# A simple proof of the Craig-Sakamoto Theorem

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**Abstract**

We give a simple proof of the Craig-Sakamoto Theorem, which asserts that two real symmetric matrices $A$ and $B$ satisfy $\det(I - aA - bB) = \det(I - aA)\det(I - bB)$ for all real numbers $a$ and $b$ if and only if $AB = 0$.

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The Craig-Sakamoto Theorem on the independence of two quadratic forms can be stated as follows.

**Theorem 1** *Two $n \times n$ real symmetric matrices $A$ and $B$ satisfy*

$$\det(I - aA - bB) = \det(I - aA)\det(I - bB) \qquad \forall\ a, b \in \mathbf{R} \tag{1}$$

*if and only if $AB = 0$.*

One may see [1,3] for the history and the importance of this result, and see [2,4,5,6] for several proofs of it. The purpose of this note is to give a simple proof of Theorem 1. Our proof depends only on the following well known fact.

**Lemma 2** *Suppose $C = (c_{ij})$ is an $n \times n$ real symmetric matrix with the largest eigenvalue equal to $\lambda_1$. Then $c_{ii} \le \lambda_1$ for all $i = 1, \dots, n$. If $c_{ii} = \lambda_1$, then $c_{ij} = 0 = c_{ji}$ for all $j \ne i$.*

For the sake of completeness, we give a short

**Proof of Lemma 2** Suppose $C$ satisfies the hypothesis of the lemma and the largest eigenvalue of $C$ has multiplicity $m$ with $1 \le m \le n$. Then there is an orthonormal basis $\{v_1, \dots, v_n\}$ for $\mathbf{R}^n$ such that $Cv_j = \lambda_j v_j$ with $\lambda_1 = \cdots = \lambda_m > \lambda_{m+1} \ge \cdots \ge \lambda_n$. Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbf{R}^n$. For any $i$ with $1 \le i \le n$, there exist $t_1, \dots, t_n \in \mathbf{R}$ with $\sum_{j=1}^n t_j^2 = 1$ such that $e_i = \sum_{j=1}^n t_j v_j$ and $c_{ii} = e_i^t C e_i = \sum_{j=1}^n t_j^2 \lambda_j \le \lambda_1$. The equality holds if and only if $t_{m+1} = \cdots = t_n = 0$, i.e., $e_i$ is an eigenvector of $C$ corresponding to the largest eigenvalue. Thus, $Ce_i = \lambda_1 e_i$, and hence $c_{ii} = \lambda_1$ is the only nonzero entry in the $i$th column. Since $C$ is symmetric, $c_{ii}$ is also the only nonzero entry in the $i$th row. □

We are now ready to present our

**Proof of Theorem 1** The ($\Leftarrow$) part is clear. We prove the converse by induction on $n$. The result is clear if $n = 1$. Suppose $n > 1$ and the result is true for symmetric matrices of sizes smaller than $n$. Let $A$ and $B$ be nonzero $n \times n$ real symmetric matrices satisfying (1). Denote by $\rho(C)$ the spectral radius of a square matrix $C$. Replacing $A$ by $\pm A/\rho(A)$ and $B$ by $B/\rho(B)$, we may assume that $1 = \rho(A) = \rho(B)$ is the largest eigenvalue of $A$. Let $Q$ be an orthogonal matrix such that $QAQ^t = I_m \oplus \operatorname{diag}(a_{m+1}, \dots, a_n)$ with $1 > a_{m+1} \ge \cdots \ge a_n$. We shall show that $(QAQ^t)(QBQ^t) = 0$ and hence $AB = 0$.

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For simplicity, we assume that $Q = I$. Let $b = \pm 1$. If $r > 1$, then both $A/r$ and $bB/r$ have eigenvalues in the open interval $(-1, 1)$. Thus, $I - A/r$ and $I - bB/r$ are invertible, and

$$\det(I - A/r - bB/r) = \det(I - A/r)\det(I - bB/r) \neq 0.$$

Moreover, since

$$\det(I - A - bB) = \det(I - A)\det(I - bB) = 0,$$

we see that 1 is the largest eigenvalue of the matrix $A + bB$ for $b = \pm 1$.

Next, we show that $B$ is of the form $0_m \oplus B_2$. Note that all the first $m$ diagonal entries of $A$ are equal to the largest eigenvalue of $A \pm B$. If the first $m$ diagonal entries of $B$ are not all 0, then the matrix $A + B$ or $A - B$ will have a diagonal entry larger than 1, contradicting Lemma 2. So, all the first $m$ diagonal entries of the matrix $A+B$ equal the largest eigenvalue. By Lemma 2 again, $A + B$ must be of the form $I_m \oplus C_2$. Hence, $B$ is of the form $0_m \oplus B_2$ as asserted.

Now, let $A = I_m \oplus A_2$. Then for any real numbers $a$ and $b$ with $a \neq 1$, we have

$$\begin{aligned} & \det(I_{n-m} - aA_2 - bB_2) \\ = \; & \det(I_n - aA - bB)/\det(I_m - aI_m) \\ = \; & \det(I_n - aA)\det(I_n - bB)/\det(I_m - aI_m) \\ = \; & \det(I_{n-m} - aA_2)\det(I_{n-m} - bB_2). \end{aligned}$$

By continuity, we can remove the restriction that $a \neq 1$. Using the induction assumption, we see that $A_2 B_2 = 0$. Hence, we have $AB = 0$ as desired. □

# References


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