Decomposable numerical ranges of normal matrices

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Abstract

Let $\mathbf{M}_{n,k}$ ($\mathbf{M}_n$) be the set of $n \times k$ ($n \times n$) complex matrices, and $\mathrm{per}(X)$ be the permanent of a square matrix X. We study the three types of generalized numerical ranges associated with generalized matrix functions

$$\Pi_k(A) = \left\{ \prod_{j=1}^{k} (V^*AV)_{ii} : V \in \mathbf{M}_{n,k},\ V^*V = I_k \right\},$$

$$D_k(A) = \left\{ \det(V^*AV) : V \in \mathbf{M}_{n,k},\ V^*V = I_k \right\},$$

and

$$P_k(A) = \left\{ \mathrm{per}(V^*AV) : V \in \mathbf{M}_{n,k},\ V^*V = I_k \right\}.$$

We give complete descriptions of the set $\Pi_2(A)$, $D_2(A)$ and $P_2(A)$ for essentially hermitian matrices $A \in \mathbf{M}_n$. In particular, all three sets are star-shaped. For 3×3 normal matrices A, it is known that $D_2(A)$ is convex. We show that $\Pi_2(A)$ and $P_2(A)$ are star-shaped. This affirms a conjecture of Nakazato et. al.

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1. Introduction

Let $\mathbf{M}_{m,n}$ be the set of all $m \times n$ complex matrices. When $m = n$, $\mathbf{M}_{n,n}$ is abbreviated to $\mathbf{M}_n$. Suppose $1 \leq k \leq n$ and $\chi : H \to \mathbb{C}$ is an irreducible character on a subgroup H of the symmetric group S_k of order k. The generalized matrix function $d_\chi^H : \mathbf{M}_k \to \mathbb{C}$ associated with H and χ is defined, for $B \in \mathbf{M}_k$, by

$$d_\chi^H(B) = \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{k} (B)_{i\sigma(i)}.$$

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*Dedicated to Professor Yiu-Tung Poon on the occasion of his birthday.

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Here $(B)_{ij}$ is the (i,j)-th entry of B. Marcus and Wang [17] introduced the decomposable numerical range of $A \in \mathbf{M}_n$ associated with d_χ^H as

$$W_\chi^H(A) := \left\{ d_\chi^H(V^*AV) : V \in \mathbf{M}_{n,k},\ V^*V = I_k \right\}.$$

When $k = 1$, the set $W_\chi^H(A)$ reduces to the classical numerical range $W(A)$, which has been studied extensively; see [8], [9, Chapter 1], and their references.

If $H = \{e\} \subseteq S_k$ is the trivial subgroup and χ is the principal character, then $d_\chi^H(B) = \prod_{i=1}^k (B)_{ii}$. In this case, we denote $W_\chi^H(A)$ as

$$\Pi_k(A) = \left\{ \prod_{j=1}^k (V^*AV)_{ii} : V \in \mathbf{M}_{n,k},\ V^*V = I_k \right\}.$$

Geometric properties of $\Pi_k(A)$ including convexity, star-shapedness and simply connectedness were investigated in [2, 14]. In particular, the authors showed that $\Pi_k(A)$ may fail to be convex when $k \geq 2$ and fail to be simply connected when $k \geq 3$. Characterizations of $\Pi_k(A)$ for essentially hermitian matrices A, i.e., normal matrices whose eigenvalues are collinear in $\mathbb{C}$, are given in [2, 19]. Additionally, the authors in [19] showed that $\Pi_k(A)$ is star-shaped for 3×3 essentially hermitian matrices A.

Let $H = S_k$. The generalized matrix function d_χ^H associated with the alternative character χ is the determinant function $\det(\cdot)$. Alternatively, if χ equals the principal character, then d_χ^H becomes the permanent function $\operatorname{per}(\cdot)$. In these two cases, we denote the corresponding decomposable numerical ranges as

$$D_k(A) = \left\{ \det(V^*AV) : V \in \mathbf{M}_{n,k},\ V^*V = I_k \right\},$$

and

$$P_k(A) = \left\{ \operatorname{per}(V^*AV) : V \in \mathbf{M}_{n,k},\ V^*V = I_k \right\}.$$

The range $D_k(A)$ is known as the kth decomposable numerical range in the literature and has been studied extensively due to its connections with theories of determinants, exterior spaces, and other areas; see [1, 5, 16]. It is known that $D_k(A)$ generally fails to be convex when $k \geq 2$; see [15, 16]. In addition, $D_k(A)$ may fail to be simply connected when $k \geq 6$; see [5]. However, not much is known concerning the star-shapedness or the simple connectedness of $D_k(A)$ for $2 \leq k \leq 5$. The set $P_k(A)$ is referred to as the kth permanental numerical range and is applied to quantum systems of bosons; see [3, 10, 11, 12]. In general, $P_k(A)$ may fail to be convex if $k \geq 2$.

The purpose of this paper is to investigate the star-shapedness of $\Pi_2(A)$, $D_2(A)$ and $P_2(A)$. In Section 2, we give complete descriptions of the sets $\Pi_2(A)$, $D_2(A)$ and $P_2(A)$ for $n \times n$ essentially hermitian matrices A. In particular, the sets are star-shaped. In Section 3, we consider 3×3 normal matrices A. In this case, it is known that $D_2(A)$ is convex. We show that $\Pi_2(A)$ and $P_2(A)$ are star-shaped. This affirms a conjecture in [19].

2. Essentially Hermitian Matrices

Note that if a matrix $A \in \mathbf{M}_n$ is essentially hermitian then there exists $z \in \mathbb{C}$ such that $A = z(I_n + \mathrm{i}K)$ where $K \in \mathbf{M}_n$ is hermitian. Since for every $A \in \mathbf{M}_n$ and $z \in \mathbb{C}$, we have

$W_\chi^H(zA) = z^k W_\chi^H(A)$, it suffices to consider $A = I_n + \mathrm{i}K$, where K is hermitian and has eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. In the sequel, we denote by $[a,b] = \{ta+(1-t)b : 0 \leq t \leq 1\}$ the closed line segment joining $a, b \in \mathbb{C}$. For essential hermitian matrices A, we obtain the following characterizations of $\Pi_k(A)$, $D_k(A)$ and $P_k(A)$.

Theorem 2.1. *Let $A = I_n + \mathrm{i}K \in \mathbf{M}_n$ where K is a hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.*

(a) $\Pi_2(A)$ *is the region enclosed by the four line segments*

$$\left[(1+\mathrm{i}\lambda_1)(1+\mathrm{i}\lambda_2), \left(1+\mathrm{i}\frac{\lambda_1+\lambda_2}{2}\right)^2\right], \quad \left[(1+\mathrm{i}\lambda_{n-1})(1+\mathrm{i}\lambda_n), \left(1+\mathrm{i}\frac{\lambda_{n-1}+\lambda_n}{2}\right)^2\right],$$

$$\big[(1+\mathrm{i}\lambda_1)(1+\mathrm{i}\lambda_n), (1+\mathrm{i}\lambda_1)(1+\mathrm{i}\lambda_2)\big], \quad \big[(1+\mathrm{i}\lambda_1)(1+\mathrm{i}\lambda_n), (1+\mathrm{i}\lambda_{n-1})(1+\mathrm{i}\lambda_n)\big],$$

and an arc of a parabola $\left\{\left(1+\mathrm{i}\frac{\alpha}{2}\right)^2 : \alpha \in [\lambda_{n-1}+\lambda_n, \lambda_1+\lambda_2]\right\}$.

(b) $D_2(A)$ *is the region enclosed by the four line segments*

$$\big[(1+\mathrm{i}\lambda_1)(1+\mathrm{i}\lambda_2), (1+\mathrm{i}\lambda_2)^2\big], \quad \big[(1+\mathrm{i}\lambda_{n-1})(1+\mathrm{i}\lambda_n), (1+\mathrm{i}\lambda_{n-1})^2\big],$$

$$\big[(1+\mathrm{i}\lambda_1)(1+\mathrm{i}\lambda_n), (1+\mathrm{i}\lambda_1)(1+i\lambda_2)\big], \quad \big[(1+\mathrm{i}\lambda_1)(1+\mathrm{i}\lambda_n), (1+\mathrm{i}\lambda_{n-1})(1+\mathrm{i}\lambda_n)\big],$$

and an arc of a parabola $\left\{\left(1+\mathrm{i}\frac{\alpha}{2}\right)^2 : \alpha \in [2\lambda_{n-1}, 2\lambda_2]\right\}$.

(c) $P_2(A)$ *is the region enclosed by the four line segments*

$$\left[(1+\mathrm{i}\lambda_1)(1+\mathrm{i}\lambda_2), \tfrac{(1+\mathrm{i}\lambda_1)^2}{2} + \tfrac{(1+\mathrm{i}\lambda_2)^2}{2}\right], \left[(1+\mathrm{i}\lambda_{n-1})(1+\mathrm{i}\lambda_n), \tfrac{(1+\mathrm{i}\lambda_{n-1})^2}{2} + \tfrac{(1+\mathrm{i}\lambda_n)^2}{2}\right],$$

$$\left[(1+\mathrm{i}\lambda_1)(1+\mathrm{i}\lambda_n), (1+\mathrm{i}\lambda_1)(1+\mathrm{i}\lambda_2)\right], \quad \left[(1+\mathrm{i}\lambda_1)(1+\mathrm{i}\lambda_n), (1+\mathrm{i}\lambda_{n-1})(1+\mathrm{i}\lambda_n)\right],$$

and two arcs of parabolas

$$\left\{\frac{(1+\mathrm{i}\lambda_n)^2}{2} + \frac{(1+\mathrm{i}(\alpha-\lambda_n))^2}{2} : \alpha \in [\lambda_{n-1}+\lambda_n, \lambda_1+\lambda_n]\right\},$$

$$\left\{\frac{(1+\mathrm{i}\lambda_1)^2}{2} + \frac{(1+\mathrm{i}(\alpha-\lambda_1))^2}{2} : \alpha \in [\lambda_1+\lambda_n, \lambda_1+\lambda_2]\right\}.$$

The authors in [19] presented a characterization of $\Pi_2(A)$ for 3×3 essentially hermitian matrices. Theorem 2.1 (a) extends their result to $n \times n$ essentially hermitian matrices. While an algebraic characterization of $D_2(A)$ for essentially hermitian matrices A is provided in [4], Theorem 2.1 (b) offers a more geometric description of $D_2(A)$. It is known that $D_2(A)$ generally fails to be convex for $n \geq 4$; see [5, 20]. This non-convexity can also be derived from Theorem 2.1 (b). Matrices with non-convex $P_2(A)$ can be constructed by Theorem 2.1 (c) as well. In addition, Theorem 2.1 illustrates the following inclusion relations:

$$D_2(A) \subseteq \Pi_2(A) \subseteq P_2(A)$$

for every essentially hermitian matrix $A \in \mathbf{M}_n$. Note that the first inclusion relation holds for general matrices.

To prove Theorem 2.1, we begin with some observations on essentially hermitian matrices. Let $A = I_n + \mathrm{i}K \in \mathbf{M}_n$ and $V \in \mathbf{M}_{n,2}$ with $V^*V = I_2$. Then

$$\prod_{i=1}^{2}(V^*AV)_{ii} = \prod_{i=1}^{2}(I_2 + \mathrm{i}V^*KV)_{ii} = 1 - \prod_{i=1}^{2}(V^*KV)_{ii} + \mathrm{i}\,\mathrm{tr}(V^*KV).$$

As a result, we have

$$\Pi_2(A) = \left\{1 - \prod_{i=1}^{2}(V^*KV)_{ii} + \mathrm{i}\,\mathrm{tr}(V^*KV) : V \in \mathbf{M}_{n,2},\ V^*V = I_2\right\}.$$

In addition, it is straightforward to show

$$D_2(A) = \{1 - \det(V^*KV) + \mathrm{i}\,\mathrm{tr}(V^*KV) : V \in \mathbf{M}_{n,2},\ V^*V = I_2\},$$

and

$$P_2(A) = \{1 - \mathrm{per}(V^*KV) + \mathrm{i}\,\mathrm{tr}(V^*KV) : V \in \mathbf{M}_{n,2},\ V^*V = I_2\}.$$

Since the generalized interlacing inequalities of hermitian matrices, as given by Fan and Pall [6], play an essential role in this section, we present them here for the reader's reference.

Proposition 2.2. *[6] Let $1 \le k \le n$, $A \in \mathbf{M}_n$ and $B \in \mathbf{M}_k$ be hermitian matrices with eigenvalues $a_1 \ge \cdots \ge a_n$ and $b_1 \ge \cdots \ge b_k$, respectively. There exists $V \in \mathbf{M}_{n,k}$ with $V^*V = I_k$ such that $B = V^*AV$ if and only if*

$$a_j \ge b_j \ge a_{n-k+j}, \quad j = 1, \ldots, k.$$

The following lemma is a quick consequence of Proposition 2.2.

Lemma 2.3. *Let $K \in \mathbf{M}_n$ be a hermitian matrix with eigenvalues $\lambda_1 \ge \cdots \ge \lambda_n$. Then*

$$\left\{\mathrm{tr}(V^*KV) : V \in \mathbf{M}_{n,2},\ V^*V = I_2\right\} = [\lambda_{n-1} + \lambda_n, \lambda_1 + \lambda_2].$$

Lemma 2.3 illustrates that the projections of $\Pi_2(A)$, $D_2(A)$, and $P_2(A)$ on the imaginary axis equal the interval $[\lambda_{n-1} + \lambda_n, \lambda_1 + \lambda_2]$.

Miranda [18] characterized $\Pi_2(A)$ for 2×2 essentially hermitian matrices A.

Proposition 2.4. *[18] Let $A = I_2 + \mathrm{i}K \in \mathbf{M}_2$ where K is a hermitian matrix with eigenvalues $\lambda_1 \ge \lambda_2$. Then*

$$\Pi_2(A) = \left\{1 - z + \mathrm{i}(\lambda_1 + \lambda_2) : z \in \left[\lambda_1\lambda_2, \left(\frac{\lambda_1 + \lambda_2}{2}\right)^2\right]\right\}.$$

We extend Proposition 2.4 as follows.

Lemma 2.5. *Let $A = I_n + \mathrm{i}K \in \mathbf{M}_n$ where K is a hermitian matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. For each $\alpha \in [\lambda_{n-1}+\lambda_n, \lambda_1+\lambda_2]$, we define*

$$\mathcal{L}_\alpha = \begin{cases} [\lambda_n(\alpha-\lambda_n), (\alpha/2)^2\ ] & \text{if } \ \lambda_{n-1}+\lambda_n \leq \alpha \leq \lambda_1+\lambda_n; \\ [\lambda_1(\alpha-\lambda_1), (\alpha/2)^2\ ] & \text{if } \ \ \lambda_1+\lambda_n < \alpha \leq \lambda_1+\lambda_2. \end{cases}$$

Then

$$\Pi_2(A) = \bigcup_{\alpha\in[\lambda_{n-1}+\lambda_n,\lambda_1+\lambda_2]} \{1-z+\mathrm{i}\alpha : z \in \mathcal{L}_\alpha\}.$$

Proof. ($\subseteq$): Let $V \in \mathbf{M}_{n,2}$ and $V^*V = I_2$. Note that

$$\prod_{i=1}^{2}(V^*AV)_{ii} = \prod_{i=1}^{2}(I_2+\mathrm{i}V^*KV)_{ii} \in \Pi_2(I_2+\mathrm{i}V^*KV).$$

Let $\mu_1 \geq \mu_2$ be the eigenvalues of V^*KV and $\alpha = \mathrm{tr}(V^*KV) = \mu_1+\mu_2$. By Proposition 2.4,

$$\Pi_2(I_2+\mathrm{i}V^*KV) = \left\{1-z+\mathrm{i}\alpha : z \in \left[\mu_1\mu_2, (\alpha/2)^2\right]\right\}.$$

It suffices to show that $\mu_1\mu_2 \in \mathcal{L}_\alpha$.

Firstly, suppose that $\lambda_{n-1}+\lambda_n \leq \alpha \leq \lambda_1+\lambda_n$. Then $\mu_1\mu_2 \in \mathcal{L}_\alpha$ if and only if $\mu_1\mu_2 \geq \lambda_n(\alpha-\lambda_n)$. By direct computation

$$\begin{aligned} \mu_1\mu_2 - \lambda_n(\alpha-\lambda_n) &= \mu_1\mu_2 - \lambda_n(\mu_1+\mu_2-\lambda_n) \\ &= (\mu_1-\lambda_n)(\mu_2-\lambda_n) \\ &\geq 0. \end{aligned}$$

The last inequality follows from $\mu_1 \geq \mu_2 \geq \lambda_n$. Secondly, suppose that $\lambda_1+\lambda_n < \alpha \leq \lambda_1+\lambda_2$. We have to show $\mu_1\mu_2 \geq \lambda_1(\alpha-\lambda_1)$. Since $\lambda_1 \geq \mu_1 \geq \mu_2$, it follows by $\mu_1\mu_2 - \lambda_1(\alpha-\lambda_1) = (\mu_1-\lambda_1)(\mu_2-\lambda_1) \geq 0$.

($\supseteq$): Suppose that $\lambda_{n-1}+\lambda_n \leq \alpha \leq \lambda_1+\lambda_n$. By Proposition 2.2, there exists $V \in \mathbf{M}_{n,2}$ with $V^*V = I_2$ such that V^*KV has eigenvalues λ_n and $\alpha-\lambda_n$. By Proposition 2.4,

$$\left\{1-z+\mathrm{i}\alpha : z \in [\lambda_n(\alpha-\lambda_n), (\alpha/2)^2]\right\} = \Pi_2(I_2+\mathrm{i}V^*KV) = \Pi_2(V^*AV) \subseteq \Pi_2(A).$$

The case of $\lambda_1+\lambda_n < \alpha \leq \lambda_1+\lambda_2$ can be shown similarly. ∎

For $D_2(A)$, we have the following characterization.

Lemma 2.6. *Let $A = I_n + \mathrm{i}K \in \mathbf{M}_n$ where K is a hermitian matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. For every $\alpha \in [\lambda_{n-1}+\lambda_n, \lambda_1+\lambda_2]$, let $\mathcal{L}_\alpha = [\ell_\alpha, u_\alpha]$ where*

$$u_\alpha = \begin{cases} \lambda_{n-1}(\alpha-\lambda_{n-1}) & \text{if } \ \lambda_{n-1}+\lambda_n \leq \alpha < 2\lambda_{n-1}; \\ (\alpha/2)^2 & \text{if } \ \ 2\lambda_{n-1} \leq \alpha \leq 2\lambda_2; \\ \lambda_2(\alpha-\lambda_2) & \text{if } \ \ 2\lambda_2 < \alpha \leq \lambda_1+\lambda_2, \end{cases}$$

and

$$\ell_\alpha = \begin{cases} \lambda_n(\alpha-\lambda_n) & \textit{if} \quad \lambda_{n-1}+\lambda_n \le \alpha \le \lambda_1+\lambda_n; \\ \lambda_1(\alpha-\lambda_1) & \textit{if} \quad \lambda_1+\lambda_n < \alpha \le \lambda_1+\lambda_2. \end{cases}$$

Then

$$D_2(A) = \bigcup_{\alpha\in[\lambda_{n-1}+\lambda_n,\lambda_1+\lambda_2]} \{1-z+\mathrm{i}\alpha : z\in\mathcal{L}_\alpha\}.$$

Proof. ($\subseteq$): Let $V\in\mathbf{M}_{n,2}$ and $V^*V=I_2$. Recall that $\det(V^*AV)=1-\det(V^*KV)+\mathrm{i}\operatorname{tr}(V^*KV)$. Assume that $\mu_1\ge\mu_2$ are eigenvalues of V^*KV and $\alpha=\operatorname{tr}(V^*KV)=\mu_1+\mu_2$. It suffices to show that $\det(V^*KV)=\mu_1\mu_2\in\mathcal{L}_\alpha$. From the proof of Lemma 2.5, we have $\ell_\alpha\le\mu_1\mu_2\le(\alpha/2)^2$. Thus, it remains to show that $\mu_1\mu_2\le u_\alpha$ when $\lambda_{n-1}+\lambda_n\le\alpha<2\lambda_{n-1}$ and $2\lambda_2<\alpha\le\lambda_1+\lambda_2$.

Assume that $\lambda_{n-1}+\lambda_n\le\alpha<2\lambda_{n-1}$. The quantity $\mu_1\mu_2=\mu_1(\alpha-\mu_1)$ is a concave quadratic function in μ_1 and is decreasing for $\mu_1\ge\frac{\alpha}{2}$. By Proposition 2.2, we have $\mu_1\ge\lambda_{n-1}\ge\frac{\alpha}{2}$. Therefore $\mu_1\mu_2\le\lambda_{n-1}(\alpha-\lambda_{n-1})$. The case for $2\lambda_2<\alpha\le\lambda_1+\lambda_2$ can be shown similarly.

($\supseteq$): Assume that $\lambda_{n-1}+\lambda_n\le\alpha<2\lambda_{n-1}$. We divide the proof into two cases: $\alpha\le\lambda_1+\lambda_n$ and $\alpha>\lambda_1+\lambda_n$.

Case 1. Suppose that $\alpha\le\lambda_1+\lambda_n$. Define for $t\in[0,1]$, $\mu(t)=t\lambda_{n-1}+(1-t)(\alpha-\lambda_n)$. Note that $\mu(t)$ is in the interval $[\lambda_{n-1},\alpha-\lambda_n]$ and $\lambda_{n-1}\le\alpha-\lambda_n\le\lambda_1$. Hence we have $\lambda_{n-1}\le\mu(t)\le\lambda_1$. In addition, $\mu(t)\le\alpha-\lambda_n$ yields $\lambda_n\le\alpha-\mu(t)$. Since $\alpha<2\lambda_{n-1}$ and $\mu(t)\ge\lambda_{n-1}$, we have $\alpha-\mu(t)<2\lambda_{n-1}-\lambda_{n-1}\le\lambda_2$. By Proposition 2.2, for each $t\in[0,1]$, there exists $V_t\in\mathbf{M}_{n,2}$ with $V_t^*V_t=I_2$ such that $V_t^*KV_t$ has eigenvalues $\mu(t)$ and $\alpha-\mu(t)$. Hence $1-\det(V_t^*KV_t)+\mathrm{i}\alpha=1-\mu(t)(\alpha-\mu(t))+\mathrm{i}\alpha\in D_2(A)$. Since $\mu(0)(\alpha-\mu(0))=\ell_\alpha$ and $\mu(1)(\alpha-\mu(1))=u_\alpha$, by the continuity of the function $\mu(t)$, we have the line segment $\{1-z+\mathrm{i}\alpha : z\in\mathcal{L}_\alpha\}\subseteq D_2(A)$.

Case 2. Suppose $\alpha>\lambda_1+\lambda_n$. Define for $t\in[0,1]$, $\mu(t)=t\lambda_{n-1}+(1-t)\lambda_1$. It clear that $\lambda_{n-1}\le\mu(t)\le\lambda_1$. In addition, $\alpha-\mu(t)\ge\lambda_1+\lambda_n-\lambda_1\ge\lambda_n$ and $\alpha-\mu(t)\le2\lambda_{n-1}-\lambda_{n-1}\le\lambda_2$. Proposition 2.2 asserts that for each $t\in[0,1]$, there exists $V_t\in\mathbf{M}_{n,2}$ with $V_t^*V_t=I_2$ such that $V_t^*KV_t$ has eigenvalues $\mu(t)$ and $\alpha-\mu(t)$. As $\mu(0)(\alpha-\mu(0))=\ell_\alpha$ and $\mu(1)(\alpha-\mu(1))=u_\alpha$, by the continuity of the function $\mu(t)$, we have $\{1-z+\mathrm{i}\alpha : z\in\mathcal{L}_\alpha\}\subseteq D_2(A)$.

The cases $2\lambda_{n-1}\le\alpha\le2\lambda_2$ and $2\lambda_2<\alpha\le\lambda_1+\lambda_2$ can be shown similarly. ∎

For 2×2 essentially hermitian matrices A, a characterization of $P_2(A)$ is provided in [11].

Proposition 2.7. *[11] Let $A=I_2+\mathrm{i}K\in\mathbf{M}_2$ where K is a hermitian matrix with eigenvalues $\lambda_1\ge\lambda_2$. Then*

$$P_2(A)=\left\{1-z+\mathrm{i}(\lambda_1+\lambda_2) : z\in\left[\lambda_1\lambda_2,\left(\frac{\lambda_1^2+\lambda_2^2}{2}\right)\right]\right\}.$$

For general $n\times n$ essentially hermitian matrices, we have the following result.

Lemma 2.8. *Let $A = I_n + \mathrm{i}K \in \mathbf{M}_n$ where K is a hermitian matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. For $\alpha \in [\lambda_{n-1} + \lambda_n, \lambda_1 + \lambda_2]$, define*

$$\mathcal{L}_\alpha = \begin{cases} [\lambda_n(\alpha - \lambda_n), ((\alpha - \lambda_n)^2 + \lambda_n^2)/2] & \text{if } \lambda_{n-1} + \lambda_n \leq \alpha \leq \lambda_1 + \lambda_n; \\ [\lambda_1(\alpha - \lambda_1), ((\alpha - \lambda_1)^2 + \lambda_1^2)/2] & \text{if } \lambda_1 + \lambda_n < \alpha \leq \lambda_1 + \lambda_2. \end{cases}$$

Then

$$P_2(A) = \bigcup_{\alpha \in [\lambda_{n-1}+\lambda_n, \lambda_1+\lambda_2]} \{1 - z + \mathrm{i}\alpha : z \in \mathcal{L}_\alpha\}.$$

Proof. ($\subseteq$): Let $V \in \mathbf{M}_{n,2}$ with $V^*V = I_2$. Assume that $\mu_1 \geq \mu_2$ are eigenvalues of V^*KV and $\alpha = \mathrm{tr}(V^*KV) = \mu_1 + \mu_2$. By Proposition 2.7, we have

$$\mathrm{per}(V^*AV) \in P_2(I_2 + V^*KV) = \left\{1 - z + \mathrm{i}(\mu_1 + \mu_2) : z \in \left[\mu_1\mu_2, \left(\frac{\mu_1^2 + \mu_2^2}{2}\right)\right]\right\}.$$

Thus, it suffices to show that $\left[\mu_1\mu_2, \left(\frac{\mu_1^2+\mu_2^2}{2}\right)\right] \subseteq \mathcal{L}_\alpha$. We will demonstrate this for the case $\lambda_{n-1} + \lambda_n \leq \alpha \leq \lambda_1 + \lambda_n$; the second case can be shown similarly.

By Lemma 2.5, we get $\lambda_n(\alpha - \lambda_n) \leq \mu_1\mu_2$. Consequently we have

$$\frac{\mu_1^2 + \mu_2^2}{2} = \frac{(\mu_1 + \mu_2)^2}{2} - \mu_1\mu_2 = \frac{\alpha^2}{2} - \mu_1\mu_2 \leq \frac{\alpha^2}{2} - \lambda_n(\alpha - \lambda_n) = \frac{(\alpha - \lambda_n)^2 + \lambda_n^2}{2}.$$

($\supseteq$): Suppose that $\lambda_{n-1} + \lambda_n \leq \alpha \leq \lambda_1 + \lambda_n$. Since $\lambda_{n-1} \leq \alpha - \lambda_n \leq \lambda_1$, Proposition 2.2 asserts that there exists $V \in \mathbb{C}^{n \times 2}$ with $V^*V = I_2$ such that V^*KV has eigenvalues λ_n and $\alpha - \lambda_n$. Therefore, we have

$$\{1 - z + \mathrm{i}\alpha : z \in \mathcal{L}_\alpha\} = P_2(I_2 + \mathrm{i}V^*KV) \subseteq P_2(A).$$

The remaining case can be shown similarly. ■

Proof of Theorem 2.1. We show the case of $\Pi_2(A)$ only as $D_2(A)$ and $P_2(A)$ can be derived similarly.

In Lemma 2.5, each α in the interval $[\lambda_{n-1} + \lambda_n, \lambda_1 + \lambda_2]$ determines a horizontal line segment of $\Pi_2(A)$ on the complex plane. Thus $\Pi_2(A)$ is the region bounded above and below by the line segments

$$\{1 - z + \mathrm{i}(\lambda_1 + \lambda_2) : z \in \mathcal{L}_{\lambda_1+\lambda_2}\} = \left[(1 + \mathrm{i}\lambda_1)(1 + \mathrm{i}\lambda_2), \left(1 + \mathrm{i}\frac{\lambda_1 + \lambda_2}{2}\right)^2\right]$$

and

$$\{1 - z + \mathrm{i}(\lambda_{n-1} + \lambda_n) : z \in \mathcal{L}_{\lambda_{n-1}+\lambda_n}\} = \left[(1 + \mathrm{i}\lambda_{n-1})(1 + \mathrm{i}\lambda_n), \left(1 + \mathrm{i}\frac{\lambda_{n-1} + \lambda_n}{2}\right)^2\right]$$

respectively. In addition, the lower bounds of $\mathcal{L}_\alpha$ is a continuous function on α and they bound the $\Pi_2(A)$ on the right. More precisely, $\Pi_2(A)$ is bounded on the right by the two line segments

$$\{1-\lambda_1(\alpha-\lambda_1)+\mathrm{i}\alpha : \alpha \in [\lambda_1+\lambda_n, \lambda_1+\lambda_2]\} = \big[(1+\mathrm{i}\lambda_1)(1+\mathrm{i}\lambda_n), (1+\mathrm{i}\lambda_1)(1+\mathrm{i}\lambda_2)\big]$$

and

$$\{1-\lambda_n(\alpha-\lambda_n)+\mathrm{i}\alpha : \alpha \in [\lambda_{n-1}+\lambda_n, \lambda_1+\lambda_n]\} = \big[(1+\mathrm{i}\lambda_1)(1+\mathrm{i}\lambda_n), (1+\mathrm{i}\lambda_n)(1+\mathrm{i}\lambda_{n-1})\big].$$

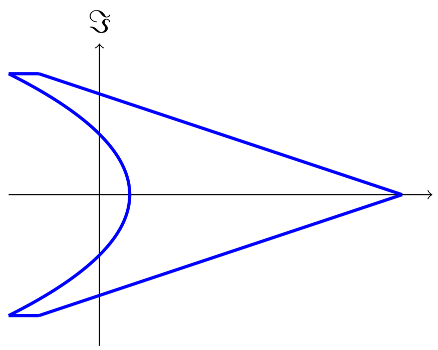
Similarly the upper bounds of $\mathcal{L}_\alpha$ is a quadratic function on α, they give an arc of the parabola which bound $\Pi_2(A)$ on the left, that is,

$$\{1-(\alpha/2)^2+\mathrm{i}\alpha : \alpha \in [\lambda_{n-1}+\lambda_n, \lambda_1+\lambda_2]\} = \left\{\left(1+\mathrm{i}\frac{\alpha}{2}\right)^2 : \alpha \in [\lambda_{n-1}+\lambda_n, \lambda_1+\lambda_2]\right\}.$$

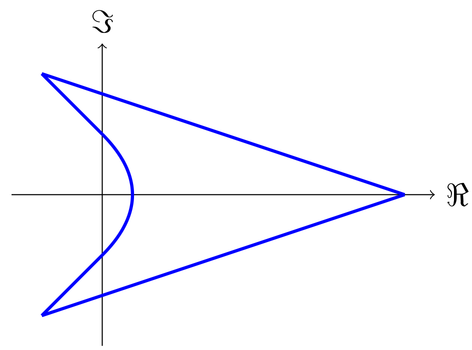
■

We plot $\Pi_2(A)$, $D_2(A)$, and $P_2(A)$ for a 5×5 essentially hermitian matrix below.

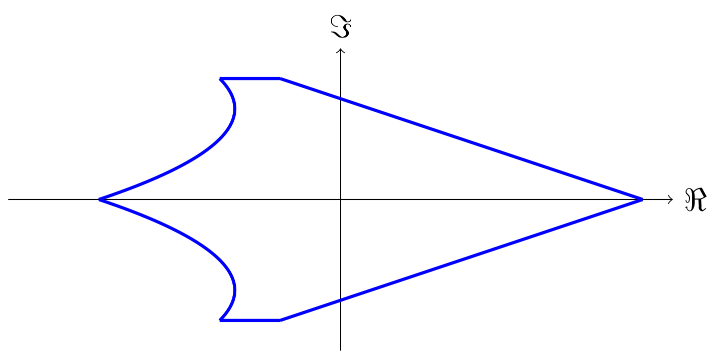
Example 1. Let $A = \mathrm{diag}(1+3\mathrm{i}, 1+\mathrm{i}, 1, 1-\mathrm{i}, 1-3\mathrm{i}) \in \mathbf{M}_5$.



(a) $\Pi_2(A)$



(b) $D_2(A)$



(c) $P_2(A)$

Recall that a set S is star-shaped with a star-center $c \in S$ if $tx + (1-t)c \in S$ for every $x \in S$ and $0 \le t \le 1$. The characterizations in Theorem 2.1 lead to the star-shapedness of $\Pi_2(A)$, $D_2(A)$ and $P_2(A)$.

Theorem 2.9. *Let $A = I_n + \mathrm{i}K \in \mathbf{M}_n$ where K is a hermitian matrix with eigenvalues $\lambda_1 \ge \cdots \ge \lambda_n$. Then $\Pi_2(A)$, $D_2(A)$ and $P_2(A)$ are star-shaped with a star-center $(1+\mathrm{i}\lambda_1)(1+\mathrm{i}\lambda_n)$.*

3. 3×3 Normal matrices

In this section, we consider 3×3 normal matrices A with eigenvalues λ_1, λ_2 and λ_3. We have the following result regarding the star-shapedness of $\Pi_2(A)$.

Theorem 3.1. *Let $A \in \mathbf{M}_3$ be a normal matrix with eigenvalues λ_1, λ_2 and λ_3. Then $\Pi_2(A)$ is star-shaped with star-center $\frac{1}{3}(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3)$.*

In [19], the authors conjectured $\Pi_2(A)$ is always star-shaped for 3×3 normal matrices without proposing a star-center; see also [13, Question 3.3]. Theorem 3.1 settles the conjecture in the affirmative.

We need some notations to prove Theorem 3.1. Define

$$\Lambda_3 = \left\{(\alpha_1, \alpha_2, \alpha_3)^\top \in \mathbb{R}^3 : \alpha_1, \alpha_2, \alpha_3 \geq 0,\, \alpha_1 + \alpha_2 + \alpha_3 = 1\right\}.$$

Each $\alpha \in \Lambda_3$ determines a subset of $\Pi_2(A)$ as follows. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \Lambda_3$ and $x_\alpha = \left(\sqrt{\alpha_1}, \sqrt{\alpha_2}, \sqrt{\alpha_3}\right)^\top \in \mathbb{C}^3$. We define

$$S_A(\alpha) = \left\{(y_1^* A y_1)(y_2^* A y_2) : x_\alpha, y_1, y_2 \in \mathbb{C}^3 \text{ are orthonormal}\right\} \subseteq \Pi_2(A).$$

It is not hard to see that if $Y \in \mathbf{M}_{3,2}$ with $Y^*Y = I_2$ whose range space is $\{x_\alpha\}^\perp$, then $S_A(\alpha) = \Pi_2(Y^*AY)$.

The following lemma characterizes $\Pi_2(A)$ for 3×3 normal matrices A in terms of $S_A(\alpha)$.

Lemma 3.2. *Let $A \in \mathbf{M}_3$ be a normal matrix. Then*

$$\Pi_2(A) = \bigcup_{\alpha \in \Lambda_3} S_A(\alpha).$$

Proof. ($\supseteq$) It is clear by the definition of $S_A(\alpha)$.

($\subseteq$) By unitarily similarity, we assume without loss of generality $A = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$. Let $z \in \Pi_2(A)$. There exist orthonormal $y_1, y_2 \in \mathbb{C}^3$ such that $z = (y_1^* A y_1)(y_2^* A y_2)$. Let $x = (x_1, x_2, x_3)^\top \in \mathbb{C}^3$ be a unit vector orthogonal to y_1 and y_2. We shall show that $z \in S_A(\alpha)$ where $\alpha = (|x_1|^2, |x_2|^2, |x_3|^2) \in \Lambda_3$.

Let $|x| = (|x_1|, |x_2|, |x_3|)^\top$ and D be a diagonal unitary matrix in which $Dx = |x|$. Then $|x|, Dy_1, Dy_2$ are orthonormal vectors in $\mathbb{C}^3$. By the definition of $S_A(\alpha)$, we have

$$z = (y_1^* A y_1)(y_2^* A y_2) = (y_1^* D^* A D y_1)(y_2^* D^* A D y_2) = ((Dy_1)^* A (Dy_1))\,((Dy_2)^* A (Dy_2)) \in S_A(\alpha).$$

$\blacksquare$

A characterization of $\Pi_2(A)$ for any $A \in \mathbf{M}_2$ was given by Hu and Tam [12].

Proposition 3.3. *[12] Let $A \in \mathbf{M}_2$ with eigenvalues μ_1 and μ_2. Then $\Pi_2(A)$ is an elliptical disk with foci at $\mu_1\mu_2$ and $\left(\frac{\mu_1+\mu_2}{2}\right)^2$, and major axis of length $\frac{1}{2}\operatorname{tr}(A^*A) - \left|\frac{\mu_1+\mu_2}{2}\right|^2$.*

To characterize $S_A(\alpha)$, we need the following result.

Lemma 3.4. *Let $A = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$ and $x = (x_1, x_2, x_3)^\top \in \mathbb{C}^3$ be a unit vector. Suppose $Y \in \mathbf{M}_{3,2}$ with $Y^*Y = I_2$ whose range space is $\{x\}^\perp$. If μ_1, μ_2 are eigenvalues of Y^*AY, then $\mu_1\mu_2 = |x_1|^2\lambda_2\lambda_3 + |x_2|^2\lambda_1\lambda_3 + |x_3|^2\lambda_1\lambda_2$ and $\frac{\mu_1+\mu_2}{2} = \frac{1-|x_1|^2}{2}\lambda_1 + \frac{1-|x_2|^2}{2}\lambda_2 + \frac{1+|x_3|}{2}\lambda_3$.*

Proof. Let $\alpha_i = |x_i|^2$, $i = 1, 2, 3$. By [7, Theorem 1], the characteristic polynomial of Y^*AY is

$$\begin{aligned} p(z) &= \alpha_1(z-\lambda_2)(z-\lambda_3) + \alpha_2(z-\lambda_1)(z-\lambda_3) + \alpha_3(z-\lambda_1)(z-\lambda_2) \\ &= z^2 - ((\alpha_2+\alpha_3)\lambda_1 + (\alpha_1+\alpha_3)\lambda_2 + (\alpha_1+\alpha_2)\lambda_3)z + \alpha_1\lambda_2\lambda_3 + \alpha_2\lambda_1\lambda_3 + \alpha_3\lambda_1\lambda_2 \\ &= z^2 - \left(\frac{1-\alpha_1}{2}\lambda_1 + \frac{1-\alpha_2}{2}\lambda_2 + \frac{1+\alpha_3}{2}\lambda_3\right) z + \alpha_1\lambda_2\lambda_3 + \alpha_2\lambda_1\lambda_3 + \alpha_3\lambda_1\lambda_2 \end{aligned}$$

The result follows straightforwardly. ∎

By Proposition 3.3 and Lemma 3.4, we present a characterization of $S_A(\alpha)$.

Lemma 3.5. *Let $A = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \Lambda_3$. Then $S_A(\alpha)$ is an elliptical disk with foci at $\alpha_1\lambda_2\lambda_3 + \alpha_2\lambda_1\lambda_3 + \alpha_3\lambda_1\lambda_2$ and $\left(\frac{1-\alpha_1}{2}\lambda_1 + \frac{1-\alpha_2}{2}\lambda_2 + \frac{1-\alpha_3}{2}\lambda_3\right)^2$, and major axis of length $\frac{1}{2}\sum_{i=1}^3 (1-\alpha_i)^2|\lambda_i|^2 + \sum_{1\le i<j\le 3} \alpha_i\alpha_j \operatorname{Re}\left(\lambda_i\overline{\lambda}_j\right) - \left|\frac{1-\alpha_1}{2}\lambda_1 + \frac{1-\alpha_2}{2}\lambda_2 + \frac{1-\alpha_3}{2}\lambda_3\right|^2$.*

Proof. Given $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \Lambda_3$, we let $x_\alpha = (\sqrt{\alpha_1}, \sqrt{\alpha_2}, \sqrt{\alpha_3})^\top \in \mathbb{C}^3$. Note that $S_A(\alpha) = \Pi_2(Y^*AY)$ where $Y \in \mathbf{M}_{3,2}$ with $Y^*Y = I_2$ whose range space is $\{x_\alpha\}^\perp$. Let μ_1, μ_2 be eigenvalues of Y^*AY. Lemma 3.4 shows $\mu_1\mu_2 = \alpha_1\lambda_2\lambda_3 + \alpha_2\lambda_1\lambda_3 + \alpha_3\lambda_1\lambda_2$ and $\frac{\mu_1+\mu_2}{2} = \frac{1-\alpha_1}{2}\lambda_1 + \frac{1-\alpha_2}{2}\lambda_2 + \frac{1+\alpha_3}{2}\lambda_3$.

To find the length of major axis, we have

$$YY^* = I_3 - x_\alpha x_\alpha^* = \begin{bmatrix} 1-\alpha_1 & -\sqrt{\alpha_1\alpha_2} & -\sqrt{\alpha_1\alpha_3} \\ -\sqrt{\alpha_1\alpha_2} & 1-\alpha_2 & -\sqrt{\alpha_2\alpha_3} \\ -\sqrt{\alpha_1\alpha_3} & -\sqrt{\alpha_2\alpha_3} & 1-\alpha_3 \end{bmatrix},$$

and

$$\begin{aligned} \operatorname{tr}(Y^*A^*YY^*AY) &= \operatorname{tr}(YY^*A^*YY^*A) \\ &= \sum_{i=1}^3 (1-\alpha_i)^2|\lambda_i|^2 + \sum_{1\le i<j\le 3} \alpha_i\alpha_j \operatorname{Re}(\lambda_i\overline{\lambda}_j). \end{aligned}$$

Then the result follows from Proposition 3.3. ∎

For a normal matrix $A \in \mathbf{M}_3$ with eigenvalues λ_1, λ_2 and λ_3, we define

$$e_2(A) = \frac{1}{3}(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3).$$

The main technical result of this section is as follows.

Theorem 3.6. *Let $A = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$ and $J = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Then for every $\alpha \in \Lambda_3$ and $0 \le t \le 1$, the following inclusion relation holds*

$$tS_A(\alpha) + (1-t)e_2(A) \subseteq S_A(t\alpha + (1-t)J).$$

We need the two following lemmas to prove our main result.

Lemma 3.7. *Let $A=\operatorname{diag}(\lambda_1,\lambda_2,\lambda_3)$ and $J=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$. For every $\alpha\in\Lambda_3$, $0\le t\le 1$ and $\lambda\in\mathbb{C}$, if*

$$tS_{A-\lambda I}(\alpha)+(1-t)e_2(A-\lambda I)\subseteq S_{A-\lambda I}(t\alpha+(1-t)J),$$

then

$$tS_A(\alpha)+(1-t)e_2(A)\subseteq S_A(t\alpha+(1-t)J).$$

Proof. Let $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in\Lambda_3$, $0\le t\le 1$, and $\lambda\in\mathbb{C}$. Define $x=(x_1,x_2,x_3)^\top$, $\tilde{x}=(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)^\top\in\mathbb{C}^3$ where $x_i=\sqrt{\alpha_i}$ and $\tilde{x}_i=\sqrt{t\alpha_i+(1-t)/3}$, $i=1,2,3$. For every $\eta\in S_A(\alpha)$, there exist $y_1,y_2\in\mathbb{C}^3$ in which y_1,y_2,x are orthonormal and $\eta=(y_1^*Ay_1)(y_2^*Ay_2)$. By the inclusion relation assumption, there exist $\tilde{y}_1,\tilde{y}_2\in\mathbb{C}^3$ such that $\tilde{y}_1,\tilde{y}_2,\tilde{x}$ are orthonormal and

$$t\left(y_1^*(A-\lambda I)y_1\right)\left(y_2^*(A-\lambda I)y_2\right)+(1-t)e_2(A-\lambda I)=\left(\tilde{y}_1^*(A-\lambda I)\tilde{y}_1\right)\left(\tilde{y}_2^*(A-\lambda I)\tilde{y}_2\right).$$

For simplicity, denote $z_1=y_1^*Ay_1$, $z_2=y_2^*Ay_2$, $\tilde{z}_1=\tilde{y}_1^*A\tilde{y}_1$ and $\tilde{z}_2=\tilde{y}_2^*A\tilde{y}_2$. Since

$$x^*Ax+z_1+z_2=\operatorname{tr}A=\tilde{x}^*A\tilde{x}+\tilde{z}_1+\tilde{z}_2\quad\text{and}\quad\tilde{x}^*A\tilde{x}=tx^*Ax+\frac{1-t}{3}\operatorname{tr}A,$$

we have

$$\tilde{z}_1+\tilde{z}_2=\operatorname{tr}A-\tilde{x}^*A\tilde{x}=t(\operatorname{tr}A-x^*Ax)+\frac{2(1-t)}{3}\operatorname{tr}A=t(z_1+z_2)+\frac{2(1-t)}{3}\operatorname{tr}A.$$

By direct computation,

$$\begin{aligned}
&t\left(y_1^*(A-\lambda I)y_1\right)\left(y_2^*(A-\lambda I)y_2\right)+(1-t)e_2(A-\lambda I)\\
=\;&t\left(z_1z_2-\lambda(z_1+z_2)+\lambda^2\right)+(1-t)e_2(A)-\frac{2(1-t)}{3}\lambda\operatorname{tr}A+(1-t)\lambda^2\\
=\;&tz_1z_2+(1-t)e_2(A)-\lambda\left(t(z_1+z_2)+\frac{2(1-t)}{3}\operatorname{tr}A\right)+\lambda^2\\
=\;&tz_1z_2+(1-t)e_2(A)-\lambda\left(\tilde{z}_1+\tilde{z}_2\right)+\lambda^2
\end{aligned}$$

and on the other hand

$$\left(\tilde{y}_1^*(A-\lambda I)\tilde{y}_1\right)\left(\tilde{y}_2^*(A-\lambda I)\tilde{y}_2\right)=\tilde{z}_1\tilde{z}_2-\lambda\left(\tilde{z}_1+\tilde{z}_2\right)+\lambda^2.$$

Hence $t\eta+(1-t)e_2(A)=tz_1z_2+(1-t)e_2(A)=\tilde{z}_1\tilde{z}_2\in S_A(t\alpha+(1-t)J)$. ∎

Lemma 3.8. *Let $a,c>0$, $b\in\mathbb{R}$, $ac\ge b^2$ and $\lambda\in\mathbb{C}$. Then*

$$|a-2b\lambda+c\lambda^2|\le a-2b\operatorname{Re}(\lambda)+c|\lambda|^2.$$

Proof. By direct computation, we have

$$\begin{aligned}
|a-2b\lambda+c\lambda^2| &= \left|\left(\sqrt{a}-\frac{b}{\sqrt{a}}\lambda\right)^2+\left(c-\frac{b^2}{a}\right)\lambda^2\right| \\
&\le \left|\left(\sqrt{a}-\frac{b}{\sqrt{a}}\lambda\right)\right|^2+\left(c-\frac{b^2}{a}\right)|\lambda|^2 \\
&= a-2b\mathrm{Re}\,(\lambda)+\frac{b^2}{a}|\lambda|^2+\left(c-\frac{b^2}{a}\right)|\lambda|^2 \\
&= a-2b\mathrm{Re}\,(\lambda)+c|\lambda|^2.
\end{aligned}$$

■

Proof of Theorem 3.6. The result is clear if A is a scalar matrix. Assume A is non-scalar. By Lemma 3.7, we may assume that $\lambda_1=0$ and $\lambda_1\neq\lambda_2$. In addition, one may replace A by A/λ_2 and further assume that $A=\mathrm{diag}(0,1,\lambda)$ with $\lambda\in\mathbb{C}$. Hence $e_2(A)=\frac{\lambda}{3}$.

By Lemma 3.5, for every $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in\Lambda_3$, the set $S_A(\alpha)$ is an elliptical disk with foci at $\alpha_1\lambda$ and $\left(\frac{1-\alpha_2}{2}+\frac{1-\alpha_3}{2}\lambda\right)^2$, and major axis of length

$$\begin{aligned}
&\frac{1}{2}(1-\alpha_2)^2+\frac{1}{2}(1-\alpha_3)^2|\lambda|^2+\alpha_2\alpha_3\mathrm{Re}\,(\lambda)-\left|\frac{1-\alpha_2}{2}+\frac{1-\alpha_3}{2}\lambda\right|^2 \\
=&\left|\frac{1-\alpha_2}{2}+\frac{1-\alpha_3}{2}\lambda\right|^2+\alpha_2\alpha_3\mathrm{Re}\,(\lambda)-(1-\alpha_2)(1-\alpha_3)\mathrm{Re}\,(\lambda) \\
=&\left|\frac{1-\alpha_2}{2}+\frac{1-\alpha_3}{2}\lambda\right|^2-\alpha_1\mathrm{Re}\,(\lambda).
\end{aligned}$$

As a consequence, the set $tS_A(\alpha)+(1-t)e_2(A)$ is an elliptical disk with foci $f_1=t\alpha_1+(1-t)\frac{\lambda}{3}$, $f_2=t\left(\frac{1-\alpha_2}{2}+\frac{1-\alpha_3}{2}\lambda\right)^2+(1-t)\frac{\lambda}{3}$, and major axis of length $M=t\left|\frac{1-\alpha_2}{2}+\frac{1-\alpha_3}{2}\lambda\right|^2-t\alpha_1\mathrm{Re}\,(\lambda)$.

On the other hand, similar computation shows that $S_A(t\alpha+(1-t)J)$ is an elliptical disk with foci $\tilde{f}_1=t\alpha_1+(1-t)\frac{\lambda}{3}$, $\tilde{f}_2=\left(\frac{1-t\alpha_2-\frac{1-t}{3}}{2}+\frac{1-t\alpha_3-\frac{1-t}{3}}{2}\lambda\right)^2$, and major axis of length $\tilde{M}=\left|\frac{1-t\alpha_2-\frac{1-t}{3}}{2}+\frac{1-t\alpha_3-\frac{1-t}{3}}{2}\lambda\right|^2-\left(t\alpha_1+\frac{1-t}{3}\right)\mathrm{Re}\,(\lambda)$.

Let $z\in tS_A(\alpha)+(1-t)\frac{\lambda}{3}$. Equivalently $|z-f_1|+|z-f_2|\le M$. To prove $z\in S_A(t\alpha+(1-t)J)$, we shall show that $|z-\tilde{f}_1|+|z-\tilde{f}_2|\le\tilde{M}$. Since $f_1=\tilde{f}_1$, we have

$$\begin{aligned}
|z-\tilde{f}_1|+|z-\tilde{f}_2| &\le M-|z-f_2|+|z-\tilde{f}_2| \\
&\le M+|\tilde{f}_2-f_2| \\
&= \tilde{M}+\left(|\tilde{f}_2-f_2|+M-\tilde{M}\right).
\end{aligned}$$

It suffices to show that $|\tilde{f}_2 - f_2| + M - \tilde{M} \le 0$. By direct computation, we have

$$
\begin{aligned}
& M - \tilde{M} \\
= \;& t\left|\tfrac{1-\alpha_2}{2} + \tfrac{1-\alpha_3}{2}\lambda\right|^2 - t\alpha_1 \mathrm{Re}\,(\lambda) - \left|\frac{1-t\alpha_2-\frac{1-t}{3}}{2} + \frac{1-t\alpha_3-\frac{1-t}{3}}{2}\lambda\right|^2 + \left(t\alpha_1 + \tfrac{1-t}{3}\right)\mathrm{Re}\,(\lambda) \\
= \;& t\left|\tfrac{1-\alpha_2}{2} + \tfrac{1-\alpha_3}{2}\lambda\right|^2 - \left|\tfrac{t(1-\alpha_2)}{2} + \tfrac{1-t}{3} + \left(\tfrac{t(1-\alpha_3)}{2} + \tfrac{1-t}{3}\right)\lambda\right|^2 + \tfrac{1-t}{3}\mathrm{Re}\,(\lambda) \\
= \;& \tfrac{t(1-\alpha_2)^2}{4} - \left(\tfrac{t(1-\alpha_2)}{2} + \tfrac{1-t}{3}\right)^2 + \tfrac{t(1-\alpha_3)^2}{4}|\lambda|^2 - \left(\tfrac{t(1-\alpha_3)}{2} + \tfrac{1-t}{3}\right)^2|\lambda|^2 \\
& \quad +2\left(t\left(\tfrac{1-\alpha_2}{2}\right)\left(\tfrac{1-\alpha_3}{2}\right) - \left(\tfrac{t(1-\alpha_2)}{2} + \tfrac{1-t}{3}\right)\left(\tfrac{t(1-\alpha_3)}{2} + \tfrac{1-t}{3}\right)\right)\mathrm{Re}\,(\lambda) + \tfrac{1-t}{3}\mathrm{Re}\,(\lambda) \\
= \;& t(1-t)\left(\tfrac{1-\alpha_2}{2} - \tfrac{1}{3}\right)^2 - \tfrac{1-t}{9} + t(1-t)\left(\tfrac{1-\alpha_3}{2} - \tfrac{1}{3}\right)^2|\lambda|^2 - \tfrac{1-t}{9}|\lambda|^2 \\
& \quad +2t(1-t)\left(\tfrac{1-\alpha_2}{2} - \tfrac{1}{3}\right)\left(\tfrac{1-\alpha_3}{2} - \tfrac{1}{3}\right)\mathrm{Re}\,(\lambda) + \tfrac{1-t}{9}\mathrm{Re}\,(\lambda) \\
= \;& t(1-t)\left|\left(\tfrac{1-\alpha_2}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1-\alpha_3}{2} - \tfrac{1}{3}\right)\lambda\right|^2 - \tfrac{1-t}{9}\left(1 + |\lambda|^2 - \mathrm{Re}\,(\lambda)\right) \\
= \;& (1-t)\left|t\left(\tfrac{1-\alpha_2}{2} + \tfrac{1-\alpha_3}{2}\lambda - \tfrac{1+\lambda}{3}\right)^2\right| - \tfrac{1-t}{9}\left(1 + |\lambda|^2 - \mathrm{Re}\,(\lambda)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
& |\tilde{f}_2 - f_2| \\
= \;& \left|\left(\frac{1-t\alpha_2-\frac{1-t}{3}}{2} + \frac{1-t\alpha_3-\frac{1-t}{3}}{2}\lambda\right)^2 - t\left(\tfrac{1-\alpha_2}{2} + \tfrac{1-\alpha_3}{2}\lambda\right)^2 - (1-t)\tfrac{\lambda}{3}\right| \\
= \;& \left|\left(t\left(\tfrac{1-\alpha_2}{2} + \tfrac{1-\alpha_3}{2}\lambda\right) + \tfrac{1-t}{3}(1+\lambda)\right)^2 - t\left(\tfrac{1-\alpha_2}{2} + \tfrac{1-\alpha_3}{2}\lambda\right)^2 - (1-t)\tfrac{\lambda}{3}\right| \\
= \;& \left|t(t-1)\left(\tfrac{1-\alpha_2}{2} + \tfrac{1-\alpha_3}{2}\lambda\right)^2 - \tfrac{2t(t-1)}{3}(1+\lambda)\left(\tfrac{1-\alpha_2}{2} + \tfrac{1-\alpha_3}{2}\lambda\right) + \tfrac{(1-t)^2}{9}(1+\lambda)^2 - (1-t)\tfrac{\lambda}{3}\right| \\
= \;& \left|t(t-1)\left(\tfrac{1-\alpha_2}{2} + \tfrac{1-\alpha_3}{2}\lambda - \tfrac{1+\lambda}{3}\right)^2 + \tfrac{1-t}{9}(1+\lambda)^2 - (1-t)\tfrac{\lambda}{3}\right| \\
= \;& (1-t)\left|t\left(\tfrac{1-\alpha_2}{2} + \tfrac{1-\alpha_3}{2}\lambda - \tfrac{1+\lambda}{3}\right)^2 - \tfrac{1-\lambda+\lambda^2}{9}\right|.
\end{aligned}
$$

Hence, $|\tilde{f}_2 - f_2| + M - \tilde{M} \le 0$ if and only if

$$
\left|t\left(\tfrac{1-\alpha_2}{2} + \tfrac{1-\alpha_3}{2}\lambda - \tfrac{1+\lambda}{3}\right)^2 - \tfrac{1-\lambda+\lambda^2}{9}\right| + \left|t\left(\tfrac{1-\alpha_2}{2} + \tfrac{1-\alpha_3}{2}\lambda - \tfrac{1+\lambda}{3}\right)^2\right| \le \tfrac{1}{9}\left(1 + |\lambda|^2 - \mathrm{Re}\,(\lambda)\right).
$$

The inequality is equivalent to the condition that the number $t\left(\tfrac{1-\alpha_2}{2} + \tfrac{1-\alpha_3}{2}\lambda - \tfrac{1+\lambda}{3}\right)^2$ lies inside the ellipse with foci $\tfrac{1-\lambda+\lambda^2}{9}$ and 0, and major axis of length $\tfrac{1}{9}\left(1 + |\lambda|^2 - \mathrm{Re}\,(\lambda)\right)$. Hence it suffices to show that this holds for $t = 1$. Note that

$$
\begin{aligned}
& \left|\left(\tfrac{1-\alpha_2}{2} + \tfrac{1-\alpha_3}{2}\lambda - \tfrac{1+\lambda}{3}\right)^2 - \tfrac{1-\lambda+\lambda^2}{9}\right| \\
= \;& \tfrac{1}{9}\left|1 - \left(\tfrac{1}{2} - \tfrac{3\alpha_2}{2}\right)^2 - \left(2\left(\tfrac{1}{2} - \tfrac{3\alpha_2}{2}\right)\left(\tfrac{1}{2} - \tfrac{3\alpha_3}{2}\right) + 1\right)\lambda + \left(1 - \left(\tfrac{1}{2} - \tfrac{3\alpha_2}{2}\right)^2\right)\lambda^2\right|
\end{aligned}
$$

and

$$\begin{aligned}
&\tfrac{1}{9}\left(1+|\lambda|^2-\mathrm{Re}\,(\lambda)\right)-\left|\left(\tfrac{1-\alpha_2}{2}+\tfrac{1-\alpha_3}{2}\lambda-\tfrac{1+\lambda}{3}\right)^2\right| \\
= \;& \tfrac{1}{9}\left(1-\left(\tfrac{1}{2}-\tfrac{3\alpha_2}{2}\right)^2-\left(2\left(\tfrac{1}{2}-\tfrac{3\alpha_2}{2}\right)\left(\tfrac{1}{2}-\tfrac{3\alpha_3}{2}\right)+1\right)\mathrm{Re}\,(\lambda)+\left(1-\left(\tfrac{1}{2}-\tfrac{3\alpha_3}{2}\right)^2\right)|\lambda|^2\right).
\end{aligned}$$

Now let $a=1-\left(\frac{1}{2}-\frac{3\alpha_2}{2}\right)^2$, $2b=2\left(\frac{1}{2}-\frac{3\alpha_2}{2}\right)\left(\frac{1}{2}-\frac{3\alpha_3}{2}\right)+1$ and $c=1-\left(\frac{1}{2}-\frac{3\alpha_3}{2}\right)^2$. The inequality is equivalent to

$$|a-2b\lambda+c\lambda^2|\le a+2b\mathrm{Re}\,(\lambda)+c|\lambda|^2.$$

Note that $a=\left(1-\left(\frac{1}{2}-\frac{3\alpha_2}{2}\right)^2\right)\ge 0$ and equality holds if and only if $\alpha_2=1$. If $\alpha_2=1$, then $\alpha_3=0=a=b$, $c=\frac{3}{4}$,

$$|a-2b\lambda+c\lambda^2|=\frac{3}{4}|\lambda|^2=a+2b\mathrm{Re}\,(\lambda)+c|\lambda|^2.$$

Similar argument works for the case $c=0$. Now assume $a,c>0$. To apply Lemma 3.8, we need to check $ac\ge b^2$. It follows by

$$\begin{aligned}
ac-b^2 &= \frac{4ac-(2b)^2}{4} \\
&= \tfrac{1}{4}\left(4\left(1-\left(\tfrac{1}{2}-\tfrac{3\alpha_2}{2}\right)^2\right)\left(1-\left(\tfrac{1}{2}-\tfrac{3\alpha_3}{2}\right)^2\right)-\left(2\left(\tfrac{1}{2}-\tfrac{3\alpha_2}{2}\right)\left(\tfrac{1}{2}-\tfrac{3\alpha_3}{2}\right)+1\right)^2\right) \\
&= \frac{1}{4}\left(3-(1-3\alpha_2)^2-(1-3\alpha_3)^2-(1-3\alpha_2)(1-3\alpha_3)\right) \\
&= \tfrac{9}{4}\left(\alpha_2+\alpha_3-(\alpha_2+\alpha_3)^2+\alpha_2\alpha_3\right) \\
&\ge 0
\end{aligned}$$

where the inequality follows by $\alpha_2+\alpha_3\le 1$. This completes the proof. ■

Proof of Theorem 3.1. Let $z\in\Pi_2(A)$, $0\le t\le 1$ and $e_2(A)=\dfrac{1}{3}(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3)$. By Lemma 3.2, $z\in S_A(\alpha)$ for some $\alpha\in\Lambda_3$. By Theorem 3.6,

$$tz+(1-t)e_2(A)\in tS_A(\alpha)+(1-t)e_2(A)\subseteq S_A(t\alpha+(1-t)J)\subseteq\Pi_2(A).$$

■

By Theorem 3.6, $P_2(A)$ is star-shaped for 3×3 normal matrices A.

Theorem 3.9. *Let $A\in\mathbf{M}_3$ be a normal matrix with eigenvalues λ_1,λ_2 and λ_3. Then $P_2(A)$ is star-shaped with star-center $e_2(A)=\dfrac{1}{3}(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3)$.*

Proof. Let $V \in \mathbf{M}_{3,2}$ with $V^*V = I_2$ and $0 \le t \le 1$. Write $V^*AV = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then

$$\operatorname{per}(V^*AV) = 2a_{11}a_{22} - \det(V^*AV).$$

Let $x = (x_1, x_2, x_3)^\top$ be a unit vector with $V^*x = 0$ and let $\alpha = (|x_1|^2, |x_2|^2, |x_3|^2) \in \Lambda_3$. Then $\det(V^*AV) = |x_1|^2\lambda_2\lambda_3 + |x_2|^2\lambda_1\lambda_3 + |x_3|^2\lambda_1\lambda_2$ and $a_{11}a_{22} \in S_A(\alpha)$. Let $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)^\top$ where $\tilde{x}_i = \sqrt{t|x_i|^2 + (1-t)\frac{1}{3}}$, $i = 1, 2, 3$. By Theorem 3.6, there exist $\tilde{y}_1, \tilde{y}_2 \in \mathbb{C}^3$ such that $\tilde{x}, \tilde{y}_1, \tilde{y}_2$ are orthonormal and $ta_{11}a_{22} + (1-t)e_2(A) = (\tilde{y}_1^* A\tilde{y}_1)(\tilde{y}_2^* A\tilde{y}_2)$. Set $Y = [\tilde{y}_1, \tilde{y}_2] \in \mathbf{M}_{3,2}$. We shall show that $t\operatorname{per}(V^*AV) + (1-t)e_2(A) = \operatorname{per}(Y^*AY)$. Since

$$\det(Y^*AY) = |\tilde{x}_1|^2\lambda_2\lambda_3 + |\tilde{x}_2|^2\lambda_1\lambda_3 + |\tilde{x}_3|^2\lambda_1\lambda_2 = t\det(V^*AV) + (1-t)e_2(A),$$

we have

$$\begin{aligned} t\operatorname{per}(V^*AV) + (1-t)e_2(A) &= 2ta_{11}a_{22} - t\det(V^*AV) + (1-t)e_2(A) \\ &= 2(y_1^*Ay_1)(y_2^*Ay_2) - (t\det(V^*AV) + (1-t)e_2(A)) \\ &= 2(y_1^*Ay_1)(y_2^*Ay_2) - \det(Y^*AY) \\ &= \operatorname{per}(Y^*AY) \\ &\in P_2(A). \end{aligned}$$

■

By the theory of compound matrices and decomposable tensor, one can show that $D_{n-1}(A)$ equals the classical numerical range of the $(n-1)$-th compound matrix of A and is always convex; see [16]. We can use Lemma 3.4 to deduce this result for 3×3 normal matrices.

Theorem 3.10. *Let $A \in \mathbf{M}_3$ be a normal matrix with eigenvalues λ_1, λ_2 and λ_3. Then*

$$D_2(A) = \operatorname{conv}\{\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3\}.$$

Proof. ($\subseteq$) Let $V \in \mathbf{M}_{3,2}$ and $V^*V = I_2$. There exists a unit vector $x = (x_1, x_2, x_3)^\top \in \mathbb{C}^3$ such that $V^*x = 0$. Suppose μ_1 and μ_2 are eigenvalues of V^*AV. By Lemma 3.4, we have

$$\det(V^*AV) = \mu_1\mu_2 = |x_1|^2\lambda_2\lambda_3 + |x_2|^2\lambda_1\lambda_3 + |x_3|^2\lambda_1\lambda_2 \in \operatorname{conv}\{\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3\}.$$

($\supseteq$) Suppose $z = \alpha_1\lambda_1\lambda_2 + \alpha_2\lambda_2\lambda_3 + \alpha_3\lambda_1\lambda_3$ where $(\alpha_1, \alpha_2, \alpha_3) \in \Lambda_3$. Then for $x = (\sqrt{\alpha_2}, \sqrt{\alpha_3}, \sqrt{\alpha_1})^\top$, there exists a matrix $V \in \mathbf{M}_{3,2}$ such that $V^*V = I_2$ and $V^*x = 0$. By Lemma 3.4, $z = \det(V^*AV) \in D_2(A)$. ■

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