Principal Submatrices of a Hermitian matrix

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Abstract

Suppose $k_1, \cdots, k_m$ and n are positive integers such that $k_1 + \cdots + k_m \leq n$. We characterize those $k_i \times k_i$ Hermitian matrices A_i, $i = 1, \cdots, m$ that can appear along the block diagonal of an $n \times n$ Hermitian matrix C with prescribed eigenvalues. The characterization will be given in terms of the eigenvalues of C and A_i, $i = 1, \cdots, m$. Our results extend those of Thompson and Freede, Horn, Fan and Pall.

Keywords: Hermitian matrices, singular values, eigenvalues.

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1 Introduction

Let $\mathcal{H}_n$ be the real linear space of $n \times n$ complex Hermitian matrices, and let $\mathcal{U}_n$ be the group of $n \times n$ unitary matrices. Given $c = (c_1, \cdots, c_n) \in \mathbb{R}^{1\times n}$, let $\mathcal{U}(c)$ be the set of matrices in $\mathcal{H}_n$ with eigenvalues $c_1, \cdots, c_n$. Schur [12] gave some necessary conditions for $d \in \mathbb{R}^n$ to be the diagonal of a matrix in $U(c)$ and Horn [7] showed that the conditions are also sufficient. For $1 \leq k \leq n$, Fan and Pall [3] gave necessary and sufficient conditions for a matrix in $\mathcal{H}_k$ to be the principal submatrix of a matrix in $\mathcal{U}(c)$. Thompson and Freede [13] obtained necessary conditions for $A_{11}, \ldots, A_{mm}$ such that $A_{jj} \in \mathcal{H}_{n_j}$, where $n_1 + \cdots + n_k = n$, are the digonal blocks of a matrix in $\mathcal{U}(c)$, i.e., there exists $A = (A_{ij})_{1\leq i,j\leq m} \in \mathcal{U}(c)$. In this paper, we give a necessary and sufficient condition for this to happen. More generally, for any $m-$tuple of positive integers $\underline{k} = (k_1, k_2, \ldots, k_m)$ such that $\sum_{i=1}^m k_i \leq n$, define

$$P_{\underline{k}}(c) = \left\{(A_{11}, \ldots, A_{mm}) \in \mathcal{H}_{k_1} \times \cdots \times \mathcal{H}_{k_m} : \text{ there exists } A = (A_{ij})_{i,j=1}^{m+1} \in \mathcal{U}(c)\right\} .$$

Using an idea of Thompson and a recent result of Fulton, we give a complete description of $P_{\underline{k}}(c)$ by showing that $(A_{11}, \ldots, A_{mm}) \in P_{\underline{k}}(c)$ if and only if the eigenvalues of $A_{11}, \ldots, A_{mm}$ satisfy a certain collection of inequalities. Moreover, we discuss how to reduce the number of inequalities in the collection, and demonstrate how to use our results to obtain those of Horn, Fan and Pall.

Although our discussion is on complex Hermitian matrices, all our results are valid (with the same proofs) for real symmetric and Hermitian matrices over real quaternions.

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In our discussion, the sets of eigenvalues or singular values always mean the multi-sets of eigenvalues or singular values counting multiplicities.

Before concluding this section, we mention a different formulation of our problem used by other authors. Suppose $C \in \mathcal{H}_n$ has eigenvalues $c_1 \geq \cdots \geq c_n$. Then $P_{\underline{k}}(C)$ can be identified with the set of matrices $P_1CP_1 + \cdots + P_mCP_m \in \mathcal{H}_n$ for some idempotents $P_1, \ldots, P_m \in \mathcal{H}_n$, where rank $P_i = k_i$ for $1 \leq i \leq m$, and $P_iP_j = 0$ if $i \neq j$. If $P_1 + \cdots + P_m = I_n$, then $P_1CP_1 + \cdots + P_mCP_m$ is a *pinching* of the matrix C, which is a useful concept in studying matrix and operator inequalities; see [1]. One can easily reformulate our results under this setting.

2 Principal Submatrices

Let $\mathcal{S}_r^n = \{(j_1, \ldots, j_r) : 1 \leq j_1 < j_2 < \ldots < j_r \leq n\}$. For $J = (j_1, \ldots, j_r) \in \mathcal{S}_r^n$, define

$$\lambda(J) = (j_r - r, \ldots, j_1 - 1). \tag{1}$$

Denote by $LR_r^n(m)$ the set of $(m+1)$-tuple $(J_0, J_1, \ldots, J_m)$, $J_i \in \mathcal{S}_r$, such that the Littlewood-Richardson coefficient of the partitions $\lambda(J_0), \lambda(J_1), \ldots, \lambda(J_m)$ is positive, i.e., one can generate the young diagram of $\lambda(J_0)$ from those of $\lambda(J_1), \ldots, \lambda(J_m)$ according to the Littlewood-Richardson rule; see [4]. We have the following result; see [5] and also [9].

Lemma 2.1 *There exist $A_i \in \mathcal{H}_n$ with eigenvalues $a_1^{(i)} \geq \cdots \geq a_n^{(i)}$, $i = 1, 2, \ldots, m$ and $C \in \mathcal{H}_n$ with eigenvalues $c_1 \geq \cdots \geq c_n$ such that*

$$C = A_1 + A_2 + \cdots + A_m$$

if and only if

$$\sum_{j=1}^{n} c_j = \sum_{i=1}^{m} \sum_{j=1}^{n} a_j^{(i)} \tag{2}$$

and for each $r \in \{1, \ldots, n\}$ and $(J_0, J_1, \ldots, J_m) \in LR_r^n(m)$,

$$\sum_{j \in J_0} c_j \leq \sum_{i=1}^{m} \sum_{j \in J_i} a_j^{(i)} . \tag{3}$$

Let $C \in \mathcal{H}_n$ and let $\underline{k} = (k_1, k_2, \ldots, k_m)$ be an $m-$tuple of positive integers such that $\sum_{i=1}^m k_i \leq n$.

Now, we can use Lemma 2.1 and an idea of Thompson and Freede [13] to characterize the set

$$P_{\underline{k}}(c) = \left\{(A_{11}, \ldots, A_{mm}) \in \mathcal{H}_{k_1} \times \cdots \times \mathcal{H}_{k_m} : \text{ there exists } A = (A_{ij})_{i,j=1}^{m+1} \in \mathcal{U}(c)\right\} .$$

Since $(A_1, \ldots, A_m) \in P_{\underline{k}}(c)$ if and only if $(A_1 - \alpha I_{n_1}, \ldots, A_m - \alpha I_{n_m}) \in P_{\underline{k}}(c - \alpha 1_n)$, there is no harm to restrict our attention to positive semi-definite matrices $C, A_1, \ldots, A_m$.

Theorem 2.2 *Suppose $C \in \mathcal{H}_n$ has eigenvalues $c_1 \geq \cdots \geq c_n \geq 0$. Let $\underline{k} = (k_1, k_2, \ldots, k_m)$ be an m-tuple of positive integers such that $\ell = n - \sum_{i=1}^m k_i \geq 0$.*

(a) *Suppose $\ell = 0$. For $1 \leq i \leq m$, let $A_i \in \mathcal{H}_{k_i}$ have eigenvalues $a_1^{(i)} \geq \cdots \geq a_{k_i}^{(i)} \geq 0$. Define $a_j^{(i)} = 0$ for $1 \leq i \leq m$ and $k_i + 1 \leq j \leq n$. Then $(A_1, \ldots, A_m) \in P_{\underline{k}}(c)$ if and only if*

$$\sum_{i=1}^{n} c_j = \sum_{i=1}^{m} \sum_{j=1}^{n} a_j^{(i)} \tag{4}$$

and for each $r \in \{1, \ldots, n-1\}$ and $(J_0, J_1, \ldots, J_m) \in LR_r^n(m)$,

$$\sum_{j \in J_0} c_j \leq \sum_{i=1}^{m} \sum_{j \in J_i} a_j^{(i)}. \tag{5}$$

(b) *Suppose $\ell > 0$. Let $d = (\operatorname{tr} C - \sum_{i=1}^m \operatorname{tr} A_i)/\ell$, and $\underline{k'} = (k_1, \ldots, k_m, \underbrace{1, \ldots, 1}_{\ell})$. Then $(A_1, \ldots, A_m) \in P_{\underline{k}}(c)$ if and only if $(A_1, \ldots, A_m, \underbrace{[d], \ldots, [d]}_{\ell}) \in P_{\underline{k'}}(c)$. In this case, we can apply* (a) *to check the last condition.*

Proof. (a) Suppose $\ell = 0$ and $(A_1, \ldots, A_m) \in P_{\underline{k}}(c)$. Let $A = (A_{ij})_{i,j=1}^m \in \mathcal{U}(c)$ with $A_{ii} = A_i$. Write $A = [B_1|\cdots|B_m]^*[B_1|\cdots|B_m]$ so that B_i is $n \times n_i$ satisfying $B_i^* B_i = A_i$ for $i = 1, \ldots, m$. Hence,

$$A = [B_1|\cdots|B_m]^*[B_1|\cdots|B_m] = (B_i^* B_j)_{i,j=1}^m. \tag{6}$$

Then the matrix

$$\tilde{A} = [B_1|\cdots|B_m][B_1|\cdots|B_m]^* = B_1 B_1^* + \cdots + B_m B_m^* \tag{7}$$

has eigenvalues $c_1 \geq \cdots \geq c_n$, and each $B_i B_i^*$ has eigenvalues $a_1^{(i)} \geq \cdots \geq a_n^{(i)}$, $i = 1, \ldots, m$. We can apply Lemma 2.1 to get (4) and (5).

Conversely, suppose $A_i \in \mathcal{H}_{k_i}$, $i = 1, \ldots, m$, satisfy (4) and (5). Then by Lemma 2.1, for $i = 1, \ldots, m$, there exists $\tilde{A}_i \in \mathcal{H}_n$ with eigenvalues $a_1^{(i)} \geq \cdots \geq a_n^{(i)}$ so that $\tilde{A} = \tilde{A}_1 + \cdots + \tilde{A}_m$ has eigenvalues $c_1 \geq \cdots \geq c_n$. Moreover, since $\tilde{A}_i$ has rank at most k_i, we can write $\tilde{A}_i = B_i B_i^*$ such that $B_i \in \mathbb{C}^{n \times k_i}$ and $\tilde{A} \in \mathcal{U}(c)$ has the form (7). Then there exist $U_i \in \mathcal{U}_{k_i}$, $i = 1, \ldots, m$ such that $U_i^* B_i^* B_i U_i = A_i$. Let A be the matrix in the form (6) and $U = \oplus_{i=1}^m U_i$, then $U^* A U \in \mathcal{U}(c)$. Therefore, $(A_1, \ldots, A_m) \in P_{\underline{k}}(c)$.

(b) Suppose $\ell > 0$. Then we have $(A_{11}, \ldots, A_{mm}) \in P_{\underline{k}}(c)$ if and only if there exists $A = (A_{ij})_{1 \le i,j \le m+1} \in \mathcal{U}(c)$. By [8, Theorem 1.3.4], we may assume that $A_{m+1,m+1}$ has constant diagonal entries equal to $d = (\operatorname{tr} C - \sum_{i=1}^m \operatorname{tr} A_i)/\ell$; otherwise, replace A by the matrix $(I_{n-\ell} \oplus W^*) A (I_{n-\ell} \oplus W)$ for a suitable $W \in \mathcal{U}_\ell$. The first assertion follows.

Now, since $\sum_{i=1}^m k_i + \ell = n$, we can use (a) to check the condition

$$(A_1, \ldots, A_m, \underbrace{[d], \ldots, [d]}_{\ell}) \in P_{\underline{k}'}(c).$$

□

There are numerous inequalities in (5). To reduce the list of inequalities, one can replace $LR_r^n(m)$ by the set $\widetilde{LR}_r^n(m)$ consisting of the set of $(J_0, \ldots, J_m)$ such that the Littlewood-Richardson coefficient of the partitions $\lambda(J_0), \ldots, \lambda(J_m)$ is one. In the next section, we will consider other reduction schemes.

3 Reducing the List of Inequalities

In the following, we show that one can reduce the number of inequalities needed to be checked in (5). We need two lemmas; the first one is the solution of the saturation conjecture; see [2, 5, 10, 11].

Lemma 3.1 *Suppose* $1 \le m \le n$ *and* $J_i \in \mathcal{S}_r^n$ *for* $0 \le i \le m$. *Then* $(J_0, J_1, \ldots, J_m) \in LR_r^n(m)$ *if and only if there exist* $B_0, B_1, \ldots, B_m \in \mathcal{H}_r$ *so that* $B_0 = B_1 + \cdots + B_m$, *and for* $i \in \{0, 1, \ldots, m\}$, $\lambda(J_i)$, *as defined in (1), is the vector of eigenvalues of* B_i *with entries arranged in descending order. In particular, if* $\lambda(J_0) = \sum_{i=1}^m \lambda(J_i)$, *then* $(J_0, J_1, \ldots, J_m) \in LR_r^n(m)$.

Lemma 3.2 *Let* $(J_0, J_1, \ldots, J_m) \in LR_r^n(m)$ *with* $\lambda(J_i) = (\alpha_1^{(i)}, \ldots, \alpha_r^{(i)})$ *for* $i = 0, 1, \ldots, m$. *Suppose there exists* $i_0 \in \{1, \ldots, m\}$ *such that*

$$\alpha_{j_0}^{(i_0)} > \alpha_{j_0+1}^{(i_0)} \quad \text{with} \quad j_0 \in \{1, \ldots, r-1\} \qquad \text{or} \qquad \alpha_{j_0}^{(i_0)} > 0 \quad \text{with} \quad j_0 = r.$$

Let $\tilde{J}_{i_0}$ *be obtained from* J_{i_0} *by subtracting one from the* j_0 *entry. Then there exists* $\tilde{J}_0$ *such that* $J_0 - \tilde{J}_0$ *has nonnegative entries and*

$$(\tilde{J}_0, J_1, \ldots, J_{i_0-1}, \tilde{J}_{i_0}, J_{i_0+1}, \ldots, J_m) \in LR_r^n(m).$$

Proof. Without loss of generality, we may assume that $i_0 = 1$. We are going to prove by induction on r. The result is trivial for $r = 1$. Suppose $r > 1$ and the result holds for all dimension less than r. Since $(J_0, J_1, \ldots, J_m) \in LR_r^n(m)$. By Lemma 3.1, there exist $B_0, B_1, \ldots, B_m \in \mathcal{H}_r$ so that $B_0 = B_1 + \cdots + B_m$, and for $i \in \{0, 1, \ldots, m\}$, $\lambda(J_i)$ is the vector of eigenvalues of B_i with entries arranged in descending order. By Lemma 2.1,

$$\sum_{k=1}^{r} \alpha_k^{(0)} = \sum_{i=1}^{m} \sum_{j=1}^{r} \alpha_j^{(i)}, \tag{8}$$

and

$$\sum_{j \in I_0} \alpha_j^{(0)} \le \sum_{i=1}^{m} \sum_{j \in I_i} \alpha_j^{(i)}, \tag{9}$$

whenever $(I_0, I_1, \ldots, I_m) \in LR_p^r(m)$ with $1 \le p < r$.

If strict inequality holds in (9) for all $(I_0, I_1, \ldots, I_m) \in LR_p^r(m)$ with $p \in \{1, \ldots, r-1\}$, let $\tilde{J}_0$ be such that $\lambda(\tilde{J}_0) = (\tilde{\alpha}_1^{(0)}, \ldots, \tilde{\alpha}_r^{(0)})$ equals the decreasing rearrangement of $(\alpha_1^{(0)} - 1, \alpha_2^{(0)}, \ldots, \alpha_r^{(0)})$. Then

1. $\sum_{s=1}^r \tilde{\alpha}_s^{(0)} = \sum_{s=1}^r \alpha_s^{(0)} - 1 = \sum_{i=1}^m \sum_{j=1}^r \alpha_j^{(i)} - 1 = \sum_{j=1}^r \tilde{\alpha}_j^{(1)} + \sum_{2=1}^m \sum_{j=1}^r \alpha_j^{(i)}$, and
2. $\sum_{s \in I_0} \tilde{\alpha}_s^{(0)} \le \sum_{s \in I_0} \alpha_s^{(0)} \le \sum_{i=1}^m \sum_{j \in I_i} \alpha_j^{(i)} - 1 \le \sum_{j \in I_1} \tilde{\alpha}_j^{(1)} + \sum_{i=2}^m \sum_{j \in I_i} \alpha_j^{(i)}$,

   whenever $(I_0, I_1, \ldots, I_m) \in LR_p^r(m)$ with $1 \le p < r$. Note that in the second inequality, we use the fact that all the inequalities in (9) are strict.

Therefore, $(\tilde{J}_0, \tilde{J}_1, J_2, \ldots, J_m) \in LR_r^n(m)$.

Suppose there exists $(I_0, I_1, \ldots, I_m) \in LR_p^r(m)$ with $p \in \{1, \ldots, r-1\}$ such that

$$\sum_{s \in I_0} \alpha_s^{(0)} = \sum_{i=1}^{m} \sum_{j \in I_i} \alpha_j^{(i)}.$$

Let $I_i^c = \{1, 2, \cdots, r\} \setminus I_i$ for $i \in \{0, 1, \ldots, m\}$. By an (easy) extension of [6, Theorem 5] to the sum of m Hermitian matrices, there are Hermitian matrices $B_0, B_1, \ldots, B_m \in \mathcal{H}_p$ and $C_0, C_1, \ldots, C_m \in \mathcal{H}_{r-p}$ such that

(i) $B_0 = B_1 + \cdots + B_m$, $C_0 = C_1 + \cdots + C_m$, and

(ii) for $i \in \{0, 1, \ldots, m\}$, $(\alpha_s^{(i)})_{s \in I_i}$ (respectively, $(\alpha_s^{(i)})_{s \in I_i^c}$) is the vector of eigenvalues of B_i (respectively, C_i) with entries arranged in descending order.

If $j_0^{(1)} \in I_1$, we can apply induction assumption to $B_0, \ldots, B_m$ to get $\tilde{B}_0, \ldots, \tilde{B}_m \in \mathcal{H}_p$ so that $\tilde{B}_i$ and B_i have the same eigenvalues for $i = 2, \ldots, n$, the eigenvalue of $\tilde{B}_0$ are those of

B_0 except one of them is reduced by one, and the eigenvalues of $\tilde{B}_1$ are those of B_1 except the one equal to $\alpha_{j_0}^{(1)}$ is reduced by one. Now, if $A_i = \tilde{B}_i \oplus C_i \in \mathcal{H}_k$, we have $A_0 = A_1 + \cdots + A_m$. We can then obtain $(\tilde{J}_0, \tilde{J}_1, J_2, \ldots, J_m) \in LR_r^n(m)$ from the eigenvalues of $A_0, \ldots, A_m$. If $j_0^{(1)} \in I_1^c$, we can apply induction assumption on $C_0, \ldots, C_m$, and get the conclusion by a similar argument. □

Theorem 3.3 *In checking condition* (5) *in Theorem 2.2 (a), it suffices to check those sequences* $(J_0, J_1, \ldots, J_m) \in LR_r^n(m)$ *satisfying the following condition:*

For all $1 \le i \le m$, *if* $J_i = (j_1^{(i)}, \ldots, j_r^{(i)})$ *and* $j_{p-1}^{(i)} \le k_i < j_p^{(i)}$ *for some* $1 \le p \le r$, *then*

$$(j_p^{(i)}, j_{p+1}^{(i)}, \ldots, j_r^{(i)}) = (k_i + 1, k_i + 2, \ldots, k_i + 1 + r - p).$$

Proof. Suppose condition (5) in Theorem 2.2 (a) holds for all $(J_0, J_1, \ldots, J_m) \in LR_r^n(m)$ satisfying the above condition. For each $J_i = (j_1^{(i)}, \ldots, j_r^{(i)})$ such that $j_{p-1}^{(i)} \le k_i < j_p^{(i)}$ for some $1 \le p \le r$, define $\tilde{J}_i = (\tilde{j}_1^{(i)}, \ldots, \tilde{j}_r^{(i)})$, where $\tilde{j}_\ell^{(i)} = j_\ell^{(i)}$ for $1 \le \ell < p$ and $\tilde{j}_{p+t}^{(i)} = k_i + t + 1$ for all $0 \le t \le r - p$. Since $\lambda(J_i) = (j_r^{(i)} - r, \ldots, j_1^{(i)} - 1)$ and

$$j_p^{(i)} - p > j_{p-1}^{(i)} - (p-1) \Leftrightarrow j_p^{(i)} > j_{p-1}^{(i)} + 1,$$

we can use Lemma 3.2 and replace J_0 with $\tilde{J}_0 = (\tilde{j}_1^{(0)}, \ldots, \tilde{j}_r^{(0)})$ such that $\tilde{j}_\ell^{(0)} \le j_\ell^{(0)}$. We have

$$\sum_{j \in J_0} c_j \le \sum_{j \in \tilde{J}_0} c_j \le \sum_{i=1}^m \sum_{j \in \tilde{J}_i}^n a_j^{(i)} = \sum_{i=1}^m \sum_{j \in J_i}^n a_j^{(i)}.$$

□

We illustrate how to use the above theorem in the following.

Theorem 3.4 *Let* $C \in \mathcal{H}_n$ *have eigenvalues* $c_1 \ge \cdots \ge c_n$, *and* $\underline{k} = (\underbrace{2, \ldots, 2}_{m}, \underbrace{1, \ldots, 1}_{n-2m})$. *Suppose* $A_i \in \mathcal{H}_2$ *have eigenvalues* $a_1^{(i)} \ge a_2^{(i)}$ *for* $1 \le i \le m$, *and* $A_i = [a_1^{(i)}] \in \mathcal{H}_1$ *for* $m + 1 \le i \le n - m$. *Let* $(i_1, \cdots, i_m)$ *be a permutation of* $(1, \cdots, m)$ *such that* $a_2^{(i_1)} \ge \cdots \ge a_2^{(i_m)}$. *For any subset* $R \subseteq \{1, \cdots, m\}$ *with* $|R| = r$, *let* $b_1^R \ge \cdots \ge b_{n-m-2r}^R$ *be the eigenvalues of* $\oplus_{i \notin R} A_i$. *Then* $(A_1, \ldots, A_{n-m}) \in P_{\underline{k}}(c)$ *if and only if*

$$\sum_{i=1}^n c_i = \sum_{i=1}^{n-m} \operatorname{tr} A_i, \tag{10}$$

and for any $s \in \{0, \cdots, m\}$ *and* $t \in \{0, \ldots, n-2s\}$*, with* $0 < s+t < n$*, we have*

$$\sum_{i=1}^{t} c_i + \sum_{i=t+2}^{s+t+1} c_i \geq \sum_{j \in S} a_2^{(j)} + \sum_{i=1}^{t} b_i^S\,, \tag{11}$$

for any s *element subset* $S \subseteq \{i_1, \cdots, i_\ell\}$ *where* $\ell = \min\{m,\, s+t\}$.

Proof. We may assume that $C, A_1, \ldots, A_{n-m}$ are positive semi-definite; otherwise, apply the translation $X \mapsto X + \gamma I$ to them for a sufficiently large $\gamma > 0$. Furthermore, we may assume that $i_j = j$ for $j = 1, \cdots, m$.

Suppose $(A_1, \ldots, A_{n-m}) \in P_{\underline{k}}(c)$. Then (10) clearly holds. Let $s \in \{0, \cdots, m\}$ and $t \in \{0, \ldots, n-2s\}$, with $0 < s+t < n$ and $\ell = \min\{m,\, s+t\}$. For any s element subset $S \subseteq \{1, \cdots, \ell\}$, choose disjoint subsets $P \subseteq \{1, \cdots, n-m\} \setminus S$ and $Q \subseteq \{1, \cdots, m\} \setminus S$ such that $|P| + 2|Q| = t$ and $\sum_{i \in P} a_1^{(i)} + \sum_{i \in Q} \left(a_1^{(i)} + a_2^{(i)}\right) = \sum_{i=1}^{t} b_i^S$.

Set $r = n - t - s$. Define

$$J_0 = (t+1, s+t+2, s+t+3, \ldots, n), \quad \text{and}$$

$$J_i = \begin{cases} (2, \ldots, r+1) & \text{for } i \in P, \\ (3, 4, \ldots, r+2) & \text{for } i \in Q, \\ (1, 3, \ldots, r+1) & \text{for } i \in S, \\ (1, 2, \ldots, r) & \text{for } i \in \{1, 2, \ldots, n-m\} \setminus (P \cup Q \cup S). \end{cases}$$

Then

$$\lambda(J_0) = (s+t, s+t, \ldots, s+t,\, t), \quad \text{and}$$

$$\lambda(J_i) = \begin{cases} (1, \ldots, 1) & \text{for } i \in P, \\ (2, 2, \ldots, 2) & \text{for } i \in Q, \\ (1, \ldots, 1, 0) & \text{for } i \in S, \\ (0, \ldots, 0) & \text{for } i \in \{1, 2, \ldots, n-m\} \setminus (P \cup Q \cup S). \end{cases}$$

Since $\lambda(J_0) = \lambda(J_1) + \cdots + \lambda(J_{n-m})$, by Lemma 3.1 $(J_0, J_1, \ldots, J_{n-m}) \in LR_r^n(n-m)$. So, we have

$$\begin{aligned} \sum_{i=1}^{t} c_i + \sum_{i=t+2}^{s+t+1} c_i &= \sum_{i=1}^{n} c_i - \sum_{j \in J_0} c_j \\ &\geq \sum_{i=1}^{n-m} \operatorname{tr} A_j - \sum_{i=1}^{n-m} \sum_{j \in J_i} a_j^{(i)} \\ &= \sum_{i \in S} a_2^{(i)} + \sum_{i \in P} a_1^{(i)} + \sum_{i \in Q} (a_1^{(i)} + a_2^{(i)}) \\ &= \sum_{j \in S} a_2^{(j)} + \sum_{i=1}^{t} b_i^S. \end{aligned}$$

Conversely, suppose (10) and the inequalities (11) hold and $(J_0, J_1, \ldots, J_{n-m}) \in LR_r^n(n-\ m)$ with $J_i = (j_1^{(i)}, \ldots, j_r^{(i)})$. We need to show that

$$\sum_{j\in J_0} c_j \le \sum_{i=1}^{n-m} \sum_{j\in J_i} a_j^{(i)}. \tag{12}$$

Let

$$\begin{aligned} P &= \{i : 1 \le i \le m,\ j_1^{(i)} = 2\} \cup \{i : m+1 \le i \le n-m,\ j_1^{(i)} \ge 2\}, \\ Q &= \{i : 1 \le i \le m,\ j_1^{(i)} > 2\}, \\ S &= \{i : 1 \le i \le m,\ j_1^{(i)} = 1,\ j_2^{(i)} > 2\}. \end{aligned}$$

By Theorem 3.3, we can assume that

$$J_i = \begin{cases} (2, \ldots, r+1) & \text{for } i \in P, \\ (3, 4, \ldots, r+2) & \text{for } i \in Q, \\ (1, 3, \ldots, r+1) & \text{for } i \in S, \\ (1, 2, \ldots, r) & \text{for } i \in \{1, 2, \ldots, n-m\} \setminus (P \cup Q \cup S). \end{cases}$$

Let $t = |P| + 2|Q|$ and $s = |S|$. If $s + t = 0$ or n, (12) follows from (10). So we may assume that $0 < s + t < n$. Since

$$\lambda(J_i) = \begin{cases} (1, \ldots, 1) & \text{for } i \in P, \\ (2, 2, \ldots, 2) & \text{for } i \in Q, \\ (1, \ldots, 1, 0) & \text{for } i \in S, \\ (0, \ldots, 0) & \text{for } i \in \{1, 2, \ldots, n-m\} \setminus (P \cup Q \cup S). \end{cases}$$

we have

$$J_0 = (t+1, s+t+2, s+t+3, \ldots, r+s+t)\,.$$

Therefore, we have

$$\sum_{j\in J_0} c_j = c_{t+1} + \sum_{i=s+t+2}^{r+s+t} c_i \le \sum_{i=1}^{n} c_i - \left(\sum_{i=1}^{t} c_i + \sum_{i=t+2}^{s+t+1} c_i \right)$$

and

$$\begin{aligned} \textstyle\sum_{i=1}^{n-m} \sum_{j\in J_i} a_j^{(i)} &= \operatorname{tr} A_j - \left(\textstyle\sum_{i\in S} a_2^{(i)} + \sum_{i\in P} a_1^{(i)} + \sum_{i\in Q} \left(a_1^{(i)} + a_2^{(i)} \right) \right) \\ &\ge \operatorname{tr} A_j - \left(\textstyle\sum_{i\in S} a_2^{(i)} + \sum_{i=1}^{t} b_i^S \right) \end{aligned}$$

If $S \subseteq \{1, \cdots, \ell\}$, then (12) follows from (10) and (11).

If $0 < s + t < m$ and $S \not\subseteq \{1, \cdots, s+t\}$, then we can choose disjoint subsets $P \subseteq \{1, \cdots, n-m\} \setminus S$ and $Q \subseteq \{1, \cdots, m\} \setminus S$ such that

$$|P| + 2|Q| = t \quad \text{and} \quad \sum_{i\in P} a_1^{(i)} + \sum_{i\in Q} (a_1^{(i)} + a_2^{(i)}) = \sum_{i=1}^{t} b_i^S\,.$$

Since $S \not\subseteq \{1, \ldots, s+t\}$, we have

$$|(P \cup Q \cup S) \cap \{1, \cdots, s+t\}| \leq s+t-1 \,.$$

Choose $j \in \{1, \cdots, s+t\} \setminus (P \cup Q \cup S)$, $j' \in S \setminus \{1, \cdots, s+t\}$ and let $S_1 = S \cup \{j\} \setminus \{j'\}$. Then $|S_1| = s$ and $|S_1 \setminus \{1, \cdots, s+t\}| < |S \setminus \{1, \cdots, s+t\}|$. Thus,

$$\sum_{i \in S_1} a_2^{(i)} + \sum_{i=1}^{t} b_i^{S_1} \geq \sum_{i \in S} a_2^{(i)} + \sum_{i=1}^{t} b_i^{S}$$

and hence

$$\operatorname{tr} A_j - \left(\sum_{i \in S_1} a_2^{(i)} + \sum_{i=1}^{t} b_i^{S_1} \right) \leq \operatorname{tr} A_j - \left(\sum_{i \in S} a_2^{(i)} + \sum_{i=1}^{t} b_i^{S} \right) .$$

Repeating the above procedure, we can get an s element subset $\tilde{S} \subseteq \{1, \cdots, s+t\}$ such that

$$\operatorname{tr} A_j - \left(\sum_{i \in \tilde{S}} a_2^{(i)} + \sum_{i=1}^{t} b_i^{\tilde{S}} \right) \leq \operatorname{tr} A_j - \left(\sum_{i \in S} a_2^{(i)} + \sum_{i=1}^{t} b_i^{S} \right) ,$$

from which (12) follows. $\square$

Note that for each choice of s element set $S \subseteq \{1, \ldots, \ell\}$, there are $\binom{\min\{m,\, s+t\}}{s}$ inequalities in (11). Excluding the cases $(s,t) = (0,0)$ and $(s,t) = (m, n-2m)$, where the inequalities follows from (10), we see that the number of inequalities in (11) is given by

$$\sum_{s=0}^{m} \sum_{t=0}^{n-2s} \binom{\min\{m,\, s+t\}}{s} - 2$$

By taking $m = 0$, we have Horn's result [7]: There exists $A \in \mathcal{U}(c)$ with diagonal entries $d_1 \geq \cdots \geq d_n$ if and only if

$$\sum_{j=1}^{n} c_j = \sum_{j=1}^{n} d_j \quad \text{and} \quad \sum_{j=1}^{s} c_j \geq \sum_{j=1}^{s} d_j \quad \text{for all } 1 \leq s < n \,.$$

Using a similar argument, we can prove the following result of Fan and Pall [3]:

For $1 \leq k \leq n-1$, an $k \times k$ Hermitian matrix A is the principal submatrix of an $n \times n$ Hermitian matrices C with eigenvalues $c_1 \geq \cdots \geq c_n$ if and only if A has eigenvalues $a_1 \geq \cdots \geq a_k$ such that

$$c_j \geq a_j \geq c_{n-k+j}, \text{ for } \; j = 1, \ldots, k.$$

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