Diagonals and Partial Diagonals of Sum of Matrices

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Abstract

Given a matrix A, let $\mathcal{O}(A)$ denote the orbit of A under a certain group action such as

(1) $U(m) \otimes U(n)$ *acting on* $m \times n$ *complex matrices* A *by* $(U, V) * A = UAV^t$,

(2) $O(m) \otimes O(n)$ *or* $SO(m) \otimes SO(n)$ *acting on* $m \times n$ *real matrices* A *by* $(U, V) * A = UAV^t$,

(3) $U(n)$ *acting on* $n \times n$ *complex symmetric or skew-symmetric matrices* A *by* $U * A = UAU^t$,

(4) $O(n)$ *or* $SO(n)$ *acting on* $n \times n$ *real symmetric or skew-symmetric matrices* A *by* $U * A = UAU^t$.

Denote by

$$\mathcal{O}(A_1, \ldots, A_k) = \{X_1 + \cdots + X_k : X_i \in \mathcal{O}(A_i),\ i = 1, \ldots, k\}$$

the joint orbit of the matrices $A_1, \ldots, A_k$. *We study the set of diagonals or partial diagonals of matrices in* $\mathcal{O}(A_1, \ldots, A_k)$, *i.e., the set of vectors* $(d_1, \ldots, d_r)$ *whose entries lie in the* $(1, j_1), \ldots, (r, j_r)$ *positions of a matrix in* $\mathcal{O}(A_1, \ldots, A_k)$ *for some distinct column indices* $j_1, \ldots, j_r$. *In many cases, complete description of these sets is given in terms of the inequalities involving the singular values of* $A_1, \ldots, A_k$. *We also characterize those extreme matrices for which the equality cases hold. Furthermore, some convexity properties of the joint orbits are considered. These extend many classical results on matrix inequalities, and answer some questions by Miranda. Related results on the joint orbit* $\mathcal{O}(A_1, \ldots, A_k)$ *of complex Hermitian matrices under the action of unitary similarities are also discussed.*

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1 Introduction

Let $M_{m,n}(\mathbf{F})$ (respectively, $M_n(\mathbf{F})$) be the set of $m \times n$ (respectively, $n \times n$) matrices over $\mathbf{F}$, where $\mathbf{F}$ is the complex field $\mathbf{C}$ or the real field $\mathbf{R}$. Let $U(n)$ denote the unitary group in $M_n(\mathbf{C})$, and let $O(n)$ and $SO(n)$ denote the orthogonal and special orthogonal group in $M_n(\mathbf{R})$. For notational convenience, we sometimes use $U_n(\mathbf{F})$ to denote the unitary or real orthogonal group depending on $\mathbf{F} = \mathbf{C}$ or $\mathbf{R}$.

Given a matrix A, let $\mathcal{O}(A)$ denote the orbit of A under a certain group action such as

(1) $U(m) \otimes U(n)$ acting on $M_{m,n}(\mathbf{C})$ by $(U, V) * A = UAV^t$,

(2) $O(m) \otimes O(n)$ or $SO(m) \otimes SO(n)$ acting on $M_{m,n}(\mathbf{R})$ by $(U, V) * A = UAV^t$,

(3) $U(n)$ acting on $n \times n$ complex symmetric or skew-symmetric matrices A by $U * A = UAU^t$,

(4) $O(n)$ or $SO(n)$ acting on $n \times n$ real symmetric or skew-symmetric matrices A by $U * A = UAU^t$.

The *joint (additive) orbit* of the matrices $A_1, \ldots, A_k$ is defined by

$$\mathcal{O}(A_1, \ldots, A_k) = \{X_1 + \cdots + X_k : X_i \in \mathcal{O}(A_i),\ i = 1, \ldots, k\},$$

which has been studied extensively in connections with many subjects including matrix inequalities, operator theory, combinatorial theory, Lie theory, and algebraic geometry, see for example, [1, 16] and their references.

In this paper we study the set of diagonal or partial diagonals of matrices in the joint orbit $\mathcal{O}(A_1, \ldots, A_k)$, i.e., the set of r tuples $(d_1, \ldots, d_r)$ whose entries lie in the $(1, j_1), \ldots, (r, j_r)$ positions of a matrix $X \in \mathcal{O}(A_1, \ldots, A_k)$ for some distinct column indices $j_1, \ldots, j_r$. In many cases, complete description of these sets is given in terms of the inequalities involving the singular values of $A_1, \ldots, A_k$. We also characterize those extreme matrices for which the equality cases hold. Furthermore, some convexity properties of the joint orbits are considered. These extend many results in matrix inequalities, and answer some questions raised by Miranda [7] (see Theorem 2.7).

In our discussion, we shall make heavy use of the theory of *majorizations*. We refer the readers to [6] for general background. Here we give some basic definitions and notations. Given two real vectors $\mathbf{x}$ and $\mathbf{y}$ we say that $\mathbf{x}$ is *weakly majorized* by $\mathbf{y}$, denoted by $\mathbf{x} \prec_w \mathbf{y}$ if the sum of the r largest entries of $\mathbf{x}$ is not larger than that of $\mathbf{y}$ for $r = 1, \ldots, n$. If $\mathbf{x} \prec_w \mathbf{y}$ and the sum of entries of $\mathbf{x}$ is the same as that of $\mathbf{y}$, we say that $\mathbf{x}$ is *majorized* by $\mathbf{y}$, denoted by $\mathbf{x} \prec \mathbf{y}$. Let $\mathbf{x} = (x_1, \cdots, x_n)$ be a complex vector, define $|\mathbf{x}| = (|x_1|, \cdots, |x_n|)$. We shall let

$$T = T_{\mathbf{F}} = \{\mu \in \mathbf{F} : |\mu| = 1\}.$$

Furthermore, if $X \in M_n(\mathbf{F})$, the vector of diagonal entries of X is denoted by $\mathrm{diag}\,(X)$. The standard basis for $M_{m,n}(\mathbf{F})$ will be denoted by $\{E_{11}, E_{12}, \ldots, E_{mn}\}$. For a given $A \in M_{m,n}(\mathbf{F})$, let $s(A) = (s_1(A), \ldots, s_m(A))$ be the vector of singular values of A with $s_1(A) \geq \cdots \geq s_m(A)$.

2 Matrices under the action of $U_m(\mathbf{F}) \otimes U_n(\mathbf{F})$

In this section, we consider matrices in $M_{m,n}(\mathbf{R})$ under the group action $U_m(\mathbf{F}) \otimes U_n(\mathbf{F})$ defined by $(U,V) * A = UAV^t$. It is well known the orbit $\mathcal{O}(A)$ of $A \in M_{m,n}(\mathbf{F})$ consists of all matrices $X \in M_{m,n}(\mathbf{F})$ such that $s(X) = s(A)$. Suppose $A_1, \ldots, A_k \in M_{m,n}(\mathbf{F})$ and

$$\mathcal{O}(A_1, \ldots, A_k) = \left\{ \sum_{i=1}^{k} X_i : X_i \in \mathcal{O}(A_i),\ i = 1, \ldots, k \right\}. \tag{2.1}$$

We are interested in studying the set $\mathcal{D}_r(A_1, \ldots, A_k)$ of r-tuples $(d_1, \ldots, d_r)$ such that d_j is the (j,j) entry of a matrix in $\mathcal{O}(A_1, \ldots, A_k)$ for $j = 1, \ldots, r$. Since $X \in \mathcal{O}(A_1, \ldots, A_k)$ if and only if $PXQ \in \mathcal{O}(A_1, \ldots, A_k)$ for any permutation matrices P and Q, the set $\mathcal{D}_r(A_1, \ldots, A_k)$ can be viewed as the set of vectors with entries lying in the $(1, j_1), \ldots, (r, j_r)$ positions of a matrix in $\mathcal{O}(A_1, \ldots, A_k)$ for any distinct column indices $j_1, \ldots, j_r$.

We divide our discussion into several subsections. First, some background is presented in Section 2.1. In Sections 2.2 and 2.3, we give a complete description of the set $\mathcal{D}_r(A_1, \ldots, A_k)$ in terms of inequalities involving the singular values of $A_1, \ldots, A_k$. Furthermore, the extremal matrices for which the inequalities become equalities are characterized. A variation of the problem is considered in Section 2.4. Then we study some convexity properties of the set $\mathcal{D}_r(A_1, \ldots, A_k)$ in Section 2.5.

2.1 Background

In [18] (see also [12]), Thompson obtained necessary and sufficient conditions for a vector $(d_1, \ldots, d_n)$ to be the vector of diagonal entries of a matrix in $\mathcal{O}(A)$ for a given $A \in M_n(\mathbf{F})$.

Proposition 2.1 *Suppose $A \in M_n(\mathbf{F})$ has singular values $s_1 \ge \cdots \ge s_n \ge 0$. Then $d_1, \ldots, d_n \in \mathbf{F}$ with $|d_1| \ge \cdots \ge |d_n|$ are the diagonal entries (in any order) of a matrix in $\mathcal{O}(A)$ if and only if*

$$\sum_{j=1}^{r} |d_j| \le \sum_{j=1}^{r} s_j,\ r = 1, \ldots, n, \tag{2.2}$$

and

$$\sum_{j=1}^{n-1} |d_j| - |d_n| \le \sum_{j=1}^{n-1} s_j - s_n. \tag{2.3}$$

This result was later extended to the product of matrices as follows, see [8, 7, 15].

Proposition 2.2 *Suppose $A_1, \ldots, A_k \in M_n(\mathbf{F})$. Let $s_j = \prod_{i=1}^{k} s_j(A_i)$ for $j = 1, \ldots, n$. Then $d_1, \ldots, d_n \in \mathbf{F}$ with $|d_1| \ge \cdots \ge |d_n|$ are the diagonal entries (in any order) of a matrix of the form $\prod_{i=1}^{k} X_i$ with $X_i \in \mathcal{O}(A_i)$ for each i if and only if (2.2) and (2.3) hold.*

It is interesting that the same conditions (2.2) and (2.3) are necessary and sufficient in the extended result with a simple (natural) modification of the definition for $s_1, \ldots, s_n$. In particular, we have the following consequence.

Corollary 2.3 *Let $A_1, \ldots, A_k \in M_n(\mathbf{F})$ and $s_1, \ldots, s_n$ satisfy the hypothesis of Proposition 2.2. Then $d_1, \ldots, d_n$ are the diagonal entries (in any order) of a matrix $\prod_{i=1}^k X_i$ with $X_i \in \mathcal{O}(A_i)$ for each i if and only if $d_1, \ldots, d_n$ are the diagonal entries of a matrix with singular values $s_1, \ldots, s_n$.*

Evidently, Proposition 2.2 can be viewed as the description of vectors of diagonal entries of matrices in the joint (multiplicative) orbit of $A_1, \ldots, A_k$ defined by

$$\mathcal{P}(A_1, \ldots, A_k) = \left\{ \prod_{j=1}^{k} X_j : X_j \in \mathcal{O}(A_j) \right\}. \tag{2.4}$$

It is natural to ask the corresponding question for the joint (additive) orbit $\mathcal{O}(A_1, \ldots, A_k)$ defined as in (2.1). In fact, the problem was raised explicitly in [7].

Problem 2.4 *Let $A_1, \ldots, A_k \in M_n(\mathbf{F})$. Determine the necessary and sufficient condition for $(d_1, \ldots, d_n)$ to be the vector of diagonal entries of a matrix in $\mathcal{O}(A_1, \ldots, A_k)$ in terms of the singular values $A_1, \ldots, A_k$.*

In view of Propositions 2.1 and 2.2, one might guess that (2.2) and (2.3) are the necessary and sufficient conditions if $s_j = \sum_{i=1}^k s_j(A_i)$. In fact, it follows easily from Proposition 2.1 that the inequalities (2.2) and (2.3) are still sufficient, and that (2.2) is necessary. However, (2.3) may not be necessary, as shown in the following example.

Example 2.5 Let $A_1 = I_2$, and $A_2 = [1] \oplus [-1]$. Then $d_1 = 2$ and $d_2 = 0$. Since $s_1 = s_2 = 2$, we see that $d_1 - d_2 \not\leq s_1 - s_2$.

We shall give a complete answer of Problem 2.4 in the next subsection. The following observation is useful in our discussion.

Lemma 2.6 *Let $A_1, \ldots, A_k \in M_{m,n}(\mathbf{F})$, and $1 \leq r \leq \min\{m, n\}$. Then $(d_1, \ldots, d_r) \in \mathcal{D}_r(A_1, \ldots, A_k)$ if and only if*

$$(\mu_1 d_{i_1}, \ldots, \mu_r d_{i_r}) \in \mathcal{D}_r(A_1, \ldots, A_k)$$

for any permutation $(i_1, \ldots, i_r)$ of $(1, \ldots, r)$ and $\mu_i \in T$ for all $i = 1, \ldots, r$.

By the above lemma, we can always focus on those $d = (d_1, \ldots, d_r) \in \mathcal{D}_r(A_1, \ldots, A_k)$ with $d_1 \geq \cdots \geq d_r \geq 0$.

2.2 Characterization of $\mathcal{D}_r(A_1, \ldots, A_r)$ and extremal matrices: The case $m = n = r$

In this subsection, we characterize the set $\mathcal{D}_n(A_1, \ldots, A_k)$ for $A_1, \ldots, A_k \in M_n(\mathbf{F})$, giving the solution for Problem 2.4. As mentioned before, the most challenging part is to find a suitable replacement for condition (2.3). It turns out that the required condition can be understood from the 1×1 case with an appropriate formulation. We shall not distinguish 1×1 matrices and scalars in our discussion. Thus for $n = 1$ and $A_i = [s_1(A_i)]$ for $i = 1, \ldots, k$, we have

$$\mathcal{D}_1(A_1, \ldots, A_k) = \mathcal{D}_1(s_1(A_1), \ldots, s_1(A_k)) = \left\{ \sum_{i=1}^{k} \mu_i s_1(A_i) : \mu_i \in T \right\}.$$

If $\mathbf{F} = \mathbf{R}$ then $\mathcal{D}_1(A_1, \ldots, A_k)$ is a finite set with at most 2^k points. If $\mathbf{F} = \mathbf{C}$, then $d \in \mathcal{D}_1(A_1, \ldots, A_k)$ if and only if $e^{it} d \in \mathcal{D}_1(A_1, \ldots, A_k)$ for all $t \in \mathbf{R}$. Since $\mathcal{D}_1(A_1, \ldots, A_k)$ is connected, it is an annulus centered at the origin in $\mathbf{C}$. It is not hard to see that the outer radius of the annulus is $s_1 = \sum_{i=1}^{k} s_1(A_i)$. If for some j, $s_1(A_j) > \sum_{i \neq j} s_1(A_i)$, then the inner radius of the annulus is $\rho_0 = s_1(A_j) - \sum_{i \neq j} s_1(A_i)$. Otherwise, $\rho_0 = 0$. Thus,

$$\rho_0 = \min\{|d| : d \in \mathcal{D}_1(A_1, \ldots, A_k)\} = \max\left(\{0\} \cup \{2s_1(A_i) - s_1 : 1 \leq i \leq k\}\right).$$

The result of the 1×1 case can be stated in terms of inequalities so that it can be extended to higher dimensions, namely, $d \in \mathcal{D}_1(A_1, \ldots, A_k)$ if and only if

$$\begin{aligned} 0 &\geq \min\{|d - \rho| : \rho \in \mathcal{D}_1(s_1(A_1), \cdots, s_1(A_k))\} \\ &= \min\{||d| - |\rho|| : \rho \in \mathcal{D}_1(s_1(A_1), \cdots, s_1(A_k))\}. \end{aligned}$$

For higher dimensions, we have the following result.

Theorem 2.7 *Let $A_1, \ldots, A_k \in M_n(\mathbf{F})$. Suppose $s_j = \sum_{i=1}^{k} s_j(A_i)$, and $d_1, \ldots, d_n \in \mathbf{F}$ satisfy $|d_1| \geq \cdots \geq |d_n|$. Then $\mathcal{D}_n(A_1, \ldots, A_k)$ contains a vector with entries $d_1, \ldots, d_n$ in any order if and only if*

$$\sum_{j=1}^{r} |d_j| \leq \sum_{j=1}^{r} s_j, \;\; r = 1, \ldots, n, \tag{2.5}$$

and

$$\sum_{j=1}^{n-1} |d_j| + \min\{||d_n| - |\rho|| : \rho \in \mathcal{D}_1(s_n(A_1), \cdots, s_n(A_n))\} \leq \sum_{j=1}^{n-1} s_j. \tag{2.6}$$

If $\mathbf{F} = \mathbf{C}$, (2.6) can be replaced by the inequality

$$\sum_{j=1}^{n-1} |d_j| - |d_n| \leq \sum_{j=1}^{n-1} s_j - \rho_0, \tag{2.7}$$

where

$$\rho_0 = \min\{|\rho| : \rho \in \mathcal{D}_1(s_n(A_1), \cdots, s_n(A_k))\}.$$

Moreover, suppose $A = \sum_{i=1}^{k} \tilde{A}_i$, where $\tilde{A}_i = \left(a_{pq}^{(i)}\right) \in \mathcal{O}(A_i)$ for $i = 1, \ldots, k$, satisfies $\mathrm{diag}\,(A) = (|d_1|, \ldots, |d_n|)$.

(a) *Let* $1 \leq r \leq n$. *The equality holds in* (2.5) *if and only if* $\tilde{A}_i = B_i \oplus C_i$ *such that* $B_i \in M_r(\mathbf{F})$ *is positive semi-definite and has eigenvalues* $s_1(A_i), \ldots, s_r(A_i)$, *for all* $i = 1, \ldots, k$.

(b) *The equality holds in* (2.6) *if and only if* $a_{jj}^{(i)} \geq 0$ *for all* $i = 1, \ldots, k$, $j = 1, \ldots, n-1$,

$$\sum_{i=1}^{k} \left| |a_{nn}^{(i)}| - s_n(A_i) \right| = \min\{|d_n - \rho| : \rho \in \mathcal{D}_1(s_n(A_1), \ldots, s_n(A_k))\},$$

and there exists a diagonal matrix $D = I_{n-1} \oplus [\mu] \in U_n(\mathbf{F})$ *such that*

(1) *if* $\left|a_{nn}^{(i)}\right| > s_n(A_i)$, *then* $\mu a_{nn}^{(i)} = |a_{nn}^{(i)}|$ *and* DA_i *is hermitian with eigenvalues* $s_1(A_i), \ldots, s_{n-1}(A_i), s_n(A_i)$;

(2) *if* $\left|a_{nn}^{(i)}\right| < s_n(A_i)$, *then* $\mu a_{nn}^{(i)} = -|a_{nn}^{(i)}|$ *and* DA_i *is hermitian with eigenvalues* $s_1(A_i), \ldots, s_{n-1}(A_i), -s_n(A_i)$;

(3) *if* $\left|a_{nn}^{(i)}\right| = s_n(A_i)$, *then* $DA_i = B_i \oplus [\mu a_{nn}^{(i)}]$, *where* B_i *is hermitian with eigenvalues* $s_1(A_i), \ldots, s_{n-1}(A_i)$.

Furthermore, if the equality holds in (2.6) *and there exists* i_0, *such that* $s_n(A_{i_0}) > \left|a_{nn}^{(i_0)}\right| = 0$, *then* $s_n(A_i) = \left|a_{nn}^{(i)}\right|$ *for all* $i \neq i_0$.

Note that one may replace (2.6) by

$$\sum_{j=1}^{n-1} |d_j| + \min\{|d_n - \rho| : \rho \in \mathcal{D}_1(s_n(A_1), \cdots, s_n(A_k))\} \leq \sum_{j=1}^{n-1} s_j. \tag{2.8}$$

For $\mathbf{F} = \mathbf{R}$, there are more numbers of the form $|d_n - \rho|$ than the numbers of the form $||d_n| - |\rho||$ with $\rho \in \mathcal{D}_1(s_n(A_1), \cdots, s_n(A_k))$. Hence, there is an advantage for considering (2.6). For $\mathbf{F} = \mathbf{C}$, if at least two of $s_n(A_1), \cdots, s_n(A_k)$ are not equal to 0, then there are infinite many numbers of the form $|d_n - \rho|$ or $||d_n| - |\rho||$ with $\rho \in \mathcal{D}_1(s_n(A_1), \cdots, s_n(A_k))$. Nonetheless, (2.6) can be replaced by the single inequality (2.7).

Proof of Theorem 2.7 Let us first prove the fact that the conditions (2.5) and (2.6) are equivalent to (2.5) and (2.7) when $\mathbf{F} = \mathbf{C}$. Let

$$c = \min\{||d_n| - |\rho|| : \rho \in \mathcal{D}_1(s_n(A_1), \cdots, s_n(A_k))\} = ||d_n| - |\rho_1||,$$

where $\rho_1 \in \mathcal{D}_1(s_n(A_1), \cdots, s_n(A_k))$.

Suppose (2.5) and (2.6) hold. We have

$$\sum_{j=1}^{n-1} |d_j| + \rho_0 - |d_n| \leq \sum_{j=1}^{n-1} |d_j| + |\rho_1| - |d_n| \leq \sum_{j=1}^{n-1} |d_j| + ||\rho_1| - |d_n|| \leq \sum_{j=1}^{n-1} s_j.$$

Hence,

$$\sum_{j=1}^{n-1} |d_j| - |d_n| \le \sum_{j=1}^{n-1} s_j - \rho_0 \,.$$

Conversely, suppose (2.5) and (2.7) hold. If $|d_n| \le \rho_0$, then $c = \rho_0 - |d_n|$; so (2.7) gives $\sum_{j=1}^{n-1} |d_j| - |d_n| \le \sum_{j=1}^{n-1} s_j - \rho_0$. It follows that $\sum_{j=1}^{n-1} |d_j| + c \le \sum_{j=1}^{n-1} s_j$.

If $\rho_0 \le |d_n| \le s_n$, then $c = 0$ and (2.6) reduces to (2.5) with $r = n-1$. If $|d_n| > s_n$, then $c = |d_n| - s_n$ and (2.6) follows from (2.5) for $r = n$.

For the sufficiency part of Theorem 2.7, let $\mathbf{d} = (d_1, \cdots, d_n)$, with $|d_1| \ge \cdots \ge |d_n|$ satisfying (2.5) and (2.6). We are going to show that $\mathbf{d} \in \mathcal{D}_n(A_1, \ldots, A_k)$.

Suppose

$$\min\{||d_n| - |\rho|| : \rho \in \mathcal{D}_1(s_n(A_1), \cdots, s_n(A_k))\} = ||d_n| - |\rho_1||,$$

where $|\rho_1| = \sum_{i=1}^{k} \mu_i s_n(A_i) \in \mathcal{D}_1(s_n(A_1), \cdots, s_n(A_k))$. We have

$$\sum_{j=1}^{n-1} |d_j| + |\rho_1| - |d_n| \le \sum_{j=1}^{n-1} |d_j| + ||d_n| - |\rho_1|| \le \sum_{j=1}^{n-1} s_j.$$

Hence,

$$\sum_{j=1}^{n-1} |d_j| - |d_n| \le \sum_{j=1}^{n-1} s_j - |\rho_1| \,.$$

So, by Proposition 2.1, there exist $U, V \in U_n(\mathbf{F})$ such that

$$\begin{aligned}
\mathbf{d} &= \operatorname{diag}\left(U \begin{pmatrix} s_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & s_{n-1} & 0 \\ 0 & \cdots & 0 & |\rho_1| \end{pmatrix} V \right) \\
&= \operatorname{diag}\left(\sum_{i=1}^{k} U \begin{pmatrix} s_1(A_i) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & s_{n-1}(A_i) & 0 \\ 0 & \cdots & 0 & \mu_i s_n(A_i) \end{pmatrix} V \right) \in \mathcal{D}_n(A_1, \ldots, A_n).
\end{aligned}$$

We finish the proof of Theorem 2.7 by proving the necessity of (2.5) – (2.7) and the equality cases in (a) and (b). To achieve that, we need the following lemma adapted from Theorems 3.1 and 3.4 in [5] (see also the proof of Lemma 5 in [18]).

Lemma 2.8 *Let $A = (a_{pq}) \in M_n(\mathbf{F})$ have nonnegative diagonal entries.*

(a) *Suppose $1 \le r \le n$. Then $\sum_{j=1}^{r} a_{jj} = \sum_{j=1}^{r} s_j(A)$ if and only if $A = B \oplus C$ such that $B \in M_r(\mathbf{F})$ satisfies $B = B^*$ with eigenvalues $s_1(A), \ldots, s_r(A)$.*

(b) *We have* $\sum_{j=1}^{n-1} a_{jj} - a_{nn} = \sum_{j=1}^{n-1} s_j(A) - s_n(A)$ *if and only if there exists* $D = I_{n-1} \oplus [\mu] \in U_n(\mathbf{F})$ *such that* $DA = A^*D^*$ *has eigenvalues* $s_1(A), \ldots, s_{n-1}(A), -s_n(A)$.

To prove the necessity of (2.5), let $\mathbf{d} = \operatorname{diag}(\sum_{i=1}^k U_iA_iV_i)$, where $U_1, \ldots, U_k, V_1, \ldots, V_k \in U_n(\mathbf{F})$, and $d_1 \geq \cdots \geq d_n \geq 0$. Let $\mathbf{d}^{(i)} = \operatorname{diag}(U_iA_iV_i)$, $\mathbf{s}^{(i)} = (s_1(A_i), \cdots, s_n(A_i))$ and $\mathbf{s} = (s_1, \cdots, s_n)$. By Proposition 2.1, we have $|\mathbf{d}^{(i)}| \prec_w \mathbf{s}^{(i)}$ for $1 \leq i \leq k$. Therefore,

$$\mathbf{d} = |\sum_{i=1}^k \mathbf{d}^{(i)}| \prec_w \sum_{i=1}^k |\mathbf{d}^{(i)}| \prec_w \sum_{i=1}^k \mathbf{s}^{(i)} = \mathbf{s}.$$

This proves (2.5).

Suppose $1 \leq r \leq n$ and the equality holds in (2.5). Then the sum of the first r diagonal entries of $U_iA_iV_i$ must equal $\sum_{j=1}^r s_j(A_i)$. By Lemma 2.8 (a), we see that $U_iA_iV_i = B_i \oplus C_i$ such that $B_i \in M_r(\mathbf{F})$ is positive semi-definite and has eigenvalues $s_1(A_i), \ldots, s_r(A_i)$. Conversely, if $U_iA_iV_i$ has the above structure, then clearly (2.5) is an equality. Thus (a) is valid.

Next, we turn to (2.6). For the convenience of notation, let $\mathcal{D}_1 = \mathcal{D}_1(s_n(A_1), \cdots, s_n(A_k))$, and let $U_iA_iV_i = \left(a_{pq}^{(i)}\right)$. Choose $\xi_i \in T$ such that $a_{nn}^{(i)} = \xi_i |a_{nn}^{(i)}|$ for $i = 1, \ldots, k$. We have

$$|a_{nn}^{(i)} - \xi_i s_n(A_i)| = \left| |a_{nn}^{(i)}| - s_n(A_i) \right| \qquad \text{for } i = 1, \ldots, k.$$

Let $\rho_1 = \sum_{i=1}^k \xi_i s_n(A_i) \in \mathcal{D}_1$. Choose a partition $\{I_1, I_2, I_3\}$ of $K = \{1, 2, \cdots, k\}$ such that

$$I_1 = \{i \in K : |a_{nn}^{(i)}| > s_n(A_i)\},\ I_2 = \{i \in K : |a_{nn}^{(i)}| < s_n(A_i)\},\ I_3 = \{i \in K : |a_{nn}^{(i)}| = s_n(A_i)\}.$$

Then

$$\begin{aligned}
& \sum_{j=1}^{n-1} |d_j| + \min\{||d_n| - |\rho|| : \rho \in \mathcal{D}_1\} \\
= & \sum_{j=1}^{n-1} |d_j| + \min\{|d_n - \rho| : \rho \in \mathcal{D}_1\} \\
\leq & \sum_{j=1}^{n-1} |d_j| + |d_n - \rho_1| \\
= & \sum_{j=1}^{n-1} |\sum_{i=1}^k a_{jj}^{(i)}| + |\sum_{i=1}^k \xi_i \left(\left| a_{nn}^{(i)} \right| - s_n(A_i) \right)| \\
\leq & \sum_{i=1}^k \left\{ \sum_{j=1}^{n-1} |a_{jj}^{(i)}| + \left| \xi_i \left(\left| a_{nn}^{(i)} \right| - s_n(A_i) \right) \right| \right\} \\
= & \sum_{i \in I_1} \left\{ \sum_{j=1}^{n-1} |a_{jj}^{(i)}| + |a_{nn}^{(i)}| - s_n(A_i) \right\} + \sum_{i \in I_2} \left\{ \sum_{j=1}^{n-1} |a_{jj}^{(i)}| - |a_{nn}^{(i)}| + s_n(A_i) \right\} + \sum_{i \in I_3} \sum_{j=1}^{n-1} |a_{jj}^{(i)}|
\end{aligned}$$

$$\leq \sum_{\ell=1}^{3}\left(\sum_{i\in I_\ell}\sum_{j=1}^{n-1} s_j(A_i)\right) \qquad \text{(applying Proposition 2.1 to each } U_iA_iV_i)$$
$$= \sum_{i=1}^{k}\sum_{j=1}^{n-1} s_j(A_i) = \sum_{j=1}^{n-1} s_j.$$

Furthermore, the equality holds in (2.6) if and only if all the above inequalities become equalities. Thus, $a_{jj}^{(i)}$ is nonnegative for all $i = 1, \ldots, k$, and $j = 1, \ldots, n-1$, and

$$\begin{aligned}
&\min\{|d_n - \rho| : \rho \in \mathcal{D}_1\} \\
&= \left|\sum_{i\in I_1} \xi_i \left(\left|a_{nn}^{(i)}\right| - s_n(A_i)\right) + \sum_{i\in I_2} -\xi_i \left(s_n(A_i) - \left|a_{nn}^{(i)}\right|\right)\right| \\
&= \sum_{i\in I_1} \left(\left|a_{nn}^{(i)}\right| - s_n(A_i)\right) + \sum_{i\in I_2} \left(s_n(A_i) - \left|a_{nn}^{(i)}\right|\right) \\
&= \sum_{i=1}^{n} \left|\left|a_{nn}^{(i)}\right| - s_n(A_i)\right| . \qquad (2.9)
\end{aligned}$$

If $I_1 \cup I_2 = \emptyset$, set $\mu = 1$; otherwise, we can set $\mu = \xi_i = -\xi_{i'}$ for all $i \in I_1$ and $i' \in I_2$. Let $D = I_{n-1} \oplus [\mu]$. Then, by Lemma 2.8,

1. for each $i \in I_1$, we have $\mu a_{nn}^i = |a_{nn}^{(i)}|$ and $\operatorname{tr} D\tilde{A}_i = \sum_{j=1}^n \left|a_{jj}^{(i)}\right| = \sum_{j=1}^n s_j(A_i)$, hence $D\tilde{A}_i$ is hermitian with eigenvalues $s_1(A_i), \ldots, s_{n-1}(A_i), s_n(A_i)$;

2. for each $i \in I_2$, we have $\mu a_{nn}^{(i)} = -|a_{nn}^{(i)}|$ and

$$\sum_{j=1}^{n-1} \left|a_{jj}^{(i)}\right| - \left|a_{nn}^{(i)}\right| = \operatorname{tr} D\tilde{A}_i = \sum_{j=1}^{n-1} s_j(A_i) - s_n(A_i),$$

   hence $D\tilde{A}_i$ is hermitian with eigenvalues $s_1(A_i), \ldots, s_{n-1}(A_i), -s_n(A_i)$;

3. for each $i \in I_3$, we have $\sum_{j=1}^{n-1} \left|a_{jj}^{(i)}\right| = \sum_{j=1}^{n-1} s_j(A_i)$, and hence $\tilde{A}_i = B_i \oplus [a_{nn}^{(i)}]$.

Finally, if inequalities (2.6) becomes an equality and there exists i_0 such that $s_n(A_{i_0}) > \left|a_{nn}^{(i_0)}\right| = 0$, then the equalities hold in (2.9) for any choice of $\xi_{i_0} \in T$. It follows that $s_n(A_i) = \left|a_{nn}^{(i)}\right|$ for all $i \neq i_0$. □

Remark 2.9 In [10] the authors give a necessary and sufficient condition for $s_1 \geq \cdots \geq s_n$ to be the singular values of a matrix in $\mathcal{O}(A_1, \ldots, A_k)$. In principle, one can solve Problem 2.4 by studying all the possible singular values of matrices in $\mathcal{O}(A_1, \ldots, A_k)$ and then applying Proposition 2.1. However, the condition in [10] involves a large set of inequalities, which are difficult to write down especially for the real case. It does not seem to be possible to deduce our result using this method.

2.3 Characterization of $\mathcal{D}_r(A_1, \ldots, A_r)$ and extremal matrices: The remaining cases

Next, we turn to $\mathcal{D}_r(A_1, \ldots, A_k)$ with $A_1, \ldots, A_k \in M_{m,n}(\mathbf{F})$ when $m \neq n$ or $r < m = n$, i.e., all other cases not covered by Theorem 2.7.

Theorem 2.10 *Let $A_1, \ldots, A_k \in M_{m,n}(\mathbf{F})$. Suppose $1 \le r \le \min\{m, n\}$ such that $m \neq n$ or $r < m = n$. Let $s_j = \sum_{i=1}^k s_j(A_i)$ for $i = 1, \ldots, n$. The following conditions are equivalent.*

(a) *Up to a (any) permutation of the entries, the vector $(d_1, \ldots, d_r)$ is in $\mathcal{D}_r(A_1, \ldots, A_r)$.*

(b) *Up to a (any) permutation of the entries, the vector $(d_1, \ldots, d_r)$ is in $\mathcal{D}_r(A)$ for some $A \in M_n(\mathbf{F})$ with $s(A) = (s_1, \ldots, s_m)$.*

(c) *$(|d_1|, \ldots, |d_r|) \prec_w (s_1, \ldots, s_r)$.*

Furthermore, suppose $X_i \in \mathcal{O}(A_i)$ for $i = 1, \ldots, k$, and so that the first r diagonal entries of $X_1 + \cdots + X_k$ equal $d_1, \ldots, d_r$. Then $|d_1| + \cdots + |d_r| = s_1 + \cdots + s_r$ if and only if there exists a diagonal matrix $D \in U_m(\mathbf{F})$ such that

$$DX_i = \begin{pmatrix} Y_i & 0 \\ 0 & Z_i \end{pmatrix}, \quad i = 1, \ldots, k,$$

where $Y_i \in M_r(\mathbf{F})$ is hermitian with eigenvalues $s_1(A_i), \ldots, s_r(A_i)$.

Proof. The implication (a) $\Rightarrow$ (c) follows from Theorem 2.7. It is clear that (b) $\Rightarrow$ (a).

Consider the implication (c) $\Rightarrow$ (b). First, assume that $r < m = n$. Suppose (c) holds. Let $d_j = \min\{s_j, |d_r|\}$ for $j = r+1, \ldots, n$. One easily checks that $d_1, \ldots, d_n$ satisfy (2.5) and (2.6). By Theorem 2.7, there exists a matrix of the form $\sum_{j=1}^k U_j A_j V_j$ with diagonal entries $d_1, \ldots, d_n$. Thus condition (a) holds.

Next, consider the case when $m \neq n$. Without loss of generality, we may assume that $m > n$. We prove (c) $\Rightarrow$ (b) by induction on n. We may assume that $r = n$ by setting $d_j = 0$ for $r < j \le n$. Let $A_0 = \sum_{j=1}^n s_j E_{jj} \in M_{m,n}(\mathbf{F})$. It suffices to show that one can construct a matrix of the form $A = UA_0V$ with $(1,1), \ldots, (n,n)$ entries equal to $|d_1|, \ldots, |d_n|$.

If $n = 1$, then $A = |d_1|E_{11} + \sqrt{s_1^2 - |d_1|^2}E_{21}$ is a required matrix. Suppose the result holds for matrices with fewer than n columns. Let k be the largest integer such that $s_k \ge |d_1|$.

If $k < n$, then there exists a 2×2 unitary matrix U such that $U^*\mathrm{diag}\,(s_k, s_{k+1})U$ has diagonal entries $|d_1|$ and $t = s_k + s_{k+1} - |d_1|$. Let $B = \mathrm{diag}\,(s_k, s_{k+1}, s_1, s_2, \ldots, s_{k-1}, s_{k+2}, \ldots, s_n) \oplus 0_{m-n}$. Then

$$(U^* \oplus I_{m-2}) \begin{pmatrix} B \\ 0_{m-n,n} \end{pmatrix} (U \oplus I_{n-2}) = \begin{pmatrix} |d_1| & * \\ * & B_0 \end{pmatrix},$$

where B_0 has vector of singular values $(s_1, \ldots, s_{k-1}, t, s_{k+2}, \ldots, s_n)$. One easily checks that $(|d_2|, \ldots, |d_n|) \prec_w (s_1, \ldots, s_{k-1}, t, s_{k+2}, \ldots, s_n)$. By induction assumption, there exist unitary W and V so that WB_0V has diagonal entries $|d_2|, \ldots |d_n|$. Consequently, the matrix

$$A = ([1] \oplus W)(U^* \oplus I_{m-2})B(U \oplus I_{n-2})([1] \oplus V)$$

has $(1,1),\ldots,(n,n)$ entries equal to $|d_1|,\ldots,|d_n|$.

If $k=n$, then $(|d_1|,|d_2|)\prec_w (s_1,s_2)$ and $(|d_2|,\ldots,|d_n|)\prec_w (|d_2|,s_3,\ldots,s_n)$. So the result follows from an argument similar to the one in the previous paragraph.

Next, we turn to the last assertion. The sufficiency part is clear. Conversely, suppose $\sum_{j=1}^r s_j = \sum_{j=1}^r |d_j|$. Let D be a diagonal matrix in $U_m(\mathbf{F})$ such that the first r diagonal entries of $D(X_1+\cdots+X_k)$ are nonnegative. Let $DX_i = \begin{pmatrix} Y_i & * \\ * & Z_i \end{pmatrix}$ with $Y_i \in M_r(\mathbf{F})$. Since $s_j(Y_i)\le s_j(A_i)$ for $j=1,\ldots,r$, we have

$$\sum_{j=1}^r s_j = \sum_{i=1}^k \operatorname{tr} Y_i \le \sum_{i=1}^k\sum_{j=1}^r s_j(A_i) = \sum_{j=1}^r s_j.$$

This implies that $s_j(Y_i)=s_j(A_i)$ for $j=1,\ldots,r$, and $\operatorname{tr} Y_i = \sum_{j=1}^r s_j(Y_i)$. Append rows or columns to the matrix DA_i to get a square matrix if necessary. By Lemma 2.8, the resulting matrix is a direct sum of Y_i and another matrix. The result follows. □

2.3 A variation arising from Lie theory

In this subsection, we consider the set

$$\operatorname{Re}\mathcal{D}_r(A_1,\ldots,A_k) = \{(\operatorname{Re} z_1,\ldots,\operatorname{Re} z_r) : (z_1,\ldots,z_r)\in \mathcal{D}_r(A_1,\ldots,A_k)\},$$

which arises naturally if one uses the Lie theory approach to matrix inequalities (see [16]). By the results in the last two subsections , one easily deduce the following statement using the approach in [16, Theorem 6].

Proposition 2.11 *Let $A_1,\ldots,A_k\in M_{m,n}(\mathbf{C})$, and $1\le r\le \min\{m,n\}$. Suppose $s_j = \sum_{i=1}^k s_j(A_i)$ for all $j=1,\ldots,m$.*

(a) *Up to a (any) permutation of the entries, the vector $(d_1,\ldots,d_r)$ is in $\operatorname{Re}\mathcal{D}_r(A_1,\ldots,A_k)$.*

(b) *Up to a (any) permutation of the entries, the vector $(d_1,\ldots,d_r)$ is in $\operatorname{Re}\mathcal{D}_r(A)$ with $A=\sum_{i=1}^p s_iE_{ii}$ with $p=\min\{m,n\}$.*

(c) $(|d_1|,\ldots,|d_r|)\prec_w (s_1,\ldots,s_r)$.

Furthermore, suppose $X_i\in\mathcal{O}(A_i)$ for all $i=1,\ldots,k$, so that the first r diagonal entries of $X_1+\cdots+X_k$ have real parts $d_1,\ldots,d_r$. Then $|d_1|+\cdots+|d_r| = s_1+\cdots+s_r$ if and only if there exists a diagonal matrix $D\in U_m(\mathbf{C})$ such that

$$DX_i = \begin{pmatrix} Y_i & 0 \\ 0 & Z_i \end{pmatrix}, \quad i=1,\ldots,k,$$

where $Y_i\in M_r(\mathbf{C})$ is hermitian with eigenvalues $s_1(A_i),\ldots,s_r(A_i)$.

Proof. The implication (b) $\Rightarrow$ (a) is clear. Suppose $(d_1, \ldots, d_r) = (\operatorname{Re} z_1, \ldots, \operatorname{Re} z_r)$ where $(z_1, \ldots, z_r) \in \mathcal{D}_r(A_1, \ldots, A_k)$. Then by the previous results, we have

$$(|d_1|, \ldots, |d_r|) \prec_w (|z_1|, \ldots, |z_r|) \prec_w (s_1, \ldots, s_r).$$

Thus the implication (a) $\Rightarrow$ (c) is proved.

Suppose (c) holds. Then (see [6]) there exists a nonnegative vector $(c_1, \ldots, c_r)$ such that $|d_i| \le c_i$ for $1 \le i \le r$ and $(c_1, \ldots, c_r) \prec (s_1, \ldots, s_r)$. By the result in [2], there exists a real symmetric matrix A_0 with eigenvalues $s_1, \ldots, s_r$ and diagonal entries $c_1, \ldots, c_r$. One can multiply A_0 by a diagonal unitary matrix D_0 on the left so that the diagonal entries of $D_0 A_0$ are $z_1, \ldots, z_r$ with $\operatorname{Re} z_i = d_i$. Suppose $A \in M_{m,n}(\mathbf{R})$ such that the (i,i) entry equal s_i for $i = 1, \ldots, \min\{m,n\}$, and zero otherwise. Then there exists a matrix of the form $\begin{pmatrix} D_0 A_0 & 0 \\ 0 & * \end{pmatrix} \in \mathcal{O}(A)$, and hence $(d_1, \ldots, d_r) \in \operatorname{Re} \mathcal{D}_r(A)$. So, condition (b) holds.

The proof of the last assertion is similar to that of Theorem 2.10. □

2.4 Convexity properties

A subset S of $\mathbf{F}^n$ is said to be star-shaped with star-center $c \in S$ if $t\,s + (1-t)c \in S$ for all $s \in S$ and $0 \le t \le 1$. It follows from Proposition 2.1 that if $n \ge 2$, $\mathcal{D}_r(A_1, \ldots, A_k)$ and $\operatorname{Re} \mathcal{D}_r(A_1, \ldots, A_k)$ are star-shaped with $(0, \ldots, 0)$ as a star-center. Next, we consider the convexity property of $\mathcal{D}_r(A_1, \ldots, A_k)$ and $\operatorname{Re} \mathcal{D}_r(A_1, \ldots, A_k)$.

Theorem 2.12 *Let $A_1, \ldots, A_k \in M_{m,n}(\mathbf{F})$, and $1 \le r \le \min\{m,n\}$.*

(a) *The set $\operatorname{Re} \mathcal{D}_r(A_1, \ldots, A_k)$ is always convex.*

(b) *Except for the case when $m = n = r$ the set $\mathcal{D}_r(A_1, \ldots, A_k)$ is always convex.*

(c) *For $m = n$, and $\mathbf{F} = \mathbf{C}$, $\mathcal{D}_n(A_1, \ldots, A_k)$ is convex if and only if*

$$\min\{|\rho| : \rho \in \mathcal{D}_1(s_n(A_1), \cdots, s_n(A_n))\} = 0.$$

(d) *For $m = n$, and $\mathbf{F} = \mathbf{R}$, $\mathcal{D}_n(A_1, \ldots, A_k)$ is convex if and only if $s_n = 0$.*

(e) *In all cases, the convex hull of $\mathcal{D}_r(A_1, \ldots, A_k)$ is the set*

$$\mathcal{C} = \{\mathbf{d} \in \mathbf{F}^r : |\mathbf{d}| \prec_w (s_1, \ldots, s_r)\}$$

(f) *For every $\mathbf{d} \in \mathcal{C}$, there exist $\mathbf{a}$, $\mathbf{b} \in \mathcal{D}_r(A_1, \ldots, A_k)$ and $0 \le t \le 1$ such that $\mathbf{d} = t\,\mathbf{a} + (1-t)\mathbf{b}$. If $\mathbf{F} = \mathbf{C}$, we can choose $t = 1/2$.*

Proof. Assertions (a) and (b) follow from Proposition 2.11 and Theorem 2.10 .

For (c), suppose that $m = n$, and $\mathcal{D}_n(A_1, \ldots, A_k)$ is convex. $\mathcal{D}_n(A_1, \ldots, A_k)$. Since $(s_1, \cdots, s_n)$ and $(s_1, \cdots, -s_n)$ both are in $\mathcal{D}_n(A_1, \ldots, A_k)$, we have

$$(s_1, \cdots, s_{n-1}, 0) \in \mathcal{D}_n(A_1, \ldots, A_k).$$

Hence,

$$\sum_{j=1}^{n-1} s_j + \min\{|0 - |\rho|| : \rho \in \mathcal{D}_1(s_n(A_1), \cdots, s_n(A_n))\} \leq \sum_{j=1}^{n-1} s_j$$

and thus,

$$\rho_0 = \min\{|0 - |\rho|| : \rho \in \mathcal{D}_1(s_n(A_1), \cdots, s_n(A_n))\} = 0.$$

Conversely, suppose $\min\{|\rho| : \rho \in \mathcal{D}_1(s_n(A_1), \cdots, s_n(A_n))\} = 0$. Then (2.7) follows from (2.5). Hence, $\mathcal{D}_n(A_1, \ldots, A_k)$ is convex.

For (d), suppose that $m = n$, $\mathbf{F} = \mathbf{R}$, and $\mathcal{D}_n(A_1, \ldots, A_k)$ is convex. Therefore, $(s_1, \cdots, s_{n-1}, t) \in \mathcal{D}_n(A_1, \ldots, A_k)$ for all $0 \leq t \leq s_n$. Hence, $\mathcal{D}_1(s_n(A_1), \cdots, s_n(A_k)) = [0, s_n]$. Since $\mathcal{D}_1(s_n(A_1), \ldots, s_n(A_k))$ is finte, we have $s_n = 0$. Conversely, if $s_n = 0$, then $s_n(A_1) = \cdots = s_n(A_k) = 0$. Hence, $\mathcal{D}_1(s_n(A_1), \ldots, s_n(A_k)) = \{0\}$, and (2.6) reduces to (2.5) when $r = n$. Therefore, $\mathcal{D}_n(A_1, \ldots, A_k)$ is convex.

(e) follows from Theorem 2.7, Corollary 5 of [18] and [17]

(f) It suffices to prove the case when $m = n = r$. Suppose $\mathbf{d} = (d_1, \ldots, d_n) \in \mathcal{C}$ with $|d_1| \geq \ldots \geq |d_n|$. If $|d_n| > s_n$, then $\mathbf{d} \in \mathcal{D}_n$. So we may assume that $|d_n| < s_n$.

Let $d = \min(|d_{n-1}|, s_n)$. For $\mu \in T$, let $\mathbf{d}(\mu) = (d_1, \ldots, d_{n-1}, \mu d)$. Since $|d_n| \leq |d|$, there exist $\mu_1, \mu_2 \in T$ and $0 \leq t \leq 1$ such that $d_n = t\mu_1 d + (1-t)\mu_2 d$. Hence, $\mathbf{d} = t\mathbf{d}(\mu_1) + (1-t)\mathbf{d}(\mu_2)$. Furthermore, if $\mathbf{F} = \mathbf{C}$, we can choose $t = 1/2$. Clearly, $\mathbf{d}(\mu) \prec_w (s_1, \cdots, s_n)$. It remains to prove that $\mathbf{d}(\mu)$ satisfies (2.6) for all $\mu \in T$.

If $|d_{n-1}| > s_n$, we have $d = s_n$ and (2.6) is clearly satisfied.

If $|d_{n-1}| \leq s_n$, we have $d = |d_{n-1}|$ and

$$\sum_{j=1}^{n-1} |d_j| + s_n - |\mu d_{n-1}| = \sum_{j=1}^{n-2} |d_j| + s_n \leq \sum_{j=1}^{n-2} s_j + s_n \leq \sum_{j=1}^{n-1} s_j .$$

Therefore, (2.6) is satisfied. □

Let p, q be positive integers satisfying $p \leq m, q \leq n$ and $r = \min\{p, q\}$. Define

$$\Phi_{p,q}(A_1, \ldots, A_k) = \{X \in M_{p,q}(\mathbf{F}) : X \text{ is a submatrix of } Y \in \mathcal{O}(A_1, \ldots, A_k)\}.$$

By an argument similar to the one in (e) in the last theorem, we can prove the following extension of Theorem 10 in [18].

Proposition 2.13 *Let $A_1, \ldots, A_n \in M_{m,n}(\mathbf{F})$, let p, q be positive integers satifying $p \leq m, q \leq n$, and $r = \min\{p, q\}$. The convex hull of $\Phi_{p,q}(A_1, \ldots, A_k)$ is the set*

$$\{X \in M_{p,q}(\mathbf{F}) : \sigma(X) \prec_w (s_1, \ldots, s_r)\}.$$

3 Real matrices under the action of $SO(m) \otimes SO(n)$

In this section, we consider $A \in M_{m,n}(\mathbf{R})$ under the action of $SO(m) \otimes SO(n)$. Let

$$SO(A) = \{UAV^t : U \in SO(m),\ V \in SO(n)\}.$$

The joint orbit of $A_1, \ldots, A_k \in M_{m,n}(\mathbf{R})$ is

$$SO(A_1, \ldots, A_k) = \left\{ \sum_{i=1}^k X_i : X_i \in SO(A_i),\ i = 1, \ldots, k \right\}.$$

We are interested in the set $D_r(A_1, \ldots, A_k)$ of r-tuples $(d_1, \ldots, d_r)$ whose entries are the first r diagonal entries of a matrix in $SO(A_1, \ldots, A_k)$. Again, the set $D_r(A_1, \ldots, A_k)$ can be viewed as the set of vectors with entries lying in the $(1, j_1), \ldots, (r, j_r)$ positions of a matrix in $SO(A_1, \ldots, A_k)$ for any distinct column indices $j_1, \ldots, j_r$.

3.1 Background

In [18, Theorems 2 and 7] Thompson gave a complete description of $D_n(A)$ for a given $A \in M_n(\mathbf{R})$ by proving the following result.

Proposition 3.1 *Let $A \in M_n(\mathbf{R})$ have nonnegative determinant and singular values $s_1 \geq \cdots \geq s_n$. Then there exists $X \in SO(A)$ with diagonal entries $d_1, \ldots, d_n$ in any order such that q of them are negative and $|d_1| \geq \cdots \geq |d_n|$ if and only if*

$$\sum_{j=1}^r |d_j| \leq \sum_{j=1}^r s_j,\ r = 1, \ldots, n, \tag{3.1}$$

and

$$\sum_{j=1}^{n-1} |d_j| - (-1)^q |d_n| \leq \sum_{j=1}^{n-1} s_j - s_n. \tag{3.2}$$

In particular, $D_n(A)$ is the convex hull of all vectors $(\pm s_{\pi(1)}, \cdots, \pm s_{\pi(n)})$ with an even number of negative signs and with π any permutation.

In [15, Theorem 2], the author showed that if $A_1, \ldots, A_k \in M_n(\mathbf{R})$ with $\det(A_1 \cdots A_k) \geq 0$, and if $s_j = \prod_{i=1}^k s_j(A_i)$ for $j = 1, \ldots, n$, then $d_1, \ldots, d_n \in \mathbf{R}$ are the diagonal entries of a matrix of the form $\prod_{i=1}^k X_j$ with $X_j \in SO(A_j)$ for $j = 1, \ldots, k$, if and only if (3.1) and (3.2) hold.

Once again, the product version of Proposition 3.1 is relatively easy to prove, and the summation version is not so simple. It is worth mentioning that the sets $SO(A_1, \ldots, A_k)$ and $D_r(A_1, \ldots, A_k)$ arise naturally in the Lie group setting as pointed out in [16].

In the next subsection, we give a complete description of the set $D_r(A_1, \ldots, A_k)$. One easily checks (see also [16, Section 4]) that if $m \neq n$ or $r < m = n$ then

$$D_r(A_1, \ldots, A_k) = \mathcal{D}_r(A_1, \ldots, A_k),$$

which is studied in the previous section. So, we only need to consider the case $r = m = n$.

3.2 Characterization of $D_r(A_1, \ldots, A_k)$ and extremal matrices

The following results of Thompson [18] (Lemma 5' and the proof of Theorem 2) play a crucial role in our discussion.

Lemma 3.2 *Let $A \in M_n(\mathbf{R})$ with diagonal entries $d_1, \ldots, d_n$ satisfy $\det(A) \geq 0$. Then*

$$\sum_{j=1}^{n-1} d_j - d_n \leq \sum_{j=1}^{n-1} s_j(A) - s_n(A).$$

The equality holds if and only if $(I_{n-1} \oplus [-1])A$ is symmetric with eigenvalues

$$s_1(A), \ldots, s_{n-1}(A), -s_n(A).$$

Let $J = I_{n-1} \oplus [-1]$. We note that $\sum_{j=1}^{n-1} d_j - d_n = \operatorname{tr} JA = \operatorname{tr} AJ$.

Lemma 3.3 *Let $A \in M_n(\mathbf{R})$ and $(d_1, \ldots, d_n) \in D_n(A)$. Then $(d_1, \ldots, d_n)P \in D_n(A)$ for any permutation matrix P or diagonal matrix $P \in SO(n)$.*

The next result treats the special case when all A_i have nonnegative determinants. It turns out that the same set of conditions (3.1) and (3.2) are necessary and sufficient if one defines $s_1, \ldots, s_n$, appropriately.

Proposition 3.4 *Let $A_1, \ldots, A_k \in M_n(\mathbf{R})$ have nonnegative determinants, and $d_1, \ldots, d_n$ be real numbers such that q of them are negative and $|d_1| \geq \cdots \geq |d_n|$. Suppose $s_j = \sum_{i=1}^k s_j(A_i)$ for $j = 1, \ldots, n$. The following conditions are equivalent.*

(a) *Up to a (any) permutation of the entries, the vector $(d_1, \ldots, d_n)$ is in $D_n(A_1, \ldots, A_k)$.*

(b) *Up to a (any) permutation of the entries, the vector $(d_1, \ldots, d_n)$ is in $D_n(A)$ with $A = \sum_{i=1}^n s_i E_{ii}$.*

(c) *The inequalities (3.1) and (3.2) hold.*

Proof. Note that $A \in M_n(\mathbf{R})$ has nonnegative determinant if and only if there exist $U, V \in SO(n)$ such that $A = U(\sum_{j=1}^n s_j(A)E_{jj})V$. So, without loss of generality, we may assume that $A_i = \sum_{j=1}^n s_j(A_i)E_{jj}$ for each i.

The equivalence of (b) and (c) follows from Proposition 3.1. The implication (b) $\Rightarrow$ (a) is clear. Suppose (a) holds. Then (3.1) follows from Theorem 2.7. To prove (3.2), let $(d_1, \ldots, d_n) = \operatorname{diag}(\sum_{i=1}^k X_i)$ with $X_i \in SO(A_i)$ for each i.

Applying Lemma 3.2 to each X_i, we see that $\operatorname{tr} JX_i \leq \sum_{j=1}^{n-1} s_j(A_i) - s_n(A_i)$. Thus, we have

$$\sum_{j=1}^{n-1} d_j - d_n = \operatorname{tr} J(X_1 + \cdots + X_k) \leq \sum_{j=1}^{n-1} s_j - s_n. \tag{3.3}$$

Suppose there is an even number of negative terms among $d_1, \ldots, d_n$. By Lemma 3.3, we may assume that all d_j are nonnegative and (3.2) follows from (3.3). If there is an odd number of negative terms, we may assume that $d_j \geq 0$ for $j = 1, \cdots, n-1$ and $d_n \leq 0$. Again, (3.2) follows from (3.3). □

By a similar arguments, one may treat the case when $A_1, \ldots, A_k \in M_n(\mathbf{R})$ have negative determinants. For the general case, we have the following result.

Theorem 3.5 *Suppose $A_1, \ldots, A_k, A_{k+1}, \ldots, A_m \in M_n(\mathbf{R})$ such that $\det A_i \geq 0$ for $1 \leq i \leq k$ and $\det A_i < 0$ for $k+1 \leq i \leq m$. Suppose $s_j = \sum_{i=1}^m s_j(A_i)$ for $j = 1, \ldots, n-1$ and $s_n = \sum_{i=1}^k s_n(A_i) - \sum_{i=k+1}^m s_n(A_i)$ (s_n may be negative). Let $d_1, \ldots, d_n \in \mathbf{R}$ be such that q of the numbers are negative and $|d_1| \geq \cdots \geq |d_n|$. The following conditions are equivalent.*

(a) *Up to a (any) permutation of the entries, the vector $(d_1, \ldots, d_n)$ is in $D_n(A_1, \ldots, A_m)$.*

(b) *Up to a (any) permutation of the entries, the vector $(d_1, \ldots, d_n)$ is in $D_n(B)$, where $B = \sum_{j=1}^n s_j E_{jj}$.*

(c) *The following inequalities hold:*

$$\sum_{j=1}^{r} |d_j| \leq \sum_{j=1}^{r} s_j, \quad r = 1, \ldots, n-1, \tag{3.4}$$

$$\sum_{j=1}^{n-1} |d_j| + (-1)^q |d_n| \leq \sum_{j=1}^{n-1} s_j + s_n, \tag{3.5}$$

and

$$\sum_{j=1}^{n-1} |d_j| - (-1)^q |d_n| \leq \sum_{j=1}^{n-1} s_j - s_n. \tag{3.6}$$

Furthermore, suppose $X_i \in SO(A_i)$ for all $i = 1, \ldots, k$, so that the diagonal entries of $X = X_1 + \cdots + X_k$ equal $d_1, \ldots, d_n$.

(1) *For $1 \leq r < n$, the equality holds in (3.4) if and only if there exists a diagonal matrix $D \in U_n(\mathbf{R})$ such that*

$$DX_i = \begin{pmatrix} Y_i & 0 \\ 0 & Z_i \end{pmatrix}, \quad i = 1, \ldots, m,$$

where $Y_i \in M_r(\mathbf{R})$ is symmetric with eigenvalues $s_1(A_i), \ldots, s_r(A_i)$.

(2) *The equality holds in (3.5) if and only if there exists a diagonal matrix $D \in SO(n)$ such that DX has diagonal entries $|d_1|, \ldots, |d_{n-1}|, (-1)^q |d_n|$, DX_i is symmetric with eigenvalues $s_1(A_i), \ldots, s_n(A_i)$ for $i = 1, \ldots, k$, and DX_j is symmetric with eigenvalues $s_1(A_j), \ldots, s_{n-1}(A_i), -s_n(A_j)$ for $j = k+1, \ldots, m$.*

(3) *The equality holds in* (3.6) *if and only if there exists a diagonal matrix* $D \in U_n(\mathbf{R})$ *with* $\det(D) = -1$ *such that* DX *has diagonal entries* $|d_1|, \ldots, |d_{n-1}|, (-1)^{q+1}|d_n|$, DX_i *is symmetric with eigenvalues* $s_1(A_i), \ldots, s_{n-1}(A_i), -s_n(A_i)$ *for* $i = 1, \ldots, k$, *and* DX_j *is symmetric with eigenvalues* $s_1(A_j), \ldots, s_n(A_j)$ *for* $j = k+1, \ldots, m$.

Proof. Let $\mathbf{d} = (d_1, \ldots, d_n)$. Clearly (b) implies (a).

Suppose (c) holds. We consider two cases:

Case 1. Suppose $s_n \geq 0$. Then B has singular values $s_1, \ldots, s_n$ and (3.2) follows from (3.6). For $r = 1, \ldots, n-1$, (3.1) follows from (3.4); for $r = n$, (3.1) follows from (3.5) or (3.6). By (b) of Proposition 3.4, $\mathbf{d} \in D_n(B)$.

Case 2. Suppose $s_n < 0$. Then BJ has singular values $s_1, \ldots, s_{n-1}, -s_n$. Applying the argument in Case 1 to $\mathbf{d}J$ and BJ, there exist $U, V \in SO_n(\mathbf{R})$ such that $\mathbf{d}J = \operatorname{diag} U(BJ)V$. Thus $\mathbf{d} = \operatorname{diag} UB(JVJ)$ with $U, JVJ \in SO_n(\mathbf{R})$.

Next, suppose condition (a) holds. By Proposition 3.4, we may assume that $\mathbf{d} = \operatorname{diag}\,(U_1B_1V_1 + U_2B_2V_2)$ where

$$B_1 = \sum_{j=1}^{n} \left(\sum_{i=1}^{k} s_j(A_i) \right) E_{jj} \quad \textit{and} \quad B_2 = \sum_{j=1}^{n-1} \left(\sum_{i=k+1}^{m} s_j(A_i) \right) E_{jj} - \left(\sum_{i=k+1}^{m} s_n(A_i) \right) E_{nn},$$

and U_1, U_2, V_1 and $V_2 \in SO_n(\mathbf{R})$. Let $\mathbf{d}^{(i)} = \operatorname{diag}\,(U_iB_iV_i)$ for $i = 1, 2$. Then, for $r = 1, \ldots, n-1$, we have

$$\begin{aligned} \sum_{j=1}^{r} |d_j| &= \sum_{j=1}^{r} |d_j^{(1)} + d_j^{(2)}| \\ &\leq \sum_{j=1}^{r} (|d_j^{(1)}| + |d_j^{(2)}|) \\ &\leq \sum_{j=1}^{r} (s_j(B_1) + s_j(B_2)) \qquad \text{by (3.1)}\ . \end{aligned}$$

Hence (3.4) follows.

To prove (3.5) and (3.6), first consider the case where an even number of the entries in $\mathbf{d}$ are negative. By Lemma 3.3, we may choose a suitable diagonal matrix D in $SO(n)$ so that $\mathbf{d}D$ have nonnegative entries. For simplicity, we assume that D is the identity matrix; otherwise, replace $\mathbf{d}$ by $\mathbf{d}D$. Applying Lemma 3.2 to $\mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}J = \operatorname{diag}\,(U_2(B_2J)(JV_2J))$, we have

$$\sum_{j=1}^{n-1} d_j^{(1)} - d_n^{(1)} \leq \sum_{j=1}^{n-1} s_j(B_1) - s_n(B_1) \quad \text{and} \quad \sum_{j=1}^{n-1} d_j^{(2)} + d_n^{(2)} \leq \sum_{j=1}^{n-1} s_j(B_2) - s_n(B_2).$$

From (3.1) we have

$$\sum_{j=1}^{n-1} d_j^{(1)} + d_n^{(1)} \leq \sum_{j=1}^{n-1} s_j(B_1) + s_n(B_1) \quad \text{and} \quad \sum_{j=1}^{n-1} d_j^{(2)} - d_n^{(2)} \leq \sum_{j=1}^{n-1} s_j(B_2) + s_n(B_2).$$

Therefore, we have

$$\sum_{j=1}^{n-1} d_j + d_n \leq \sum_{j=1}^{n-1} s_j + s_n \quad \text{and} \quad \sum_{j=1}^{n-1} d_j - d_n \leq \sum_{j=1}^{n-1} s_j - s_n,$$

from which (3.5) and (3.6) follow. If an odd number of the entries in $\mathbf{d}$ are negative, we can replace $\mathbf{d}$ by $\mathbf{d}J$. Interchanging the roles of B_1 and B_2 in the previous arguments, we get the conclusion.

Now, we turn to the characterization of the equality cases. First, condition (1) can be proven by an argument similar to that in the proof of Theorem 2.7.

Suppose $X_i \in SO(A_i)$ so that $X_0 = X_1 + \ldots + X_m$ has diagonal entries $d_1, \ldots, d_n$ attaining the equality in (3.5). Suppose q of the diagonal entries of X_0 are negative. Then there exists $D \in SO(n)$ such that $\operatorname{diag}(DX_0) = (|d_1|, \cdots, (-1)^q |d_n|)$. For $i = 1, \ldots, k$, $\operatorname{tr}(DX_i) \leq \sum_{j=1}^n s_j(A_i)$, by (2.2). For $j = k+1, \ldots, m$, $\det(JDX_i) \geq 0$ for $j = k+1, \ldots, m$, by Lemma 3.2, we have $\operatorname{tr} DX_i = \operatorname{tr} J(JDX_i) \leq \sum_{j=1}^{n-1} s_j(A_i) - s_n(A_i)$. Hence, we have

$$\sum_{j=1}^{n} s_j = \operatorname{tr}(DX_0) = \sum_{i=1}^{m} \operatorname{tr}(DX_i) \leq \sum_{i=1}^{k}\sum_{j=1}^{n} s_j(A_i) + \sum_{i=k+1}^{m} \left(\sum_{j=1}^{n-1} s_j(A_i) - s_n(A_i) \right) = \sum_{j=1}^{n} s_j.$$

It follows that $\operatorname{tr}(DX_i) = \sum_{j=1}^n s_j(A_i)$ for $i = 1, \ldots, k$, and $\operatorname{tr} DX_i = \operatorname{tr} J(JDX_i) = \sum_{j=1}^{n-1} s_j(A_i) - s_n(A_i)$ for $i = k+1, \ldots, m$. The result follows from Lemma 3.2.

The proof for the equality in (3.6) is similar. □

3.3 Convexity properties

Proposition 3.6 *Under the assumption in Theorem 3.5, $D_n(A_1, \ldots, A_m)$ is the convex hull of all vectors $(\pm s_{\pi(1)}, \cdots, \pm s_{\pi(n)})$ with an even number of negative signs and with π any permutation. Furthermore, the set*

$$\operatorname{conv}\{s(X) : X \in SO(A_1, \ldots, A_m)\} = \{s(X) : X \in \operatorname{conv} SO(A_1, \ldots, A_m)\}$$

consists of vector $s(X) = (\sigma_1, \ldots, \sigma_n)$ such that $\sigma_1, \ldots, \varepsilon\sigma_n \in D_n(A_1, \ldots, A_m)$, where $\varepsilon = 1$ if $s_n \det(X) \geq 0$, $\varepsilon = -1$ if $s_n \det(X) < 0$.

Proof. If $s_n \geq 0$, the result follows from the condition (b) in Theorem 3.5, Proposition 3.1 and [18, Corollary 9]. Suppose $s_n < 0$. Then $\mathbf{d} \in D_n(B)$ if and only if $\mathbf{d}J \in D_n(BJ)$, and $X \in SO(B)$ if and only if $XJ \in SO(BJ)$. The result follows from the previous case. □

4 Skew-symmetric matrices under the action of $U_n(\mathbf{F})$

Let $A \in M_n(\mathbf{F})$ be a skew-symmetric matrix such that $n = 2m$ or $2m+1$. There has been considerable interest (see [13, 14, 16]) in studying the set $\tilde{\mathcal{D}}_r(A)$ of r-tuples $(\mu_1, \ldots, \mu_r)$, where for $j = 1, \ldots, r$, the number μ_j is the $(j, m+j)$ entry of a matrix of the form UAU^t with $U \in U_n(\mathbf{F})$. In this section, we study the structure of the set $\tilde{\mathcal{D}}_r(A_1, \ldots, A_k)$ of r-tuples $(\mu_1, \ldots, \mu_r)$, where for $j = 1, \ldots, r$, the number μ_j is the $(j, m+j)$ entry of a matrix of the form $X_0 = X_1 + \cdots + X_k$, where $X_i = U_i A_i U_i^t$ with $U_i \in U_n(\mathbf{F})$ for all i. We begin with the following lemma.

Lemma 4.1 *Let $n = 2m$ or $2m+1$, and let $A \in M_n(\mathbf{F})$ be a skew-symmetric matrix with singular values $s_j = s_{2j-1}(A) = s_{2j}(A)$ for $j = 1, \ldots, m$, and $s_{2m+1}(A) = 0$ if $n = 2m+1$. Suppose $1 \le r \le m$ and $\mu_1, \ldots, \mu_r \in \mathbf{F}$ satisfy $|\mu_1| \ge \cdots \ge |\mu_r|$. The following conditions are equivalent.*

(a) *Up to a (any) permutation of the entries, the vector $(\mu_1, \ldots, \mu_r)$ is in $\tilde{\mathcal{D}}_r(A)$.*

(b) *There exists $U \in U_n(\mathbf{F})$ such that $UAU^t = \begin{pmatrix} 0_m & X \\ -X^t & 0_{n-m} \end{pmatrix}$ and $(\mu_1, \ldots, \mu_r)$ is in the $(1,1), \ldots, (r,r)$ positions of X.*

(c) *The following inequalities hold:*

$$(|\mu_1|, \ldots, |\mu_r|) \prec_w (s_1, \ldots, s_r),$$

and if $r = n/2$

$$\sum_{j=1}^{r-1} |\mu_j| - |\mu_r| \le \sum_{j=1}^{r-1} s_j - s_r.$$

Proof. The implication (b) $\Rightarrow$ (a) is clear. For (a) $\Rightarrow$ (c), see [13, Theorem 1] and [14, Theorems 2.1 and 2.2]. Suppose (c) holds. By Proposition 2.1 and Theorem 2.10, one can construct an $m \times (n-m)$ matrix X with singular values $s_1, \ldots, s_m$ and (j,j) entry equal to μ_j for $j = 1, \ldots, r$. Then $\begin{pmatrix} 0 & X \\ -X^t & 0 \end{pmatrix}$ satisfies condition (b). $\square$

Lemma 4.2 *Suppose $n = 2m$ or $2m+1$, $A_1, \ldots, A_k \in M_n(\mathbf{R})$ are skew-symmetric matrices and $1 \le r \le m$. The following conditions are equivalent.*

(a) *Up to a (any) permutation of the entries, the vector $(\mu_1, \ldots, \mu_r)$ is in $\tilde{\mathcal{D}}_r(A)$.*

(b) *There exist $U_1, \ldots, U_k \in U_n(\mathbf{F})$ such that $U_i A_i U_i^t = \begin{pmatrix} 0_m & X_i \\ -X_i^t & 0_{n-m} \end{pmatrix}$, $i = 1, \ldots, k$, and $(\mu_1, \ldots, \mu_r)$ is in the $(1,1), \ldots, (r,r)$ positions of $X_1 + \cdots + X_k$.*

(c) *Up to a (any) permutation of the entries, the vector* $(\mu_1, \dots, \mu_r)$ *is in* $\tilde{\mathcal{D}}_r(B_1, \dots, B_r)$ *(as defined in Section 2), where* $B_i = \sum_{j=1}^m s_{2j}(A_i)E_{jj} \in M_{m,n-m}(\mathbf{F})$ *for* $i = 1, \dots, k$.

Proof. Suppose (a) holds. Then there exist $V_1, \dots, V_k \in U_n(\mathbf{F})$ so that μ_j is the $(j, m+j)$ entry of a matrix of the form $\sum_{i=1}^k V_i A_i V_i^t$. Applying the equivalence of (a) and (b) of Lemma 4.1, we may replace each $V_i A_i V_i^t$ by a suitable matrix of the form $U_i A_i U_i^t = \begin{pmatrix} 0_m & X_i \\ -X_i^t & 0_{n-m} \end{pmatrix}$, which has the same $(1, r+1), \dots, (r, 2r)$ entries as $V_i A_i V_i^t$. Then $X_1, \dots, X_r$ satisfy (b).

Suppose $X_1, \dots X_r$ satisfy (b). Then X_i has singular values $s_2(A_i), s_4(A_i), \dots, s_{2m}(A_i)$. Thus condition (c) holds.

If (c) holds, then clearly (b) and hence (a) holds. □

By the above lemma, we can translate all the results in Section 2 to skew-symmetric matrices. We summarize the result in the following.

Theorem 4.3 *Suppose* $A_1, \dots, A_k \in M_n(\mathbf{F})$ *are skew-symmetric matrices and* $m = [n/2]$. *Let* $s_j = \sum_{i=1}^k s_{2j}(A_i)$ *for all* $j \le n/2$. *For* $1 \le r \le m$, *let* $\tilde{\mathcal{D}}_r(A_1, \dots, A_k)$ *be the set of all* r*-tuples complex numbers equal to the* $(1, m+1), (2, m+2), \dots, (r, m+r)$ *entries of matrices of the form* $\sum_{j=1}^k U_j A_j U_j^t$ *where* $U_1, \dots, U_k$ *are unitary. Then*

$$(d_1, \dots, d_r) \in \tilde{\mathcal{D}}_r(A_1, \dots, A_k)$$

if and only if

$$(|d_1|, \dots, |d_r|) \prec_w (s_1, \dots, s_r), \tag{4.1}$$

and if $r = n/2$ *we have*

$$\sum_{j=1}^{r-1} |d_j| + \min\{||d_r| - |\rho|| : \rho \in \mathcal{D}_1(s_n(A_1), \dots, s_n(A_k))\} \le \sum_{j=1}^{r-1} s_j. \tag{4.2}$$

In the complex case, (4.2) *is equivalent to*

$$\sum_{j=1}^{r-1} |d_j| - |d_r| \le \sum_{i=1}^{r-1} s_j - \rho_0,$$

where

$$\rho_0 = \min\{|\rho| : \rho \in \mathcal{D}_1(s_n(A_1), \dots, s_n(A_k))\}.$$

In particular, we have $\tilde{\mathcal{D}}_r(A_1, \dots, A_k) = \tilde{\mathcal{D}}_r(A_0)$ *where* $A_0 = \sum_{j=1}^m s_j(E_{j,m+j} - E_{m+j,j})$, *and the set is convex whenever* $r < n/2$. *For* $r = n/2$, *the set* $\tilde{\mathcal{D}}_r(A_1, \dots, A_k) = \tilde{\mathcal{D}}_r(A_0)$ *is star-shaped with* $(0, \dots, 0)$ *as a star-center; it is convex if and only if* $s_n = 0$.

Next, we consider the equality cases for (4.1) and (4.2).

Proposition 4.4 *Suppose $A_1, \ldots, A_k$, and $s_1, \ldots, s_m$ satisfy the hypotheses of Theorem 4.3. Let*

$$A = \sum_{i=1}^{k} \tilde{A}_i \quad \text{with } \tilde{A}_i = U_i A_i U_i^t = \left(a_{pq}^{(i)}\right), \qquad U_i \in U_n(\mathbf{F}) \text{ for } i = 1, \ldots, k, \tag{4.3}$$

so that $|d_j|$ is the $(j, m+j)$ entry of A for $j = 1, \ldots, m$.

(a) *Suppose $1 \le r \le m$. Then*

$$\sum_{j=1}^{r} |d_j| = \sum_{j=1}^{r} s_j \tag{4.4}$$

if and only if for each $i = 1, \ldots, k$,

$$\tilde{A}_i = \begin{pmatrix} C_i & 0_{r,m-r} & B_i & 0_{r,n-m-r} \\ 0_{m-r,r} & * & 0_{m-r,r} & * \\ -B_i^t & 0_{r,m-r} & \overline{C}_i & 0_{r,n-m-r} \\ 0_{n-m-r,r} & * & 0_{n-m-r,r} & * \end{pmatrix}$$

such that $B_i \in M_r(\mathbf{F})$ has trace $\sum_{j=1}^{r} s_{2j}(A_i)$ and the matrix

$$\begin{pmatrix} B_i & C_i \\ -\overline{C}_i & \overline{B}_i \end{pmatrix}$$

is positive semi-definite with eigenvalues $s_1(A_i), \ldots, s_{2r}(A_i)$.

(b) *Suppose $r = n/2$ and*

$$\tilde{A}_i = \begin{pmatrix} * & X_i \\ -X_i^t & * \end{pmatrix} \quad \text{with} \quad X_i \in M_r(\mathbf{F}), \qquad \text{for } i = 1, \ldots, k. \tag{4.5}$$

Then (4.2) becomes an equality if and only if for each $i = 1, \ldots, k$,

$$\operatorname{tr}(I_{r-1} \oplus [-1])X_i = \sum_{j=1}^{r-1} s_j(X_i) - s_r(X_i) = \sum_{j=1}^{r-1} s_{2j}(A_i) - s_{2r}(A_i), \tag{4.6}$$

and hence Theorem 2.7(b) is applicable to the matrix X_i.

Proof. (a) Let $1 \le r \le n$. If $\sum_{j=1}^{r} |d_j| = \sum_{j=1}^{r} s_j$, then one can multiply the $(m+j)$th row of $\tilde{A}_i$ by -1 for $j = 1, \ldots, r$, and then permute the rows and columns appropriately to get a matrix whose first $2r$ diagonal entries are nonnegative with sum equal to

$$\sum_{j=1}^{r} |a_{j,m+j}^{(i)}| + |a_{m+j,j}^{(i)}| = \sum_{j=1}^{2r} s_j(A_i).$$

By Lemma 2.8, the resulting matrix is of the form $Q_i \oplus R_i$ such that $Q_i \in M_{2r}(\mathbf{F})$ is positive semi-definite with eigenvalues $s_1(A_i), \ldots, s_{2r}(A_i)$. Thus

$$\tilde{A}_i = \begin{pmatrix} C_i & 0_{r,m-r} & B_i & 0_{r,n-m-r} \\ 0_{m-r,r} & * & 0_{m-r,r} & * \\ -B_i^t & 0_{r,m-r} & \overline{C}_i & 0_{r,n-m-r} \\ 0_{n-m-r,r} & * & 0_{n-m-r,r} & * \end{pmatrix}$$

such that $B_i \in M_r(\mathbf{F})$ has trace $\sum_{j=1}^r s_{2j}(A_i)$ and the matrix

$$Q_i = \begin{pmatrix} B_i & C_i \\ -\overline{C}_i & \overline{B}_i \end{pmatrix}$$

is positive semi-definite with eigenvalues $s_1(A_i), \ldots, s_{2r}(A_i)$, for all $i = 1, \ldots, k$. Conversely, if A_i has the said block form, then the equality (4.4) holds.

(b) Let $n = 2r$ and (4.5) holds. By Proposition 2.1 and [14, Theorem 3.1], we have

$$\operatorname{tr}(I_{r-1} \oplus [-1])X_i \le \sum_{j=1}^{r-1} s_j(X_i) - s_r(X_i) \le \sum_{j=1}^{r-1} s_{2j}(A_i) - s_{2r}(A_i).$$

Hence, (4.2) becomes an equality if and only if (4.6) holds for each $i = 1, \ldots, k$. □

Note that the convexity result on $\tilde{\mathcal{D}}_r(A_1, \ldots, A_k)$ in the last assertion of the theorem can also be used to study the singular values of submatrices in the off-diagonal blocks of a skew-symmetric matrix of the form $\sum_{i=1}^k U_i A_i U_i^t$, $U_i \in U_n(\mathbf{F})$ for $i = 1, \ldots, k$. One can easily apply a block permutation to move the $p \times q$ submatrix in the off-diagonal position to the off-diagonal position of the leading $(p+q) \times (p+q)$ principal submatrix. It is more convenient to state the result is this way, and we have the following result in terms of the principal submatrices of skew-symmetric matrices (cf. [16, Theorem 19]).

Proposition 4.5 *Let $A_1, \ldots, A_k \in M_n(\mathbf{F})$ be skew-symmetric matrices. Suppose $1 \le q \le n$ and $\tilde{\Phi}_q(A_1, \ldots, A_k)$ is the set of $q \times q$ principal submatrices of matrices of the form $\sum_{i=1}^k U_i A_i U_i^t$, $U_i \in U_n(\mathbf{F})$. Then the convex hull of $\tilde{\Phi}_q(A_1, \ldots, A_k)$ is the set of skew-symmetric matrices X satisfying*

$$\sum_{j=1}^r s_{2j}(X) \le \sum_{j=1}^r \sum_{i=1}^k s_{2j}(A_i), \quad 1 \le r \le q/2.$$

One can also consider $\operatorname{Re} \tilde{\mathcal{D}}_r(A_1, \ldots, A_k)$. In such case, only (4.1) is needed to determined whether $(d_1, \ldots, d_r) \in \operatorname{Re} \tilde{\mathcal{D}}_r(A_1, \ldots, A_k)$. Again, one easily translates results on the characterization of equality cases, convexity, etc.

When $\mathbf{F} = \mathbf{R}$, given skew-symmetric matrices $A_1, \ldots, A_k$, and $1 \le r \le n/2$, one can consider $\tilde{D}_r(A_1, \ldots, A_k)$, the set of vectors $(d_1, \ldots, d_r)$ lying in the $(1, m+1), \ldots, (r, m+r)$

positions of a matrix of the form $\sum_{i=1}^k U_iA_iU_i^t$, $U_i \in SO(n)$ for $i = 1, \ldots, k$. Again, one can translate the results in Section 3 to $\tilde{D}_r(A_1, \ldots, A_k)$. We have the following lemma, which is crucial for the translation.

Lemma 4.6 *Let $n = 2m$ or $2m+1$, and let $A \in M_n(\mathbf{R})$ with singular values $s_j = s_{2j-1}(A) = s_{2j}(A)$ for $j = 1, \ldots, m$, and $s_{2m+1}(A) = 0$ if $n = 2m+1$. Then there exists $V \in SO(n)$ such that*

(1) $VAV^t = \sum_{j=1}^m s_j(E_{j,m+j} - E_{m+j,j})$, *or*

(2) $VAV^t = \sum_{j=1}^{m-1} s_j(E_{j,m+j} - E_{m+j,j}) - s_m(E_{m,n} - E_{n,m})$ *in case $n = 2m$.*

Suppose $1 \le r \le m$ and $\mu_1, \ldots, \mu_r \in \mathbf{R}$ satisfy $|\mu_1| \ge \cdots \ge |\mu_r|$. The following conditions are equivalent.

(a) *Up to a (any) permutation, the vector $(\mu_1, \ldots, \mu_r)$ is in $\tilde{D}_r(A)$.*

(b) *There exists $U \in SO(n)$ such that $UAU^t = \begin{pmatrix} 0_m & X \\ -X^t & 0_{n-m} \end{pmatrix}$ and $(\mu_1, \ldots, \mu_r)$ lies in the $(1,1), \ldots, (r,r)$ positions of X.*

(c) *The following inequalities hold:*

$$(|\mu_1|, \ldots, |\mu_r|) \prec_w (s_1, \ldots, s_r),$$

and

$$\sum_{j=1}^{r-1} |\mu_j| - (-1)^{p+q}|\mu_r| \le \sum_{j=1}^{r-1} s_j - s_r \quad if\ r = n/2,$$

where $p = 0$ or 1 according to (1) or (2) holds, and q is the number of negative terms in $\mu_1, \ldots, \mu_r$.

Using the above lemma one can obtain results on $\tilde{D}_r(A_1, \ldots, A_k)$. In particular, one sees that except when $r = n/2$, $\tilde{D}_r(A_1, \ldots, A_k) = \tilde{\mathcal{D}}_r(A_1, \ldots, A_k)$. We omit the details.

5 Related Results and Problems

There are results and questions on other joint orbits of matrices under different types of group actions. First, we consider complex Hermitian or real symmetric matrices under the action of unitary and orthogonal similarity, respectively. It is not difficult to prove the following.

Proposition 5.1 *Suppose $A_1, \ldots, A_k$ are complex Hermitian or real symmetric matrices. For $1 \le r \le n$, let $\mathcal{D}_r(A_1, \ldots, A_k)$ be the set of all r-tuples real numbers equal to the first r diagonal entries of matrices of the form $\sum_{j=1}^k U_jA_jU_j^*$ where $U_1, \ldots, U_n \in U_n(\mathbf{F})$. Then*

$(d_1,\ldots,d_r) \in \mathcal{D}_r(A_1,\ldots,A_k)$ if and only if $(d_1,\ldots,d_r) \prec_w (\lambda_1,\ldots,\lambda_r)$ where λ_j is the sum of the jth largest eigenvalues of the $A_1,\ldots,A_k$, and if $r=n$ then $\sum_{j=1}^n d_j = \sum_{i=1}^k \operatorname{tr} A_i$. In particular, $\mathcal{D}_r(A_1,\ldots,A_k) = \mathcal{D}_r(\sum_{j=1}^n \lambda_j E_{jj})$ is convex.

Given complex symmetric matrices $A_1,\ldots,A_k \in M_n(\mathbf{C})$ and $1 \le r \le n$, one may consider the set of all r-tuples of complex numbers lying on the $(1,1),\ldots,(r,r)$ positions of matrices of the form $\sum_{j=1}^k U_jA_jU_j^t$ where $U_1,\ldots,U_n$ are unitary. The result is rather complicated even when $k=1$ and $r=n$. Thompson [19] showed that the complex numbers $d_1,\ldots,d_n$ with $|d_1| \ge \cdots \ge |d_n|$ can be the diagonal entries of a complex symmetric matrix with singular values $s_1 \ge \cdots \ge s_n$ if and only if

$$(|d_1|,\ldots,|d_n|) \prec_w (s_1,\ldots,s_n),$$

$$\sum_{j=1}^{r-1}|d_j| - \sum_{j=r}^{n}|d_j| \le \sum_{j=1}^{n} s_j - 2s_r, \quad r=1,\ldots,n,$$

and if $n \ge 3$,

$$\sum_{j=1}^{n-3}|d_j| - \sum_{j=n-2}^{n}|d_j| \le \sum_{j=1}^{n-2} s_j - s_{n-1} - s_n.$$

It is challenging to solve the problem for general k and r. An even more difficult problem is to restrict the choice of U_j's to special unitary matrices, see [16, 20]. In both cases, if one consider the real parts of the entries, then the problem is easy (cf. [16, Theorem 18]) as shown in the following.

Proposition 5.2 *Given complex symmetric matrices $A_1,\ldots,A_k \in M_n(\mathbf{C})$ and $1 \le r \le n$. Let $s_j = \sum_{i=1}^k s_j(A_i)$ for $j=1,\ldots,n$. The following conditions are equivalent.*

(a) *Up to a permutation, $(d_1,\ldots,d_r)$ is the real parts of an r-tuple of complex numbers lying on the $(1,1),\ldots,(r,r)$ positions of matrices of the form $\sum_{j=1}^k U_jA_jU_j^t$, where $U_1,\ldots,U_n$ are unitary.*

(b) *Up to a permutation, $(d_1,\ldots,d_r)$ is the real parts of an r-tuple of complex numbers lying on the $(1,1),\ldots,(r,r)$ positions of matrices of the form $\sum_{j=1}^k U_jA_jU_j^t$, where $U_1,\ldots,U_n$ are special unitary.*

(c) $(|d_1|,\ldots,|d_r|) \prec_w (s_1,\ldots,s_r)$.

Consequently, the set of vectors in (a) or (b) is convex.

Also, it is easy to describe the convex hull of the $q \times q$ principal submatrices of matrices of the form $\sum_{j=1}^k U_jA_jU_j^t$, where $U_1,\ldots,U_n$ are unitary or special unitary (cf. [16, Theorem 19]), namely, they are just the set of symmetric matrices X in $M_q(\mathbf{C})$ satisfying

$$\sum_{j=1}^{r} s_j(X) \le \sum_{j=1}^{r}\sum_{i=1}^{k} s_j(A_i), \quad r=1,\ldots,q.$$

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