.$$

Furthermore, suppose $T = \{t_1, \ldots, t_k\}$ with $0 \leq t_1 < \cdots < t_k < 2\pi$ and $k > 2$ such that $t_k - t_1 > \pi$. Then

$$P_T^O(A) = \cap_{t\in T} P_t \qquad \text{and} \qquad P_T^I(A) = \text{conv}\,\{\mu_t : t \in T\}$$

are two polygons in $\mathbb{C}$ such that

$$P_T^I(A) \subseteq W(A) \subseteq P_T^O(A).$$

Moreover, both the area $W(A) \setminus P_T^I(A)$ and the area of $P_T^O(A) \setminus W(A)$ converge to 0 as $\max\{t_j - t_{j-1} : 1 \leq j \leq k+1\}$ converges to 0, where $t_0 = 0, t_{k+1} = 2\pi$.

Also, we have the following Gershgorin type of result for the numerical range.

Theorem 4.2 *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. For each $j = 1, \ldots, n$, let*

$$g_j = \sum_{i\neq j}(|a_{ij}| + |a_{ji}|)/2 \quad \text{and} \quad G_j(A) = \{\mu \in \mathbb{C} : |\mu - a_{jj}| \leq g_j\}.$$

Then

$$W(A) \subseteq \text{conv} \cup_{j=1}^{n} G_j(A).$$

Another important quantity associated with $W(A)$ is its distance to the origin, i.e.,

$$\widetilde{w}(A) = \min\{|\mu| : \mu \in W(A)\}.$$

For example, it is useful in connection to the existence of a real linear combination of a pair of Hermitian matrices to be positive definite, and for the simultaneous diagonalizability of two Hermitian matrices by congruence. Recall that a pair of Hermitian matrices $H, G \in \mathbb{C}^{n\times n}$ are simultaneously diagonalizable by congruence if there is an invertible S such that both S^*HS and S^*GS are in diagonal forms. Evidently, H and G are simultaneously diagonalizable by congruence if and only if $H + iG$ is diagonalizable by congruence. The numerical range and its generalizations are useful in studying simultaneous diagonalization of Hermitian matrices; see [9, 59, 51]. Here is a basic result.

Theorem 4.3 *Let $H, G \in \mathbb{C}^{n\times n}$ be Hermitian matrices. Then $0 \notin W(H + iG)$ if and only if there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha H + \beta G$ is positive definite. If $d(H + iG) > 0$, then $H + iG$ is diagonalizable by congruence, and*

$$d(H + iG) = \min\Big\{w(E + iF) : E + iF \in \mathbb{C}^{n\times n},$$

$$(H + E) + i(G + F) \text{ is not diagonalizable by congruence}\Big\}.$$

One can estimate $w(A)$ and $\widetilde{w}(A)$ as follows.

Theorem 4.4 *Let $A = H + iG$ where $H, G \in \mathbb{C}^{n\times n}$ are Hermitian. Then*

$$w(H + iG) = \max\{\lambda_1(\cos tH + \sin tG) : t \in [0, 2\pi)\}$$

and

$$d(H + iG) = \max\{\{\lambda_n(\cos tH + \sin tG) : t \in [0, 2\pi)\} \cup \{0\}\}.$$

We close this section with the following description of $W(A)$ in terms of algebraic curves; see [36].

Theorem 4.5 *Suppose $A = H + iG$ where $H, G \in \mathbb{C}^{n\times n}$ are Hermitian. Then $W(A)$ is the convex hull of the algebraic curve $\Gamma(A)$ with line equations*

$$\det(xH + yG + zI) = 0.$$

5 Numerical radius

Let $A \in \mathbb{C}^{n\times n}$. The numerical radius of A is defined by

$$w(A) = \max\{|\mu| : \mu \in W(A)\},$$

and the spectral radius of A is defined by

$$\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}.$$

Let N be a norm on $\mathbb{C}^{n\times n}$. It is unitarily invariant if $N(UAV) = N(A)$ for all $A \in \mathbb{C}^{n\times n}$ and unitary $U, V \in \mathbb{C}^{n\times n}$. It is unitary similarity invariant (also known as weakly unitarily invariant) if $N(U^*AU) = N(A)$ for all $A \in \mathbb{C}^{n\times n}$ and unitary $U \in \mathbb{C}^{n\times n}$; see [37].

Theorem 5.1 *The numerical radius $w(\cdot)$ is a unitary similarity invariant norm on $\mathbb{C}^{n\times n}$, but it is not unitarily invariant. For any $A \in \mathbb{C}^{n\times n}$, we have*

$$\rho(A) \le w(A) \le \|A\| \le 2w(A). \tag{5.1}$$

Let $A \in \mathbb{C}^{n\times n}$. It is a radialoid if $\|A\| = \rho(A)$; it is a spectraloid if $w(A) = \rho(A)$.

Theorem 5.2 *Suppose $A \in \mathbb{C}^{n\times n}$ and the minimal polynomial of A has degree m. The following conditions are equivalent.*

(a) *A is a spectraloid, i.e., $\rho(A) = w(A)$.*

(b) *There exists $k \ge 1$ such that A is unitarily similar to $\gamma U \oplus A_2$ for a unitary $U \in \mathbb{C}^{k\times k}$ and $A_2 \in \mathbb{C}^{(n-k)\times(n-k)}$ with $w(A_2) \le w(A) = \gamma$.*

(c) *There exists $k \ge m$ such that $w(A^k) = w(A)^k$.*

Theorem 5.3 *Suppose $A \in \mathbb{C}^{n\times n}$ and the minimal polynomial of A has degree m. The following conditions are equivalent.*

(a) *A is a radialoid, i.e., $\rho(A) = \|A\|$.*

(b) *$w(A) = \|A\|$.*

(c) *There exists $k \ge 1$ such that A is unitarily similar to $\gamma U \oplus A_2$ for a unitary $U \in \mathbb{C}^{k\times k}$ and a $A_2 \in \mathbb{C}^{(n-k)\times(n-k)}$ with $\|A_2\| \le \|A\| = \gamma$.*

(d) *There exists $k \ge m$ such that $\|A^k\| = \|A\|^k$.*

Theorem 5.4 *Suppose $A \in \mathbb{C}^{n\times n}$. The following conditions are equivalent.*

(a) *$\|A\| = 2w(A)$.*

(b) *$W(A)$ is a circular disk centered at origin with radius $\|A\|/2$.*

(c) *A is unitarily similar to $\|A\|(A_1 \oplus A_2)$ such that $A_1 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and $w(A_2) \le 1$.*

A norm N on $\mathbb{C}^{n\times n}$ is multiplicative if $N(AB) \le N(A)N(B)$ for all $A, B \in \mathbb{C}^{n\times n}$.

Theorem 5.5 *Let $A, B \in \mathbb{C}^{n\times n}$. Then*

$$w(AB) \le 2\|A\|w(B), 2w(A)\|B\| \le 4w(A)w(B).$$

So, $4r$ is multiplicative, i.e.,

$$4w(XY) \le (4w(X))(4w(Y)) \qquad \text{for all } X, Y \in \mathbb{C}^{n\times n}.$$

The equality holds if

$$X = Y^t = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

Even the numerical radius is not multiplicative, it satisfies the following power inequality; see [29, Problem 221].

Theorem 5.6 *Let $A \in \mathbb{C}^{n\times n}$ and k be a positive integer. Then*

$$w(A^k) \le w(A)^k.$$

Let N be unitarily invariant norm on $\mathbb{C}^{n\times n}$. Then N is multiplicative if and only if $N(A) \ge \|A\|$ for all $A \in \mathbb{C}^{n\times n}$; these conditions are equivalent to the fact that

$$N(A^k) \le N(A)^k$$

for any $A \in \mathbb{C}^{n\times n}$ and positive integer k. We have the following result for unitary similarity invariant norms.

Theorem 5.7 *Let N be a unitary similarity invariant norm on $\mathbb{C}^{n\times n}$ such that $N(A^k) \le N(A)^k$ for any $A \in \mathbb{C}^{n\times n}$ and positive integer k. Then*

$$N(A) \ge w(A) \qquad \text{for all } A \in \mathbb{C}^{n\times n}.$$

Any two norms N_1 and N_2 on $\mathbb{C}^{n\times n}$ are equivalent, i.e., there are positive numbers α, β such that

$$\alpha N_1(A) \le N_2(A) \le \beta N_1(A) \quad \text{for all } A \in \mathbb{C}^{n\times n}. \tag{5.2}$$

In particular, if N_2 is multiplicative, then $(\beta^2/\alpha)N_1$ is also multiplicative because for any $A, B \in \mathbb{C}^{n\times n}$,

$$(\beta^2/\alpha)N_1(AB) \le (\beta^2/\alpha^2)N_2(AB) \le (\beta^2/\alpha^2)N_2(A)N_2(B) \le (\beta^2/\alpha)N_1(A)(\beta^2/\alpha)N_1(B).$$

So, the fact that $4r$ is multiplicative follows from the inequalities $w(A) \le N(A) \le 2w(A)$ for all $A \in \mathbb{C}^{n\times n}$. One would like to find the best constants, i.e., smallest β and largest α, in (5.2) related to u.s.i. norms. We have the following.

Theorem 5.8 *Suppose N is a unitary similarity invariant norm on $\mathbb{C}^{n\times n}$. Let*

$$D = \begin{cases} 2I_k \oplus 0_k & \text{if } n = 2k, \\ 2I_k \oplus I_1 \oplus 0_k) & \text{if } n = 2k+1. \end{cases}$$

Then

$$\alpha = \min\{N(I_1 \oplus \gamma I_{n-1}) : \gamma \in \mathbb{C},\ |\gamma| \le 1\}$$

and

$$\beta = \max\{N(DU) : U \in \mathbb{C}^{n\times n} \text{ is unitary }\}$$

are the best constants such that

$$\alpha w(A) \le N(A) \le \beta w(A) \qquad \text{for all } A \in \mathbb{C}^{n\times n}.$$

In particular, if N is unitarily invariant, then $\alpha = N(E_{11})$ and $\beta = N(D)$.

Theorem 5.9 *Let $T \in \mathbb{C}^{m\times m}$ be nonzero. Define*

$$N(A) = w(T \otimes A) \qquad \text{for all } A \in \mathbb{C}^{n\times n}. \tag{5.3}$$

Then N is a u.s.i. norm on $\mathbb{C}^{n\times n}$ satisfying

$$N(A_1 \oplus A_2) = \max\{N(A_1 \oplus 0_k), N(0_{n-k} \oplus A_2\}, \tag{5.4}$$

for any $A = A_1 \oplus A_2 \in \mathbb{C}^{n\times n}$ with $A_1 \in \mathbb{C}^{k\times k}$. Moreover, for any $R, A \in \mathbb{C}^{n\times n}$, we have

$$N(R^*AR) \le \|R^*R\| N(A).$$

In fact, if N is a u.s.i. norm on bounded linear operators acting on ℓ_2 such that (5.4) holds for any operator of the form $A = A_1 \oplus A_2$, then N has the form (5.3); see [35].

We close this section with the following fact concerning the matrices with numerical radius at most one; [2, 5] and also [50].

Theorem 5.10 *Let $A \in \mathbb{C}^{n\times n}$. The following are equivalent.*

(a) $w(A) \le 1$.

(b) $\lambda_1(e^{it}A + e^{-it}A^*)/2 \le 1$ *for all* $t \in [0, 2\pi)$.

(c) *There is $Z \in \mathbb{C}^{n\times n}$ such that* $\begin{pmatrix} I_n + Z & A \\ A^* & I_n - Z \end{pmatrix}$ *is positive semidefinite.*

(d) *There exists $X \in \mathbb{C}^{2n\times n}$ satisfying $X^*X = I_n$ and*

$$A = X^* \begin{pmatrix} 0_n & 2I_n \\ 0_n & 0_n \end{pmatrix} X.$$

(e) *There exists an infinite-dimensional Hilbert space $\mathcal{H}$, a unitary operator U acting on $\mathcal{H}$, and an isometry $X : \mathbb{C}^n \to \mathcal{H}$ such that*

$$A^k = 2X^*U^kX \qquad \text{for all } k = 1, 2, \ldots.$$

6 Product of matrices

While $W(A+B) \subseteq W(A)+W(B)$, we do not have $W(AB) \subseteq W(A)W(B)$. We do not have $W(AB) \subseteq \operatorname{conv}\{W(A)W(B)\}$ even if $A = B$.

Example 6.1 Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $\sigma(AB) = \{i, -i\}$, $W(A) = \{\mu \in \mathbb{C} : |\mu - 1| \le 1/2\}$,

$$W(A^2) = \{\mu \in \mathbb{C} : |\mu - 1| \le 1\}$$

where as

$$\operatorname{conv} W(A)^2 \subseteq \{se^{it} \in \mathbb{C} : s \in [0.25, 2.25],\ t \in [-\pi/3, \pi/3]\}.$$

The next example shows that $\sigma(AB) \not\subseteq \operatorname{conv}\{W(A)W(B)\}$.

Example 6.2 Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$W(A) = W(B) = W(A)W(B) = [-1, 1] \qquad \text{and} \qquad W(AB) = i[-1, 1].$$

We have more information about $\sigma(AB)$ if one of the matrices has numerical range not containing the origin; see [32, Chapter 1].

Theorem 6.3 *Let $A, B \in \mathbb{C}^{n\times n}$ be such that $\widetilde{w}(A) > 0$. Then*

$$\sigma(A^{-1}B) \subseteq \{b/a : a \in W(A), b \in W(B)\}$$

and

$$\sigma(AB) \subseteq \{rab : r \ge 0,\ a \in W(A),\ b \in W(B)\}.$$

Theorem 6.4 *Let $0 \le t_1 < t_2 < t_1 + \pi$, and $S = \{re^{it} : r > 0,\ t \in [t_1, t_2]\}$. Then $\sigma(A) \subseteq S$ if and only if there is a positive definite $B \in \mathbb{C}^{n\times n}$ such that $W(AB) \subseteq S$.*

For any $A, B \in \mathbb{C}^{n\times n}$, we have $w(AB) \le 4w(A)w(B)$. We can improve the inequality for special pairs of A and B.

Theorem 6.5 *Let $A, B \in \mathbb{C}^{n\times n}$.*

(a) *If $AB = BA$, then $w(AB) \le 2w(A)w(B)$.*

(b) *If A or B is normal such that $AB = BA$, then $w(AB) \le w(A)w(B)$.*

(c) *If $A^2 = aI$ and $AB = BA$, then $w(AB) \le \|A\|w(B)$.*

(d) *If $AB = BA$ and $AB^* = B^*A$, then $w(AB) \le \min\{w(A)\|B\|, \|A\|w(B)\}$.*

Note that even if $AB = BA$, we may not have $w(AB) \le \min\{w(A)\|B\|, \|A\|w(B)\}$.

Example 6.6 Let $A = E_{12} + E_{23} + \cdots + E_{8,9} \in \mathbb{C}^{9\times 9}$, and $B = A^3 + A^7$. Then $w(A) = w(B) = \cos(\pi/10) < 1$ and $w(AB) = 1 > \|A\|w(B)$.

Let A and B be square matrices. The Hadamard (entrywise) product $A \circ B$ can be viewed as a principal submatrix of the Kronecker product $A \otimes B$. Thus, we have the following.

Theorem 6.7 *Let A and B be square matrices such that A or B is normal. Then*

$$W(A \circ B) \subseteq W(A \otimes B) = \mathrm{conv}\,\{W(A)W(B)\}.$$

Consequently,

$$w(A \circ B) \le w(A \otimes B) = w(A)w(B).$$

For every matrix $A \in \mathbb{C}^{n\times n}$, the matrix

$$\tilde{A} = \begin{pmatrix} A & \sqrt{I_n - AA^*} \\ \sqrt{I_n - A^*A} & -A^* \end{pmatrix} \tag{6.1}$$

satisfies $\tilde{A}^*\tilde{A} = \|A\|^2 I_{2n}$. Thus, we have the following.

Theorem 6.8 *Let A and B be square matrices. Then*

$$w(A \circ B) \le w(A \otimes B) \le \min\{w(A)\|B\|, \|A\|w(B)\} \le 2w(A)w(B).$$

In [4], the authors obtained the following.

Theorem 6.9 *Let $A \in \mathbb{C}^{n\times n}$. Then*

$$w(A \circ X) \le w(X) \quad \text{for all } X \in \mathbb{C}^{n\times n}$$

if and only if $A = B^*WB$ such that W is a contraction and all diagonal entries of B^*B are bonded by 1.

7 Real and nonnegative matrices

Theorem 7.1 *Let $A \in \mathbb{C}^{n\times n}$. Then*

$$W(A^*) = \{\bar{\mu} : \mu \in W(A)\} = \overline{W(A)}.$$

In particular, if A is real, then $W(A)$ is symmetric about the real axis.

Here are some Perron-Frobenius type of theorem of the numerical range; see [33], and [44, 49] for recent developments.

Theorem 7.2 *Let $A \in \mathbb{C}^{n\times n}$ have nonnegative entries. Then $w(A) = \rho(A + A^t)/2$ and there is a unit vector $x \in \mathbb{C}^n$ with nonnegative entries such that $x^tAx = w(A)$. In particular, if A is doubly stochastic, then $w(A) = 1$.*

A nonnegative matrix $A \in \mathbb{C}^{n\times n}$ is irreducible if $n = 1$ or $n \geq 2$ and there does not exist a permutation matrix $P \in \mathbb{C}^{n\times n}$ such that $P^tAP = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$, where A_{11}, A_{22} are square matrices of sizes smaller than n. Suppose $A \in \mathbb{C}^{n\times n}$ is nonnegative irreducible. Then so is $A + A^t$. Further, the primitive index of A is m if A is permutationally similar to $(A_{ij})_{1\leq i,j\leq m}$ so that the diagonal blocks are square matrices (possibly of different sizes) and A_{ij} is zero if $(i,j) \notin \{(1,2),(2,3),\ldots,(m-1,m),(m,1)\}$. We have the following.

Theorem 7.3 *Let $A \in \mathbb{C}^{n\times n}$ be nonnegative and irreducible. Then the set of elements in $W(A)$ attaining $w(A)$ has the form*

$$\{w(A)e^{i2k\pi/m} : k = 0, 1, \ldots, m-1\},$$

where m is the primitive index of A.

8 Dilations and norm estimation

A matrix $A \in \mathbb{C}^{n\times n}$ has a *dilation* $B \in \mathbb{C}^{m\times m}$ if there is $X \in \mathbb{C}^{m\times n}$ such that $X^*X = I_n$ and $X^*BX = A$. Equivalently, B is unitarily similar to a matrix of the form

$$\begin{pmatrix} A & * \\ * & * \end{pmatrix}.$$

For a given matrix $A \in \mathbb{C}^{n\times n}$, one would like to find a dilation B with simple structure and then deduce information on A from that of B.

By Theorem 1.1 (c), if A has a dilation B then

$$W(A) \subseteq W(B).$$

It turns out that this simple relation can be used to find a dilation B of a given $A \in \mathbb{C}^{n\times n}$ with simple structure. Ando [2] (see also [5]) showed that if $W(A) \subseteq W(B) = \{\mu \in \mathbb{C} : |\mu| \leq 1\}$, where $B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, then A has a dilation of the form

$$B \otimes I_n = \begin{pmatrix} 0_n & 2I_n \\ 0_n & 0_n \end{pmatrix}.$$

Mirman [52] (see also [53]) showed that if $W(A) \subseteq W(B) = \operatorname{conv}\{\mu_1, \mu_2, \mu_3\}$, where $B = \operatorname{diag}(\mu_1, \mu_2, \mu_3)$, then A has a dilation of the form $B \otimes I_n$, which is unitarily similar to $\mu_1 I_n \oplus \mu_2 I_n \oplus \mu_3 I_n$. More generally, we have the following; see [16].

Theorem 8.1 *Suppose $B \in \mathbb{C}^{2\times 2}$ or $B \in \mathbb{C}^{3\times 3}$ has a reducing eigenvector. If $W(A) \subseteq W(B)$ then A has a dilation of the form $B \otimes I_m$.*

There is no hope to further extend the result to arbitrary $B \in \mathbb{C}^{3\times 3}$ or normal matrix $B \in \mathbb{C}^{4\times 4}$ as shown in the following.

Example 8.2 Let $A = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$. Suppose

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

Then $W(A) \subseteq W(B)$. However, A does not have a dilation of the form $B \otimes I_m$ for either of the matrices because

$$\|A\| = \sqrt{2} > 1 = \|B\| = \|B \otimes I_m\|.$$

Recall that $A \in \mathbb{C}^{n\times n}$ is a contraction if $\|A\| \le 1$. Using a duality argument on completely positive linear maps, we have the following theorem concerning unitary dilation of a contraction with a certain constraint; see [17].

Theorem 8.3 *Let $r \in [-1, 1]$. Suppose $A \in \mathbb{C}^{n\times n}$ is a contraction with*

$$W(A) \subseteq \mathcal{S} = \{\mu \in \mathbb{C} : \mu + \bar{\mu} \le 2r\}.$$

Then A has a unitary dilation $U \in \mathbb{C}^{2n\times 2n}$ such that $W(U) \subseteq \mathcal{S}$.

Note that without the constraint on the numerical range, every contraction $A \in \mathbb{C}^{n\times n}$ has a unitary dilation of the form (6.1). Using Facts 8.3 and 4.1, one gets the following result conjectured by Halmos; see [17, 28].

Theorem 8.4 *Let $A \in \mathbb{C}^{n\times n}$ be a contraction. Then*

$$W(A) = \cap\{W(U) : U \in \mathbb{C}^{2n\times 2n} \text{ is a unitary dilation of } A\}.$$

By Theorem 8.4, we see that for any $A \in \mathbb{C}^{n\times n}$,

$$W(A) = \cap\{W(B) : B \in \mathbb{C}^{2n\times 2n} \text{ is a normal dilation of } A\}.$$

If A has a dilation of the form $I \otimes B$, then $\|A\| \le \|B\|$. Thus, using the dilation results, one can estimate the norms of matrices using the location of their numerical ranges.

Theorem 8.5 *Let $A \in \mathbb{C}^{n\times n}$.*

(a) *If $W(A)$ lies in an triangle with vertices z_1, z_2, z_3, then*

$$\|A\| \le \max\{|z_1|, |z_2|, |z_3|\}.$$

(b) *If $W(A)$ lies in an ellipse $\mathcal{E}$ with foci λ_1, λ_2, and minor axis of length b, then*

$$\|A\| \le \{\sqrt{(|\lambda_1| + |\lambda_2|)^2 + b^2} + \sqrt{(|\lambda_1| - |\lambda_2|)^2 + b^2}\}/2.$$

More generally, if $W(A)$ lies in the convex hull of the ellipse $\mathcal{E}$ and the point z_0, then

$$\|A\| \le \max\left\{|z_0|, \{\sqrt{(|\lambda_1| + |\lambda_2|)^2 + b^2} + \sqrt{(|\lambda_1| - |\lambda_2|)^2 + b^2}\}/2\right\}.$$

One may also use the location of the numerical ranges to estimate norms of matrices without going through the deep results of dilations. Here is a sample result; see [18].

Theorem 8.6 *Let $A \in \mathbb{C}^{n\times n}$. Suppose there is $t \in [0, 2\pi)$ such that $e^{it}W(A)$ lies in a rectangle R centered at $z_0 \in \mathbb{C}$ with vertices $z_0 \pm \alpha \pm i\beta$ and $z_0 \pm \alpha \mp i\beta$, where $\alpha, \beta > 0$, so that $z_1 = z_0 + \alpha + i\beta$ has the largest magnitude. Then*

$$\|A\| \le \begin{cases} |z_1| & \text{if } R \subseteq \operatorname{conv}\{z_1, \bar{z}_1, -\bar{z}_1\}, \\ \alpha + \beta & \text{otherwise.} \end{cases}.$$

The bound in each case is attainable.

9 Mappings on matrices

Theorem 9.1 *Suppose $f(x + iy) = (ax + by + c) + i(dx + ey + f)$ for some real numbers a, b, c, d, e, f. Define $f(H + iG) = (aH + bG + cI) + i(dH + eG + fI)$ for any two Hermitian matrices $H, G \in \mathbb{C}^{n\times n}$. We have*

$$W(f(H + iG)) = f(W(A)) = \{f(x + iy) : x + iy \in W(A)\}.$$

Theorem 9.2 *Let $\mathbf{D} = \{\mu \in \mathbb{C} : |\mu| \le 1\}$. Suppose $f : \mathbf{D} \to \mathbb{C}$ is analytic in the interior of $\mathbf{D}$ and continuous on the boundary of $\mathbf{D}$.*

(a) *If $f(\mathbf{D}) \subseteq \mathbf{D}$ and $f(0) = 0$, then $W(f(A)) \subseteq \mathbf{D}$ whenever $W(A) \subset \mathbf{D}$.*

(b) *If $f(\mathbf{D}) \subseteq \mathbb{C}_+ = \{\mu \in \mathbb{C} : \mu + \bar{\mu} \ge 0\}$, then $W(f(A)) \subseteq \mathbb{C}_+ \setminus \{(f(0) + \overline{f(0)})/2\}$ whenever $W(A) \subseteq \mathbf{D}$.*

Taking $f(A) = A^k$ in (a), we get the power inequality for the numerical radius.

Suppose $A = PU$ is such that P is positive semidefinite and U is unitary. Then the mapping $A \mapsto P^{1/2}UP^{1/2}$ is called the Aluthge transform of A. One may see [3] for some recent study and related references. Here is a basic result.

Theorem 9.3 *Suppose $\phi(A)$ is the Aluthge transform of $A \in \mathbb{C}^{n\times n}$. Then $W(\phi(A)) \subseteq W(A)$. Moreover, $\lim_{k\to\infty} W(\phi^k(A)) = \operatorname{conv}\sigma(A)$.*

Let $\phi : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ be a linear map. It is unital if $\phi(I_n) = I_m$, it is positive if $\phi(A)$ is positive semidefinite whenever A is positive semidefinite; it is completely positive if the block matrix $(\phi(E_{ij}))_{1\le i,j\le n}$ is positive semidefinite.

Theorem 9.4 *Suppose $\phi : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is a unital positive linear map. Then $W(\phi(A)) \subseteq W(A)$ for all $A \in \mathbb{C}^{n\times n}$.*

Theorem 9.5 *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{m\times m}$. Define a linear map $\phi : \mathrm{span}\,\{B, B^*, I_m\} \to \mathrm{span}\,\{A, A^*, I_n\}$ by*

$$\alpha B + \beta B^* + \gamma I_m \mapsto \alpha A + \beta A^* + \gamma I_n.$$

Then (a) $\iff$ (b) $\Rightarrow$ (c) $\iff$ (d) *hold for the following conditions.*

(a) *ϕ is completely positive,*

(b) *A has a dilation of the form $I \otimes B$.*

(c) *$W(A) \subseteq W(B)$.*

(d) *ϕ is a positive linear map.*

Theorem 9.6 *Let $\phi : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ be linear. Then*

$$W(A) = W(\phi(A)) \quad \text{for all } A \in \mathbb{C}^{n\times n}$$

if and only if there is a unitary $U \in \mathbb{C}^{n\times n}$ such that ϕ has the form

$$X \mapsto U^*XU \qquad \text{or} \qquad X \mapsto U^*X^tU.$$

One can also deduce the result from the following; see [40].

Theorem 9.7 *Let $\phi : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ be linear. The following conditions are equivalent.*

(a) *$w(A) = w(\phi(A))$ for all $A \in \mathbb{C}^{n\times n}$.*

(b) *$\tilde{w}(A) = d(\phi(A))$ for all $A \in \mathbb{C}^{n\times n}$.*

(c) *There is a unitary $U \in \mathbb{C}^{n\times n}$ and a complex unit μ such that ϕ has the form*

$$X \mapsto \mu U^*XU \qquad \text{or} \qquad X \mapsto \mu U^*X^tU.$$

Also, one may consider mappings that are not assumed to be linear *a priori;* see [43].

Theorem 9.8 *A map $\phi : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ satisfies*

$$w(\phi(A) - \phi(B)) = w(A - B) \qquad \text{for all } A, B \in \mathbb{C}^{n\times n}$$

if and only if there is a unitary $U \in \mathbb{C}^{n\times n}$, a matrix $R \in \mathbb{C}^{n\times n}$, and a complex unit μ such that ϕ has the form

$$X \mapsto \mu U^*X^\dagger U + R,$$

where $X^\dagger$ represents one of the following: $X, X^t, \overline{X}, X^*$.

10 Generalizations

This section is still under construction. There are much more to be said.

There are many generalizations of the numerical range and numerical radius motivated by pure and applied areas. We mention a few of them in the following.

10.1 The maximal numerical range and the δ-numerical range

The maximal numerical range of $A \in \mathbb{C}^{n\times n}$ is defined as

$$W_{max}(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1, \|Ax\| = \|A\|\|x\|\}.$$

One can show that $W_{max}(A) = W(X^*AX)$, where X is any $n \times m$ matrix whose columns form an orthonormal basis for the eigenspace of A^*A of the largest eigenvalue. Stampfli [61] used the maximal numerical range to study the norm of derivation operator $X \mapsto AX - XB$ for some given $A, B \in \mathbb{C}^{n\times n}$. Stampfli also introduced the δ-numerical range defined by

$$W_\delta(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1, \|Ax\| \geq \delta\},$$

which is also always a compact convex set; see [20] for more results and references.

10.2 The spatial numerical range and the algebra numerical range

In the context of Banach space and Banach algebra, researchers considered the spatial numerical range and the algebra numerical range. To define these generalized numerical ranges, let ν be a norm on $\mathbb{C}^n$ and ν^* be the dual norm defined by

$$\nu^*(x) = \max\{|y^*x| : \nu(y) \leq 1\}.$$

The spatial numerical range of A (also known as the Bauer fields of values of A) is defined by

$$W_{spec}(A) = \{y^*Ax : \nu(x) = \nu^*(y) = 1 = y^*x\}.$$

Suppose N is a norm on $\mathbb{C}^{n\times n}$, and define the dual norm on linear functional on $\mathbb{C}^{n\times n}$ by

$$N^*(f) = \max\{|f(X)| : X \in \mathbb{C}^{n\times n}, N(X) \leq 1\}.$$

The algebra numerical range of A is defined by

$$W_{alg}(A) = \{f(A) : f \text{ is a linear functional on } \mathbb{C}^{n\times n}, f(I) = 1 = N^*(f)\}.$$

In particular, if N is the induced norm associated with ν on $\mathbb{C}^n$, i.e.,

$$N(A) = \max\{\nu(Ax) : \nu(x) \leq 1\}$$

and every linear functional f on $\mathbb{C}^{n\times n}$ is identified with a matrix $F \in \mathbb{C}^{n\times n}$ such that $f(X) = \operatorname{tr}(XF^*)$, then $\sigma(A) \subseteq W_{alg}(A) = \operatorname{conv} W_{spec}(A)$ for all $A \in \mathbb{C}^{n\times n}$. In particular, $W_{alg}(A)$ is always convex, but $W_{spec}(A)$ may not be. One may also use semi-inner products on a Banach space to define the spatial and algebra numerical ranges. One may see [11] and its references for many interesting results.

10.3 The numerical range with an indefinite or degenerate inner product

Let $H \in \mathbb{C}^{n\times n}$ be Hermitian. We can define an indefinite or degenerate inner product on $\mathbb{C}^{n\times n}$ by $[x,y] = [x,y]_H = y^*Hx$ for all $x, y \in \mathbb{C}^n$, and we can define the numerical range associated with this inner product by

$$W_H(A) = \{[Ax,x]/[x,x] : x \in \mathbb{C}^n, [x,x] \neq 0\}.$$

If H is positive definite, then $W_H(A) = W(H^{1/2}AH^{1/2})$ is convex. Otherwise, $W_H(A)$ may not be convex. However, $W_H(A) = W_H^+(A) \cup W_H^-(A)$, where the sets

$$W_H^+(A) = \{[Ax,x] : x \in \mathbb{C}^n, [x,x] = 1\}$$

and

$$W_H^- A = \{-[Ax,x] : x \in \mathbb{C}^n, [x,x] = -1\} = W_{-H}^+(-A)$$

are convex. One may also consider

$$W_H^0(A) = \{[Ax,x] : x \in \mathbb{C}^n, [x,x] = 0\}.$$

Researchers have used these generalized numerical ranges to study operators acting on Krein spaces, and certain quantum systems; see [7, 42, 47] and their references.

10.4 The q-numerical range, k-numerical range, and C-numerical range

For $q \in \mathbb{C}$ with $|q| \leq 1$, Marcus introduced the q-numerical range defined by

$$W_q(A) = \{y^*Ax : x^*x = 1 = y^*y, y^*x = q\}.$$

For $k \in \{1, \ldots, n\}$, Halmos introduced the k-numerical range of $A \in \mathbb{C}^{n\times n}$ defined by

$$W_k(A) = \{\mathrm{tr}\,(PA) : P^2 = P = P^* \text{ has rank } k\}.$$

For a real vector $c = (c_1, \ldots, c_n)$, Westwick introduced the c-numerical range defined by

$$W_c(A) = \left\{ \sum_{j=1}^n c_j x_j^* A x_j : \{x_1, \ldots, x_n\} \text{ is an orthonormal basis for } \mathbb{C}^n \right\}.$$

For $C \in \mathbb{C}^{n\times n}$, the C-numerical range of $A \in \mathbb{C}^{n\times n}$ is defined by

$$W_C(A) = \{\mathrm{tr}\,(CU^*AU) : U \text{ is unitary}\}.$$

This covers all the q-numerical range (putting $C = qE_{11} + \sqrt{1-|q|^2}E_{12}$), k-numerical range (putting $C = I_k \oplus 0_{n-k}$), and the c-numerical range (putting $C = \mathrm{diag}\,(c_1, \ldots, c_n)$). The C-numerical range and C-numerical radius are useful concepts in studying properties that are invariant under unitary similarities. Here are two facts; see [14, 38].

Theorem 10.1 *Let $C \in \mathbb{C}^{n\times n}$. Then $W_C(A)$ is star-shaped with $(\mathrm{tr}\, C)(\mathrm{tr}\, A)/n$ as a star-center. Moreover, if C is normal with collinear eigenvalues or if $C - \mu I$ has rank one for some $\mu \in \mathbb{C}$, then $W_C(A)$ is convex for all $A \in \mathbb{C}^{n\times n}$. If $C - \mu I$ is unitarily similar to $(C_{ij})_{1\le i,j\le m}$ for some $\mu \in \mathbb{C}$ so that $C_{11}, \ldots, C_{mm}$ are square matrices and C_{ij} is zero for $(i,j) \notin \{(1,2), \ldots, (m-1,m)\}$, then $W_C(A)$ is a circular disk centered at $\mu\mathrm{tr}\, A$ for all $A \in \mathbb{C}^{n\times n}$.*

Define the C-numerical radius of A by $w_C(A) = \max\{|\mu| : \mu \in W_C(A)\}$.

Theorem 10.2 *Let $C \in \mathbb{C}^{n\times n}$. Then w_C is a norm if and only if C is non-scalar and $\mathrm{tr}\, C \neq 0$. Moreover, for any unitary similarity invariant norm N on $\mathbb{C}^{n\times n}$, there is a compact set $\mathcal{S} \subseteq \mathbb{C}^{n\times n}$ such that*

$$N(A) = \max\{w_D(A) : D \in \mathcal{S}\}$$

for all $A \in \mathbb{C}^{n\times n}$.

10.5 The Davis-Wielandt shell and the joint numerical range

Let $A = H + iG \in \mathbb{C}^{n\times n}$ where H and G are Hermitian. The Davis-Wielandt shell is defined by

$$DW(A) = \{(x^*Hx, x^*Gx, x^*A^*Ax) : x \in \mathbb{C}^n, x^*x = 1\} \subseteq \mathbb{R}^3;$$

see [19]. One motivation for introducing this is the following result on normal matrices.

Theorem 10.3 *Let $A \in \mathbb{C}^{n\times n}$. Then following conditions are equivalent.*

(a) *A is normal.*

(b) *$DW(A)$ is the convex hull of a finite set in $\mathbb{R}^{1\times 3}$.*

(c) *$DW(A) = \mathrm{conv}\,\{(a_j, b_j, a_j^2 + b_j^2) : a_j + ib_j \in \sigma(A)\}$.*

More generally, let $A_1, \ldots, A_k \in \mathbb{C}^{n\times n}$ be Hermitian. Then joint numerical range of $A_1, \ldots, A_k$ is defined by

$$W(A_1, \ldots, A_k) = \{(x^*A_1x, \ldots, x^*A_kx) : x \in \mathbb{C}^n, x^*x = 1\}.$$

The joint numerical ranges is always convex if span $\{I_n, A_1, \ldots, A_k\}$ has dimension at most $f(n) = \begin{cases} 3 & \text{if } n > 2, \\ 2 & \text{if } n = 2. \end{cases}$ The joint numerical range is useful in studying the joint behavior of the Hermitian matrices, and it also arises in engineering applications; see [9, 10, 41].

Theorem 10.4 *Suppose $A_1, \ldots, A_k \in \mathbb{C}^{n\times n}$ are linearly independent Hermitian matrices such that and $W(A_1, \ldots, A_k)$ lies in a polyhedron with $k+1$ vertices $v_j = (v_{j1}, \ldots, v_{j,k})$ for $j = 1, \ldots, k+1$. Then $A_1, \ldots, A_k$ have a joint dilation of diagonal operators of the form $I \otimes V_1, \ldots, I \otimes V_k$ with $V_m = \mathrm{diag}\,(v_{1m}, v_{2m}, \ldots, v_{k+1,m})$, i.e., there exists X such that $X^*X = I_m$ and $X^*(I \otimes V_m)X = A_m$.*

10.6 The numerical range of a matrix polynomials

Let $P(\lambda) = A_0 + A_1\lambda + \cdots + A_m\lambda^m$ be a matrix polynomial, where $A_0, \ldots, A_m \in \mathbb{C}^{n\times n}$. The numerical range of $P(\lambda)$ is defined by

$$W(P(\lambda)) = \{\mu \in \mathbb{C} : 0 \in W(P(\mu))\}.$$

One can also define the spectrum of $P(\lambda)$ by

$$\sigma(P(\lambda)) = \{\mu \in \mathbb{C} : \det(P(\mu)) = 0\}.$$

Clearly, we have

$$\sigma(P(\lambda)) \subseteq W(P(\lambda)).$$

One can study different properties including the factorability of $P(\lambda)$ using $W(P(\lambda))$; see [55] and its references. There are also study of the numerical range of a matrix with rational functions as entries, which can be expressed in the form $f(\lambda)^{-1}P(\lambda)$ for some polynomial $f(\lambda)$ and matrix polynomial $P(\lambda)$, defined by

$$W(f(\lambda)^{-1}P(\lambda)) = \{\mu \in \mathbb{C} : f(\mu) \neq 0, \mu \in W(P(\lambda))\};$$

see [1] for recent results and further references.

10.7 The decomposable numerical range

Let $A \in \mathbb{C}^{n\times n}$. The mth decomposable numerical range of $A \in \mathbb{C}^{n\times n}$ is defined by

$$W_m^\wedge(A) = \{\det(X^*AX) : X \text{ is } n \times m,\ X^*X = I_m\}.$$

Let $E_k(X)$ be the kth elementary symmetric function of the eigenvalues of a square matrix X. The (k,m) numerical range of $A \in \mathbb{C}^{n\times n}$ is defined by

$$W_{k,m}^\wedge(A) = \{E_k(X^*AX) : X \text{ is } n \times m, X^*X = I_m\}.$$

The mth decomposable numerical range can be viewed as the decomposable numerical range of the induced operator of A acting on the mth Grassmann space. It is convex if $m = 1$ and $m = n-1$. Similarly, the (k,m) numerical range can be viewed as the decomposable numerical range of the kth derivation of the induced operator of A acting on the mth Grassmann space. When $k = 1$, $W_{k,m}(A)$ reduces to the mth numerical range, which is always convex. It may not be the case if $k > 1$.

One may consider other multilinear structures, and the decomposable numerical ranges of other induced operators and their derivations. The decomposable numerical range is useful in studying operators, and has applications in quantum physics; see [8, 48] and their references.

10.8 Additional generalizations

Note that $W_C(A)$ can be viewed as the image of the unitary similarity orbit $\mathcal{U}(A) = \{U^*AU : U \text{ is unitary}\}$ of $A \in \mathbb{C}^{n\times n}$ under the linear functional $X \mapsto \operatorname{tr}(CX)$. One may consider the image of other adjoint orbits under a linear functional induced by the Killing form arising from Lie algebra, and define the numerical ranges and joint numerical ranges; see [45] and its references.

Researchers have also studied the numerical range of quaternionic matrices (see [6, 60] and its references) to better understand quaternionic matrices; the kth generalized numerical range

$$\mathcal{W}_k(A) = \{(x_1^*Ax_1, \ldots, x_k^*Ax_k) : \{x_1, \ldots, x_k\} \text{ an orthonormal set in } \mathbb{C}^n\}$$

(see [58]) to study the diagonal entries of a matrix of the form U^*AU for different unitary U; the kth matricial range

$$\mathbf{W}_k(A) = \{X^*AX : X \text{ is } n \times k, X^*X = I_k\}$$

(see [22] and its references) to study the connection of a matrix and its compressions on k-dimensional subspaces; the quadratic numerical range of $A \in \mathbb{C}^{n\times n}$ defined by

$$W_k^{(2)}(A) = \left\{\mu \in \mathbb{C} : \det\left(\begin{pmatrix} x^*Ax & x^*Ay \\ y^*Ax & y^*Ay \end{pmatrix} - \mu I_2\right) = 0, x \in V_k, y \in V_k^\perp, x^*x = y^*y = 1\right\},$$

where $V_k = \operatorname{span}\{e_1, \ldots, e_k\}$, (see [65] and its references) to study eigenvalues of matrices.

10.9 Extension to infinite dimensional spaces and algebras

Many of the definitions, questions, and results of the generalized numerical ranges are valid for bounded operators acting on infinite dimensional Hilbert spaces or elements in Banach algebras. One may see our basic references [23, 29, 34] and also [11] for further discussion.

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